On Optimal Quantile Regression

Mei Ling Huang*and Christine Nguyen†

[a]Department of Mathematics & Statistics, Brock University,
St. Catharines, Ontario, Canada

September 13, 2016

Abstract

In recent years, studies of heavy tailed distributions have rapidly developed. For multivariate heavy tailed distributions, estimation of conditional quantiles at very high or low tailes is of interest in numerous applications. Quantile regression uses an L_1- loss function, and the optimal solution of linear program for estimating coefficients of regression. This paper proposes a weighted quantile regression method for certain extreme value sets. The Monte Carlo simulations show good results for the proposed weighted method. Comparisons of the proposed method and existing methods are given. The paper also investigates a CO_2 Emission real-world example by using the proposed weighted method.

Keywords: CO_2 Emission, conditional quantile, extreme value distribution, generalized Pareto distribution, linear programming, weighted loss function.

AMS 2010 Subject Classifications: primary: 62G32; secondary: 62J05

1. Introduction

Extreme value events are highly unusual and rare events that can cause severe harm to people and in particular, costly damages to the environment. The distribution of the response variable, y, of an extreme event or its damage is usually heavy-tailed distributed. In the cases of these extreme events, we often want to take preventative measures to reduce the risk and damage. On the other hand, the response y may be related to other variables. It is important to estimate high conditional quantiles of a random variable y given a variable vector $\mathbf{x} = (x_1, x_2, ..., x_k)^T$.

The traditional mean linear regression estimates the conditional expectation $E(y|\mathbf{x})$, where $\mathbf{x} = (1, x_1, x_2, ..., x_k)^T,\ \mathbf{x} \in R^p, p = k + 1$. The linear mean regression model assumes

$$\mu_{y|\mathbf{x}} = E\left(y|x_1, x_2, ..., x_k\right) = \mathbf{x}^T\boldsymbol{\beta} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k.$$

*Corresponding author, e-mail: mhuang@brocku.ca.
†This research is supported by the Natural Science and Engineering Research Council of Canada.

We estimate $\boldsymbol{\beta} = (\beta_0, \beta_1, ..., \beta_k)^T$, where $\boldsymbol{\beta} \in R^p$, from a random sample $\{(y_i, \mathbf{x}_i), i = 1, ..., n\}$, where $\mathbf{x}_i$ is the p-dimensional design vector and y_i is the univariate response variable from a continuous distribution with a c.d.f. $F(y)$. The least square (LS) estimator $\widehat{\boldsymbol{\beta}}_{LS}$ is a solution to the following equation

$$\widehat{\boldsymbol{\beta}}_{LS} = \arg \min_{\boldsymbol{\beta} \in R^p} \sum_{i=1}^{n} (y_i - \mathbf{x}_i^T \boldsymbol{\beta})^2, \qquad (1.2)$$

where $\widehat{\boldsymbol{\beta}}_{LS}$ is obtained by minimizing the L_2-distance.

The traditional mean linear regression estimates the relationship between a mean response variable and explanatory variables. When analyzing extreme value events, where the response variable y is heavy-tailed distributed, the traditional mean linear regression methods are inadequate and cannot be extended and applied to non-central locations.

Quantile regression (Koenker, 2005) offers an advanced model that estimates high conditional quantiles of response variable.

