The Statistical Power Of One-Sample Location Hypothesis Tests

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Abstract
This paper documents the analytical methods for calculating the power function and related statistics for two-sided and one-sided (upper/lower) one-sample location hypothesis tests. Tables of required sample sizes are included for a unit variance under the conditions of the Central Limit Theorem.

Key Words: Hypothesis Test Power, Required Sample Size, Asymptotic Distribution

1. Introduction

Let $\{X_1, X_2, \ldots, X_n\}$ be a set of n-many independent and uncorrelated samples taken from a common distribution X with unknown mean μ (the location parameter) and known variance σ^2. To test the two-sided hypothesis

$$\begin{aligned} H_0 &: \mu = \mu_0 \\ H_A &: \mu \neq \mu_0 \end{aligned}$$

the usual (one-sample) test statistic is given by

$$Z = \frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

which, under the null hypothesis, has mean

$$\begin{aligned} E[Z] &= E\left[\frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right] \\ &= \frac{1}{\sigma\sqrt{n}}\left(\sum_{k=1}^{n} E[X_k] - n\mu_0\right) \\ &= 0 \end{aligned}$$

and variance

$$\begin{aligned} E\left[(Z - E[Z])^2\right] &= E\left[\left(\frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right)^2\right] \\ &= \frac{1}{n\sigma^2} E\left[\left(\sum_{k=1}^{n} (X_k - \mu_0)\right)^2\right] \\ &= \frac{1}{n\sigma^2}\sum_{k=1}^{n} E\left[(X_k - \mu_0)^2\right] \end{aligned}$$

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$$= \frac{n\sigma^2}{n\sigma^2}$$
$$= 1$$

since the individual X_k are uncorrelated.

Furthermore, the use of the Central Limit Theorem[1] shows that the Z test distribution converges in distribution to the standard normal distribution as $n \to \infty$. Therefore, if n is large enough, we reject the null hypothesis in favor of the alternative at $100\,(1-\alpha)\,\%$ confidence if

$$|Z| > z_\alpha > 0$$

so that the probability that we reject the null hypothesis in favor of the alternative when we should accept it is given by

$$1 - \Phi\left(z_\alpha\right) + \Phi\left(-z_\alpha\right) = 2\Phi\left(-z_\alpha\right) = \alpha$$

or

$$z_\alpha = -\Phi^{-1}\left(\frac{\alpha}{2}\right)$$

where $\Phi\left(x\right)$ is the standard normal distribution cumulative function.

Without a large enough sample size to justify the use of the Central Limit Theorem, the exact distribution of Z would be required, i.e., the distribution of X would be required.

2. The Power Function For A One-Sample Two-Sided Location Test

Let the specific alternative hypothesis value be $\mu = \mu_1$. Then define

$$Z^* = \frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu_1}{\frac{\sigma}{\sqrt{n}}}$$

so that under the alternative hypothesis Z^* has mean 0 and variance 1.

Note that

$$Z^* = \frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu_0}{\frac{\sigma}{\sqrt{n}}} + \frac{\mu_0 - \mu_1}{\frac{\sigma}{\sqrt{n}}} = Z + \frac{\mu_0 - \mu_1}{\frac{\sigma}{\sqrt{n}}}$$

Under the conditions of the use of the Central Limit Theorem, we reject the alternative hypothesis in favor of the null hypothesis at $100\,(1-\beta)\,\%$ confidence if

$$|Z^*| > z_\beta > 0$$

where

$$z_\beta = -\Phi^{-1}\left(\frac{\beta}{2}\right)$$

[1]Central Limit Theorem: Suppose $\{X_k\}_{k=1,2,\ldots,n}$ are independent identically distributed random variables such that $E\left[X_k\right] = \mu$ and $E\left[\left(X_k - \mu\right)^2\right] = \sigma^2$, where $0 < \sigma < \infty$. Then

$$\lim_{n\to\infty} P\left(\frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu}{\frac{\sigma}{\sqrt{n}}} \le z\right) = \Phi\left(z\right)$$

where

$$\Phi\left(z\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}x^2}\,dx$$

for every real number $-\infty < z < \infty$.