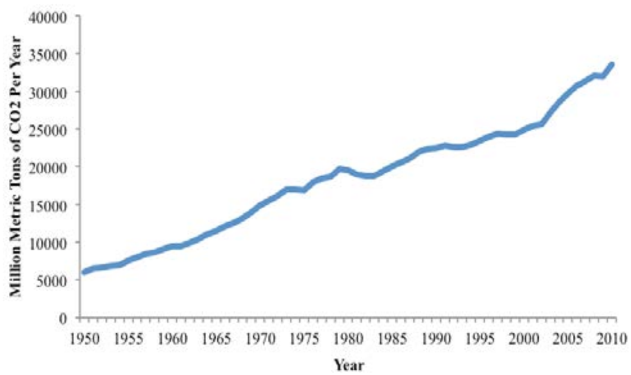


Figure 1. Global CO_2 Emissions (million metric tonnes) from 1950 to 2010

An Example: CO_2 Emission. Natural processes and human activities can cause climate change. However, the recent global warming can be largely attributed to the carbon dioxide (CO_2) and other greenhouse gases emissions. It was found that in 2009, CO_2 accounted for 82% of all European greenhouse gas emissions and about 94% of the CO_2 released to the atmosphere were from combusting fossil fuels. Although carbon dioxide is naturally present in the atmosphere, human activities have significantly altered the carbon cycle by adding more CO_2 to the atmosphere and influencing the ability of natural sinks. Figure 1 shows CO_2 emissions increases between 1950 and 2010. These increases are related with the increased energy use by an expanding economy, population and overall growth in emissions from electricity generation. It is important to estimate high conditional quantiles of the distribution of CO_2 emission in order to prevent acceleration of climate change. The data was obtained from the Carbon Dioxide Information Analysis Center at *http://cdiac.ornl.gov.* In 2010, the CO_2 emission per capita was recorded in metric tonnes. There are $n = 123$ countries remaining after the threshold of 1 tonne was applied. Table 1 lists the top 10 countries with the largest CO_2 emissions and their gross domestic product (GDP) and electricity consumption (E.C.) per capita.

Table 1. The top 10 countries with the largest CO_2 emissions and their GDP and E.C. in 2010

Country	CO_2 Emission per Capita (tonnes)	GDP per Capita (US $)	E. C. per Capita (kilowatts)
Qatar	40.31	71,510.16	86.01
Trinidad and Tobago	38.16	15,630.05	1,657.02
Kuwait	31.32	38,584.48	913.04
Brunei	22.87	30,880.34	239.40
Aruba	22.85	24,289.14	3,262.30
Luxembourg	21.36	102,856.97	2,751.26
Oman	20.41	20,922.66	1,562.59
United Arab Emitrates	19.85	33,885.93	9,007.35
Bahrain	19.34	20,545.97	10,142.73
United States	17.56	48,377.39	7,588.42

We assume a linear mean regression model is

$$\mu_{y|\mathbf{x}} = E(y|x, x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2, \qquad (1.3)$$

where y represents the CO_2 emission per capita (tonnes), x_1 represents the ln(GDP) per capita (US \$), and x_2 represents the ln(E.C.) per capita (kilowatts). The green plane in Figure 2 represents the least square mean regression $\widehat{\mu}_{y|\mathbf{x}}$ obtained by using (*1.2*) and the model (*1.3*)

$$\widehat{\mu}_{y|\mathbf{x}} = -22.5009 + 2.0708x_1 + 1.2998x_2.$$

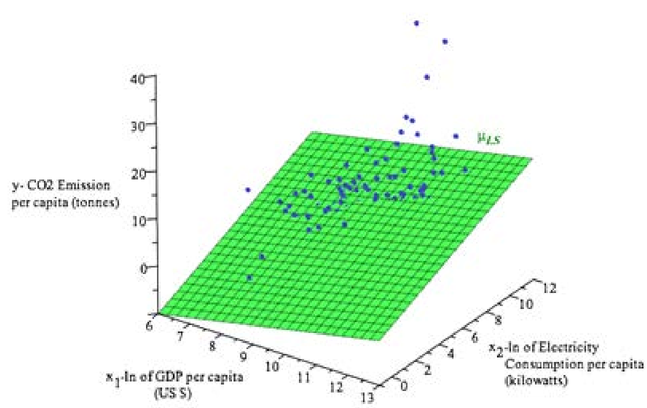


Figure 2. Scatter plot and LS mean regression $\widehat{\mu}_{y|\mathbf{x}}$ of the CO_2 emission (≥ 1 tonnes, $n = 123$) related to ln(GDP) per capita x_1 and ln(E.C.) per capita x_2.

Since the traditional mean linear regression will only provide information about the mean relationship between CO_2 emission and GDP or E.C. per capita, it cannot provide estimation for high conditional quantile of CO_2 emission. But quantile regression method (Yu, *et al.*, 2003; Wang, *et al.*, 2012; Huang, *et al.*, 2015) can estimate high CO_2 emission quantile curves related to GDP and E.C. We will discuss this example further in Section 5.

In Section 2, we review some notations. We proposed an optimal weighted quantile regression method and gives asymptotic distribution of weighted quantile estimator in Section 3. In Section 4, Monte Carlo simulations were performed by generating random samples from the bivariate Pareto distribution. We estimate conditional quantiles by the proposed weighted quantile regression and compare with classical methods. In Section 5, the three regression methods: mean regression, regular and weighted quantile regression, are applied to the CO_2 emission Example. Goodness-of-fit tests were used to assess the fit to the data. The proposed weighted quantile regression model performs better than the classical quantile regression method.

2. Notation

Pickands (1975) first introduced the Generalized Pareto Distribution (GPD).
Definition 1. The cumulative distribution function (c.d.f.) of two-parameter GPD(γ, σ) with shape parameter γ and scale parameter σ of a random variable y is given by

$$F(y) = 1 - \left(1 + \frac{\gamma y}{\sigma}\right)^{1/\gamma}, \quad \gamma,\ \sigma > 0, \quad y > 0, \tag{2.1}$$

and the probability density function

$$f(y) = \sigma^{-1}\left(1 + \frac{\gamma y}{\sigma}\right)^{\frac{1}{\gamma}-1}, \quad \gamma,\ \sigma > 0, \quad y > 0. \tag{2.2}$$

Definition 2. The τth quantile of a random variable y with c.d.f. $F(y)$ is defined as

$$Q(\tau) = F^{-1}(\tau) = \inf\{y : F(y) \geq \tau\},\ \ 0 < \tau < 1, \tag{2.3}$$

where $F(y)$ is right continuous distribution function of variable y.
Definition 3. The τth conditional linear quantile regression of y for given $\mathbf{x} = (1, x_1, x_2, ..., x_k)^T$ is defined as

$$Q_y(\tau|\mathbf{x}) = Q_\tau(y|x_1, x_2, ..., x_k) = F^{-1}(\tau|\mathbf{x}) = \mathbf{x}^T\boldsymbol{\beta}(\tau) = \beta_0(\tau) + \beta_1(\tau)x_1 + \cdots + \beta_k(\tau)x_k, \tag{2.4}$$

where $0 < \tau < 1$, $\boldsymbol{\beta}(\tau) = (\beta_0(\tau), \beta_1(\tau), ..., \beta_k(\tau))^T$.

Koenker and Bassett (1978) proposed a L_1-loss function ρ to obtain estimator $\widehat{\boldsymbol{\beta}}(\tau)$ by solving

$$\widehat{\boldsymbol{\beta}}(\tau) = \arg \min_{\boldsymbol{\beta}(\tau) \in R^p} \sum_{i=1}^{n} \rho_\tau(y_i - \mathbf{x}_i^T\boldsymbol{\beta}(\tau)),\ \ 0 < \tau < 1, \tag{2.5}$$

where ρ_τ is a loss function

$$\rho_\tau(u) = u(\tau - I(u < 0)) = \begin{cases} u(\tau - 1),\ u < 0; \\ u\tau,\ u \geq 0. \end{cases} \tag{2.6}$$

3. Proposed Weighted Quantile Regression

3.1. Proposed Weighted Quantile Regression

Huang, *et al.* (2015) proposed a weighted quantile regression (WQR) method

$$\widehat{\boldsymbol{\beta}}_w(\tau)= \arg \min_{\boldsymbol{\beta}(\tau)\in R^p} \sum_{i=1}^{n} w_i(\mathbf{x}_i,\tau)\rho_\tau(y_i - \mathbf{x}_i^T\boldsymbol{\beta}(\tau)),\ 0<\tau<1, \tag{3.1}$$

where $w_i(\mathbf{x}_i,\tau)$ is any uniformly bounded positive weight function independent of $y_i,\ i=1,...,n$, for $\mathbf{x}_i = (1, x_{i1}, x_{i2}, ..., x_{ik})^T$.