so that the probability that we reject the alternative hypothesis in favor of the null hypothesis when we should accept it is given by

$$P\left(Z^* > z_\beta \text{ or } Z^* < -z_\beta\right) = \beta$$

or

$$P\left(Z > z_\beta - \frac{\mu_0 - \mu_1}{\frac{\sigma}{\sqrt{n}}} \text{ or } Z < -z_\beta - \frac{\mu_0 - \mu_1}{\frac{\sigma}{\sqrt{n}}}\right) = \beta$$

This means the power $\pi\left(\nu; \beta, \mu_0, \sigma, n\right)$ of the one-sample location hypothesis test at $100\left(1-\beta\right)\%$ confidence (the probability we accept the alternative hypothesis, i.e., reject the null hypothesis, when we should accept it, i.e., reject the null hypothesis, is given by

$$\pi\left(\nu; \beta, \mu_0, \sigma, n\right) = \Phi\left(-\Phi^{-1}\left(\frac{\beta}{2}\right) - \frac{\mu_0 - \nu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\Phi^{-1}\left(\frac{\beta}{2}\right) - \frac{\mu_0 - \nu}{\frac{\sigma}{\sqrt{n}}}\right)$$

Without a large enough sample size to justify the use of the Central Limit Theorem, the exact distribution of Z would be required, i.e., the distribution of X would be required, and the power function would be given by

$$\pi\left(\nu; \beta, \mu_0, \sigma, n\right) = P\left(-z_\beta - \frac{\mu_0 - \nu}{\frac{\sigma}{\sqrt{n}}} \le Z \le z_\beta - \frac{\mu_0 - \nu}{\frac{\sigma}{\sqrt{n}}}\right)$$

3. The Power Function For A One-Sample One-Sided Positive Location Test

To test the general one-sided hypothesis

$$H_0 : \mu = \mu_0$$
$$H_A : \mu > \mu_0$$

the test proceeds similarly as for the two-sided test: We reject the null hypothesis in favor of the one-sided alternative at $100\left(1-\alpha\right)\%$ confidence if

$$Z > z_\alpha$$

so that, with a large enough sample size to justify the use of the Central Limit Theorem, the probability that we reject the null hypothesis in favor of the alternative when we should accept it is given by

$$1 - \Phi\left(z_\alpha\right) = \alpha$$

or

$$z_\alpha = \Phi^{-1}\left(1-\alpha\right)$$

Once again, without a large enough sample size to justify the use of the Central Limit Theorem, the exact distribution of Z would be required (the distribution of X) would be required.

Let the specific alternative hypothesis value be $\mu = \mu_1 > \mu_0$. As before, define

$$Z^* = \frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu_1}{\frac{\sigma}{\sqrt{n}}}$$

where

$$Z^* = Z - \frac{\mu_1 - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

Under the conditions of the use of the Central Limit Theorem, we reject the alternative hypothesis in favor of the null hypothesis at $100\left(1-\beta\right)\%$ confidence if

$$Z^{*}<z_{\beta}^{+}$$

where

$$z_{\beta}^{+}=-\Phi^{-1}\left(\beta\right)$$

so that the probability that we reject the alternative hypothesis in favor of the null hypothesis when we should accept it is given by

$$P\left(Z^{*}<z_{\beta}^{+}\right)=\beta$$

for

$$P\left(Z<z_{\beta}^{+}+\frac{\mu_{1}-\mu_{0}}{\frac{\sigma}{\sqrt{n}}}\right)=\beta$$

This means the power $\pi_{+}\left(\nu;\beta,\mu_{0},\sigma,n\right)$ of the one-sample one-sided location hypothesis test at $100\left(1-\beta\right)\%$ confidence (the probability we accept the alternative hypothesis, i.e., reject the null hypothesis, when we should accept it, i.e., reject the null hypothesis, is given by

$$\pi_{+}\left(\nu;\beta,\mu_{0},\sigma,n\right)=1-\Phi\left(-\Phi^{-1}\left(\beta\right)+\frac{\nu-\mu_{0}}{\frac{\sigma}{\sqrt{n}}}\right),\nu>\mu_{0}$$