Koenker (2005) suggested that when the conditional densities of the response are heterogeneous, quantile regression weights should be proportional to the local density evaluated at the quantile of interest. In this paper, we propose a weight as the local conditional density $f(y|\mathbf{x})$ of y for given $\mathbf{x}$ at the τth quantile $\xi_i(\tau)$, which is

$$w_i(\mathbf{x}_i,\tau) = f_i(\xi(\tau)),\ i=1,2,...,n,\ 0<\tau<1, \tag{3.2}$$

where $f_i(\xi(\tau))$ is uniformly bounded at the quantile points $\xi_i(\tau)$.

We will use weight *(3.2)* in Section 4 for Monte Carlo simulation. In Section 5, we use weight *(3.3)* proposed by Huang, *et al.* (2014) for the CO_2 emission example, for k regressors

$$w_i\left(\mathbf{x}_i,\tau\right) = \frac{||\mathbf{x}_i||^{-1}}{\sum_{i=1}^{n}||\mathbf{x}_i||^{-1}}, \quad 0<\tau<1, \tag{3.3}$$

where $w_i(\mathbf{x}_i,\tau) \in [0,1]$ and $\sum_{i=1}^{n} w_i(\mathbf{x}_i,\tau) = 1,\ i=1,...,n,\ ||\mathbf{x}_i|| = \sqrt{x_{i1}^2 + x_{i2}^2 + \cdots + x_{ik}^2}$.

3.2. Comparison of Quantile Regression Models

In order to compare the regular and weighted quantile regression models in *(2.5)* and *(3.1)*. We extend the idea of measure goodness of fit by Koenker and Machado (1999), and suggest to use a Relative $R(\tau)$ which is defined as

$$Relative\ R(\tau) = 1 - \frac{V_{weighted}(\tau)}{V_{regular}(\tau)}, \quad -1 \le R(\tau) \le 1, \quad \text{where} \tag{3.4}$$

$$V_{regular}(\tau) = \sum_{y_i \ge \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}(\tau)} \frac{\tau}{n}\left|y_i - \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}(\tau)\right| + \sum_{y_i < \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}(\tau)} \frac{(1-\tau)}{n}\left|y_i - \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}(\tau)\right|, \tag{3.5}$$

$$V_{weighted}(\tau) = \sum_{y_i \ge \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}_w(\tau)} w_i\tau\left|y_i - \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}_w(\tau)\right| + \sum_{y_i < \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}_w(\tau)} w_i(1-\tau)\left|y_i - \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}_w(\tau)\right|, \tag{3.6}$$

where $w_i = w_i(\mathbf{x}_i,\tau)$ is given in *(3.1)*, $\widehat{\boldsymbol{\beta}}(\tau)$ and $\widehat{\boldsymbol{\beta}}_w(\tau)$ are obtained by *(2.5)* and *(3.1)* respectively.

4. Simulations

In this Section, Monte Carlo simulations are performed. We generate m random samples with size n each from the bivariate Pareto distribution (Mardia, 1962) with c.d.f.

$$F(x, y) = 1 - x^{-\alpha} - y^{-\alpha} - (x + y + 1)^{-\alpha},\ x > 1,\ y > 1,\ \alpha > 0, \tag{4.1}$$

and the conditional quantile function of *(4.1)* is

$$\xi(\tau) = Q_y(\tau|x) = 1 - x\left(1 - \frac{1}{(1-\tau)^{-1/(\alpha+1)}}\right),\ x > 1,\ \alpha > 0, \quad 0 < \tau < 1. \tag{4.2}$$

The conditional density of y for given x is

$$f(y|x) = \frac{4x^{(\alpha+1)}}{(x + y - 1)^{(\alpha+2)}},\ x > 1,\ y > 1,\ \alpha > 0,$$

and the τth conditional density of y for given x at the τth quantile is

$$f(\xi(\tau)) = \frac{4(1-\tau)^{(\alpha+2)/(\alpha+1)}}{x},\ x > 1,\ \alpha > 0, \quad 0 < \tau < 1. \tag{4.3}$$

Assume that the true conditional quantile is $Q_y(\tau|x) = \beta_0(\tau) + \beta_1(\tau)x$. We use two quantile regression methods:

1. The regular quantile regression (QR) Q_R estimator based on *(2.5)*,

$$Q_R(\tau|x) = \widehat{\beta}_0(\tau) + \widehat{\beta}_1(\tau)x \tag{4.4}$$

2. The weighted quantile regression (WQR) Q_W estimator based on *(3.1)*

$$Q_W(\tau|x) = \widehat{\beta}_{w0}(\tau) + \widehat{\beta}_{w1}(\tau)x \tag{4.5}$$

with weight $w_i = f_i(\xi_i(\tau))$ in *(3.2)*.

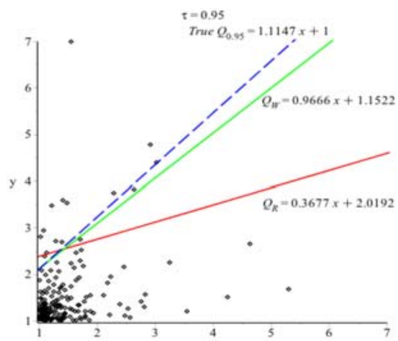


Figure 3. Simulation for $\tau = 0.95$, where Q_R is the red solid line, Q_W is the green solid line and the true conditional quantile function Q_τ is the blue dashed line.

For each method, we generate size $n = 200$, $m = 1,000$ samples, which can be estimated for each simulated sample. For ith sample, we have $Q_{R,i}(\tau|x)$ and $Q_{W,i}(\tau|x),\ i = 1, ...m$.

We use $\alpha = 3$ in *(4.3)*, then the weights for Q_W are

$$w_i(\mathbf{x}_i, \tau) = \frac{4(1-\tau)^{(5/4)}}{x_i},\ x_i > 1,\ i = 1, 2, ..., n. \qquad (4.6)$$

The two estimates $Q_R(\tau|x)$ in (*4.4*) and $Q_W(\tau|x)$ in (*4.5*) were compared with the true quantile function $Q_y(\tau|x)$ in (*4.2*). In Figure 3, it illustrates that $Q_W(\tau|x)$ is closer to the true conditional quantile function $Q_y(\tau|x)$ than $Q_R(\tau|x)$ for $\tau = 0.95$, which suggests that Q_W behaves more efficiently than Q_R.

The simulation mean squared errors (SMSE) of the estimators (*4.4*) and (*4.5*) are:

$$SMSE(Q_R(\tau)) = \frac{1}{m}\sum_{i=1}^{m}\int_1^N (Q_{R,i}(\tau|x) - Q_y(\tau|x))^2 dx; \qquad (4.7)$$

$$SMSE(Q_W(\tau)) = \frac{1}{m}\sum_{i=1}^{m}\int_1^N (Q_{W,i}(\tau|x) - Q_y(\tau|x))^2 dx, \qquad (4.8)$$

where the true τth conditional quantile of the bivariate Pareto distribution $Q_y(\tau|x)$ is defined in (*4.2*). N is a finite x value such that the c.d.f. in (*4.1*) $F(N, N) \approx 1$. In this paper, we take $N = 1000$ and the simulation efficiencies (SEFF) of $Q_W(\tau|x)$ relative to $Q_R(\tau|x)$ are given by