Without a large enough sample size to justify the use of the Central Limit Theorem, the exact distribution of Z would be required, i.e., the distribution of X would be required, and the power function would be given by

$$\pi_{+}\left(\nu;\beta,\mu_{0},\sigma,n\right)=P\left(Z\geq z_{\beta}^{+}+\frac{\nu-\mu_{0}}{\frac{\sigma}{\sqrt{n}}}\right),\nu>\mu_{0}$$

4. The Power Function For A One-Sample One-Sided Negative Location Test

To test the negative one-sided hypothesis

$$H_{0}:\mu=\mu_{0}$$
$$H_{A}:\mu<\mu_{0}$$

the test proceeds similarly as for the positive one-sided test: We reject the null hypothesis in favor of the one-sided alternative at $100\left(1-\alpha\right)\%$ confidence if

$$Z<z_{\alpha}$$

so that, with a large enough sample size to justify the use of the Central Limit Theorem, the probability that we reject the null hypothesis in favor of the alternative when we should accept it is given by

$$\Phi\left(z_{\alpha}\right)=\alpha$$

or

$$z_{\alpha}=\Phi^{-1}\left(\alpha\right)$$

Once again, without a large enough sample size to justify the use of the Central Limit Theorem, the exact distribution of Z would be required (the distribution of X) would be required.

Let the specific alternative hypothesis value be $\mu = \mu_1 < \mu_0$. As before, define

$$Z^* = \frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu_1}{\frac{\sigma}{\sqrt{n}}}$$

where

$$Z^* = Z + \frac{\mu_0 - \mu_1}{\frac{\sigma}{\sqrt{n}}}$$

Under the conditions of the use of the Central Limit Theorem, we reject the alternative hypothesis in favor of the null hypothesis at $100\,(1-\beta)\,\%$ confidence if

$$Z^* > z_\beta^-$$

where

$$z_\beta^- = \Phi^{-1}(\beta)$$

so that the probability that we reject the alternative hypothesis in favor of the null hypothesis when we should accept it is given by

$$P\left(Z^* > z_\beta^-\right) = \beta$$

or

$$P\left(Z > z_\beta^- - \frac{\mu_0 - \mu_1}{\frac{\sigma}{\sqrt{n}}}\right) = \beta$$

This means the power $\pi_-(\nu;\beta,\mu_0,\sigma,n)$ of the one-sample one-sided location hypothesis test at $100\,(1-\beta)\,\%$ confidence (the probability we accept the alternative hypothesis, i.e., reject the null hypothesis, when we should accept it, i.e., reject the null hypothesis, is given by

$$\pi_-(\nu;\beta,\mu_0,\sigma,n) = \Phi\left(\Phi^{-1}(\beta) - \frac{\mu_0 - \nu}{\frac{\sigma}{\sqrt{n}}}\right), \nu < \mu_0$$

Without a large enough sample size to justify the use of the Central Limit Theorem, the exact distribution of Z would be required, i.e., the distribution of X would be required, and the power function would be given by

$$\pi_-(\nu;\beta,\mu_0,\sigma,n) = P\left(Z \le z_\beta^- - \frac{\mu_0 - \nu}{\frac{\sigma}{\sqrt{n}}}\right), \nu < \mu_0$$

Note that

$$\begin{aligned}
\pi_+(\nu > 0;\beta,0,\sigma,n) &= P\left(Z \ge z_\beta^+ + \frac{\nu}{\frac{\sigma}{\sqrt{n}}}\right) \\
&= 1 - P\left(Z < z_\beta^+ + \frac{\nu}{\frac{\sigma}{\sqrt{n}}}\right), \nu > 0 \\
&= 1 - P\left(Z > z_\beta^- - \frac{-\nu}{\frac{\sigma}{\sqrt{n}}}\right), \nu < 0 \\
&= P\left(Z \le z_\beta^- - \frac{-\nu}{\frac{\sigma}{\sqrt{n}}}\right) \\
&= \pi_-(\nu < 0;\beta,0,\sigma,n)
\end{aligned}$$

and that both $\pi_+(\nu;\beta,\mu_0,\sigma,n)$ and $\pi_-(\nu;\beta,\mu_0,\sigma,n)$ are *increasing* functions of $\nu|\beta,\mu_0,\sigma,n$ and of $n|\nu,\beta,\mu_0,\sigma$.