$$SEFF(Q_W(\tau)) = \frac{SMSE(Q_R(\tau))}{SMSE(Q_W(\tau))}, \qquad (4.9)$$

where $SMSE(Q_R(\tau))$ and $SMSE(Q_W(\tau))$ are defined in (*4.7*) and (*4.8*) respectively. Table 2 displays the simulation efficiencies $SEFF(Q_W(\tau))$ for varying τ values by using the weight in (*4.6*). It shows that most of the $SEFF(Q_W(\tau))$ are larger than 1, which signifies that the $Q_W(\tau)$ is more efficient than $Q_R(\tau)$ when $\tau = 0.90, 0.95$ and up to 0.98.

Table 2. Simulation Efficiencies (SEFF) of Estimating $Q_y(\tau|x), m = 1000, n = 200, N = 1000$

τ	0.90	0.95	0.96	0.97	0.98
$SEFF(Q_W(\tau))$	1.5451	1.3744	1.1365	1.5598	1.2071

5. Applications

In this Section, we applied three regression models to the CO_2 emission example in Section 1:

1. The traditional mean linear regression (LS) $\widehat{\boldsymbol{\beta}}_{LS}$ in (*1.2*);
2. The regular quantile regression Q_R estimator $\widehat{\boldsymbol{\beta}}(\tau)$ in (*2.5*);
3. The proposed weight quantile regression Q_W estimator $\widehat{\boldsymbol{\beta}}_W(\tau)$ in (*3.1*) with weight (*3.3*).

5.1. Goodness-of-fit test for GPD

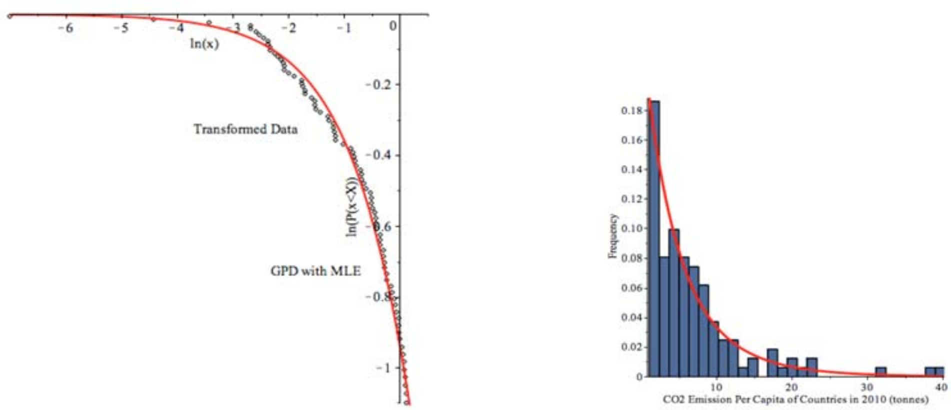
The CO_2 emission data was transformed using $y = \frac{x-\mu}{\sigma}$, where $\mu = 1$ tonne, in order to fit the data to GPD model in (*2.1*). The maximum likelihood estimators (MLE) are $\widehat{\sigma}_{MLE} = 5.3011$

and $\widehat{\gamma}_{MLE} = 0.1234$. The fit of the GPD model with the CO_2 emission data can be demonstrated in Figure 4(a),(b).

Table 3 shows three goodness-of-fit tests: the Kolmogorov-Smirnov test ($K-S$) (Kolmogorov, 1933), Anderson-Darling test ($A-D$) and Cramer-von-Mises test ($C-v-M$) (Anderson and Darling, 1952). The $K-S$ test shows that the GPD model fits the data with a probability of 83.97%. The $A-D$ and $C-v-M$ tests show that the GPD model fits the data with probability of 88.55% and 86.62% respectively.

Table 3. The Goodness of Fit Tests for CO_2 Emission Example

$K-S$		$A-D$		$C-v-M$	
Test statistic	p-value	Test statistic	p-value	Test statistic	p-value
0.0443	0.8397	0.3619	0.8855	0.0517	0.8662



(a) (b)

Figure 4. (a) Log-log plot of CO_2 emission example. The dots are the data and the red solid line is the GPD model. (b) Histogram of CO_2 emission per capita of 123 countries in 2010 (tonnes) greater than the threshold of 1 tonne with GPD model in (*2.1*).

5.2. Quantile Regression

Next, we assume a linear quantile regression model as

$$Q_y(\tau|x) = \beta_0(\tau) + \beta_1(\tau)x_1 + \beta_2(\tau)x_2, \quad 0 < \tau < 1,$$

where y is the CO_2 emission per capita (tonnes), x_1 is the country's ln(GDP) per capita and x_2 is the country's ln(E.C.) per capita. The weight in (*3.3*) will be used. Table 4 lists $\widehat{\beta}_0(\tau)$, $\widehat{\beta}_1(\tau)$ and $\widehat{\beta}_2(\tau)$ for $\tau = 0.95, 0.96, 0.97, 0.98, 0.99$ for $Q_R(\tau)$ and $Q_W(\tau)$.

Figure 5 shows the 3D scatter plot with Q_R (red) and Q_W (green) of CO_2 emission per capita given the country's GDP per capita and electricity consumption per capita at $\tau = 0.935$ and 0.96 respectively. It is important to note that the Q_R and Q_W quantile regression planes appear to fit the data. In general, the Q_W plane produces a higher estimated CO_2 emissions per capita than Q_R curve at high quantiles.

Table 4. $\widehat{\beta}_0(\tau)$, $\widehat{\beta}_1(\tau)$ and $\widehat{\beta}_2(\tau)$ Values for Regular and Weighted Quantile Regression and least squares mean regression for CO_2 emission example.

τ	Weight	$\widehat{\beta}_0(\tau)$	$\widehat{\beta}_1(\tau)$	$\widehat{\beta}_2(\tau)$
LS	–	-22.5009	2.0708	1.2998
0.95	Q_R	-41.6856	5.8924	0.5527
	Q_W	-29.3795	3.9296	1.3779
0.96	Q_R	-44.8147	5.4258	1.9505
	Q_W	-38.9137	4.7370	2.0928
0.97	Q_R	-46.7095	5.6513	2.4429
	Q_W	-37.4893	5.1502	1.4934
0.98	Q_R	-47.4004	5.7323	2.4739
	Q_W	-47.4004	5.7323	2.4739
0.99	Q_R	-51.2657	6.1856	2.6475
	Q_W	-47.4004	5.7323	2.4739

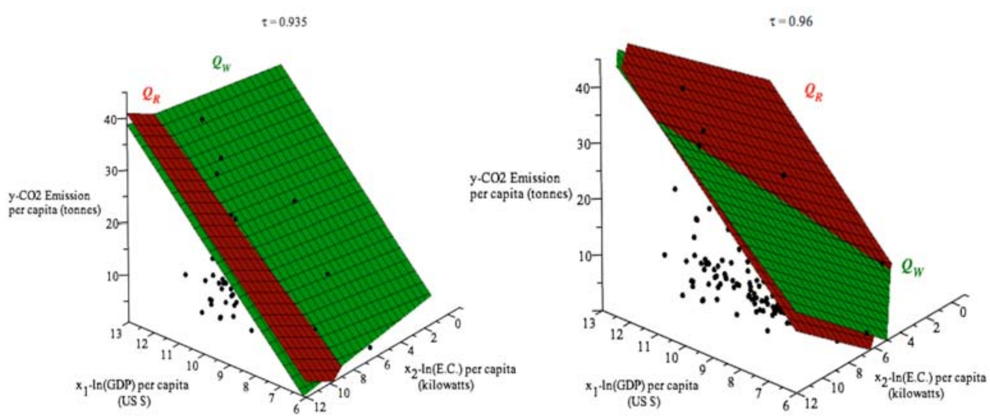


Figure 5. 3D scatter plot with Q_R (red) and Q_W (green) planes for $\tau = 0.935$ and 0.96.