5. Statistics Of The Power Function

The Power Profile aggregates power values across all β values, while the Total Power aggregates across all ν values. The Centroid Of Power is the first moment (the center of "weight") across ν values, which indicates a "balance ν point" based on the Total Power. Higher moments are also available.

5.1 The Power Profile

Define the *Power Profile Of The One-Sample Two-Sided Location Hypothesis Test* $\substack{H_0:\mu=\mu_0 \\ H_A:\mu\neq\mu_0}$ to be

$$\Pi\left(v;\mu_0\right)=\int_0^1 \pi\left(\nu;\beta,\mu_0\right)\,d\beta$$

In particular, we have

$$\Pi\left(v;\mu_0\right)=\int_0^1 P\left(-z_\beta-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\leq Z\leq z_\beta-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\,d\beta$$

and under the conditions of the use of the Central Limit Theorem, let

$$u=\Phi^{-1}\left(\frac{\beta}{2}\right),\,\beta=2\Phi\left(u\right),\,d\beta=2\frac{d}{du}\Phi\left(u\right)\,du$$

so that

$$\begin{aligned}
\Pi\left(v;\mu_0\right) &= \int_0^1\left(\Phi\left(-\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)-\Phi\left(\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\right)\,d\beta \\
&= 2\int_{-\infty}^0\left(\Phi\left(-u-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)-\Phi\left(u-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\right)\frac{d}{du}\Phi\left(u\right)\,du \\
&= 2\left(\int_{-\infty}^0\Phi\left(-u-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\,d\Phi\left(u\right)-\int_{-\infty}^0\Phi\left(u-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\,d\Phi\left(u\right)\right) \\
&= 2\left(\int_{-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}}^{\infty}\Phi\left(u\right)\,d\Phi\left(u\right)-\int_{-\infty}^{\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}}\Phi\left(u\right)\,d\Phi\left(u\right)\right) \\
&= \Phi^2\left(u\right)\Big|_{u=-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}}^{u\to\infty}-\Phi^2\left(u\right)\Big|_{u\to-\infty}^{u=\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}} \\
&= \left(1-\Phi^2\left(-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\right)-\left(\Phi^2\left(\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)-0\right) \\
&= 1-\left(1-\Phi\left(\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\right)^2-\Phi^2\left(\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right) \\
&= 1-\left(1-2\Phi\left(\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)+\Phi^2\left(\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\right)-\Phi^2\left(\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right) \\
&= 2\Phi\left(-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\Phi\left(\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)
\end{aligned}$$

5.2 The Total Power

Define the *Total Power Of The One-Sample Two-Sided Location Hypothesis Test* $\substack{H_0:\mu=\mu_0 \\ H_A:\mu\neq\mu}$ to be

$$\Omega(\beta,\mu_0)=\int_{-\infty}^{\infty}\pi(\nu;\beta,\mu_0)\ d\nu$$

In particular, we have

$$\Omega(\beta,\mu_0)=\int_{-\infty}^{\infty}P\left(-z_\beta-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\leq Z\leq z_\beta-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\ d\nu$$

and under the conditions of the use of the Central Limit Theorem, let

$$u=-\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}, v=\mu_0+\frac{\sigma}{\sqrt{n}}\left(u+\Phi^{-1}\left(\frac{\beta}{2}\right)\right), dv=\frac{\sigma}{\sqrt{n}}du$$

so that

$$\begin{aligned}\Omega(\beta,\mu_0)&=\int_{-\infty}^{\infty}\left(\Phi\left(-\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)-\Phi\left(\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\right)\ dv\\&=\frac{\sigma}{\sqrt{n}}\left(\int_{-\infty}^{\infty}\left(\Phi(u)-\Phi\left(u+2\Phi^{-1}\left(\frac{\beta}{2}\right)\right)\right)\ du\right)\\&=\frac{\sigma}{\sqrt{n}}\int_{-\infty}^{\infty}\left(\int_{u+2\Phi^{-1}\left(\frac{\beta}{2}\right)}^{u}f(x)\ dx\right)\ du\\&=\frac{\sigma}{\sqrt{n}}\int_{-\infty}^{\infty}f(x)\left(\int_{x}^{x-2\Phi^{-1}\left(\frac{\beta}{2}\right)}\ du\right)\ dx\\&=-2\frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(\frac{\beta}{2}\right)\end{aligned}$$