5.3. Comparison of Q_R and Q_W

Next, we compare Q_R and Q_W. Table 5 shows the values for Relative $R(\tau)$ in *(3.4)*. For $\tau \geq 0.95$, all values of Relative $R(\tau)$ are larger than 0, which signifies that $V_{weighted}(\tau) < V_{regular}(\tau)$ and as well, it suggests that the weighted quantile regression model Q_W is a better fit to the CO_2 emission per capita data than the regular quantile regression model Q_R.

Table 5. Relative $R(\tau)$ values for CO_2 emission example

	$\tau = 0.95$	$\tau = 0.96$	$\tau = 0.97$	$\tau = 0.98$	$\tau = 0.99$
Relative $R(\tau)$	0.05846	0.04901	0.05020	0.04385	0.03782

From study results, we can conclude that countries with higher gross domestic product per capita or are consuming high amounts of electricity per capita will produce higher CO_2 emissions per capita. Since carbon dioxide is not destroyed over time, but moves among different parts of the ocean-atmosphere-land system, it can remain in the atmosphere for thousands of years due to the very slow process by which carbon is transferred to ocean sediments. As a result, countries should monitor their CO_2 emissions per capita in order to prevent further damages to the environment. Countries can consider producing more energy from renewable sources, such as wind, solar, hydro and using fuels with lower carbon content to reduce carbon emissions.

6. Conclusions

1. The traditional mean regression estimates conditional mean of response variables. The quantile regression estimates conditional quantiles of response variable, and it is very useful.

2. The Monte Carlo simulations and the example of applications confirmed that the proposed weighted quantile regression established better results than the regular quantile regression.

References

[1] Anderson, T.W. and Darling, D.A. (1952). Asymptotic theory of certain "Goodness of fit" criteria based on stochastic processes. *The Annals of Statistics* 23(2), 193-212.

[2] Carbon Dioxide Information Analysis Centre, *http://cdiac.ornl.gov*

[3] Huang, M.L., Xu, X. and Tashnev, D. (2015). A weighted linear quantile regression. *Journal of Statistical Computation and Simulation* 85(13), 2596-2618.

[4] Huang, M.L., Xu, Y. and Yuen, W.K. (2014). On quantile regression for extremes, 2014 JSM Proceedings, Statistical Computation Section, Alexandria, VA. American Statistical Association, 561-601.

[5] Koenker, R. (2005). *Quantile regression.* Cambridge University Press: New York, NY.

[6] Koenker, R. and Bassett, G. (1978). Regression quantiles. *Econometrica* 46(1), 33-50.

[7] Koenker, R. and Machado, J.A.F. (1999). Goodness of fit and related inference processes for quantile regression. *Journal of the American Statistical Association* 96(454), 1296-1311.

[8] Kolmogorov, A.N. (1933). Sulla determinazione empirica di une legge di distribuzione. *Giornale dell'Istit. degli att.* 4, 83-91.

[9] Mardia, K. W. (1962). Multivariate Pareto distribution, *Annals of mathematical Statistics, 33,* 1008-1015.

[10] Pickands, J. (1975). Statistical inference using extreme order statistics. *The Annals of Statistics* 3, 119-131.

[11] Wang, H.J., Li, D. and He, X. (2012). Estimation of high conditional quantiles for heavy-tailed distributions. *Journal of American Statistical Association* 107(500), 1-12.

[12] Yu, K., Lu, Z. and Stander, J. (2003). Quantile regression: applications and current research areas. *The Statistician* 52(3), 331-350.