5.3 The Centroid Of Power

Define the *Centroid Of Power Of The One-Sample Two-Sided Location Hypothesis Test* $\substack{H_0:\mu=\mu_0 \\ H_A:\mu\neq\mu}$ to be

$$C(\beta,\mu_0)=\frac{\int_{-\infty}^{\infty}\nu\pi(\nu;\beta,\mu_0)\ d\nu}{\Omega(\beta,\mu_0)}$$

In particular, we have

$$C(\beta,\mu_0)=\frac{\int_{-\infty}^{\infty}\nu P\left(-z_\beta-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\leq Z\leq z_\beta-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\ d\nu}{\int_{-\infty}^{\infty}P\left(-z_\beta-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\leq Z\leq z_\beta-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\ d\nu}$$

and under the conditions of the use of the Central Limit Theorem, let

$$u=-\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}, v=\mu_0+\frac{\sigma}{\sqrt{n}}\left(u+\Phi^{-1}\left(\frac{\beta}{2}\right)\right), dv=\frac{\sigma}{\sqrt{n}}du$$

so that

$$\int_{-\infty}^{\infty}\nu\pi(\nu;\beta,\mu_0)\ d\nu=\int_{-\infty}^{\infty}v\left(\Phi\left(-\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)-\Phi\left(\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)\right)\ dv$$

$$
\begin{aligned}
&= \frac{\sigma}{\sqrt{n}} \left(\int_{-\infty}^{\infty} \left(\mu_0 + \frac{\sigma}{\sqrt{n}} \left(u + \Phi^{-1}\left(\frac{\beta}{2}\right)\right)\right) \left(\Phi\left(u\right) - \Phi\left(u + 2\Phi^{-1}\left(\frac{\beta}{2}\right)\right)\right) du \right) \\
&= \frac{\sigma}{\sqrt{n}} \begin{pmatrix} \int_{-\infty}^{\infty} \frac{\sigma}{\sqrt{n}} u \left(\Phi\left(u\right) - \Phi\left(u + 2\Phi^{-1}\left(\frac{\beta}{2}\right)\right)\right) du \\ + \left(\mu_0 + \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(\frac{\beta}{2}\right)\right) \int_{-\infty}^{\infty} \left(\Phi\left(u\right) - \Phi\left(u + 2\Phi^{-1}\left(\frac{\beta}{2}\right)\right)\right) du \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sigma^2}{n} \int_{-\infty}^{\infty} u \left(\Phi\left(u\right) - \Phi\left(u + 2\Phi^{-1}\left(\frac{\beta}{2}\right)\right)\right) du \\ -2\frac{\sigma}{\sqrt{n}} \left(\mu_0 + \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(\frac{\beta}{2}\right)\right) \Phi^{-1}\left(\frac{\beta}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sigma^2}{n} \int_{-\infty}^{\infty} f\left(x\right) \left(\int_{x}^{x-2\Phi^{-1}\left(\frac{\beta}{2}\right)} u \, du\right) dx \\ -2\frac{\sigma}{\sqrt{n}} \left(\mu_0 + \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(\frac{\beta}{2}\right)\right) \Phi^{-1}\left(\frac{\beta}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sigma^2}{2n} \int_{-\infty}^{\infty} \left(-4x\Phi^{-1}\left(\frac{\beta}{2}\right) + 4\left(\Phi^{-1}\left(\frac{\beta}{2}\right)\right)^2\right) f\left(x\right) \, dx \\ -2\frac{\sigma}{\sqrt{n}} \left(\mu_0 + \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(\frac{\beta}{2}\right)\right) \Phi^{-1}\left(\frac{\beta}{2}\right) \end{pmatrix} \\
&= \left(2\frac{\sigma^2}{n} \left(\Phi^{-1}\left(\frac{\beta}{2}\right)\right)^2 - 2\frac{\sigma}{\sqrt{n}} \left(\mu_0 + \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(\frac{\beta}{2}\right)\right) \Phi^{-1}\left(\frac{\beta}{2}\right) \right) \\
&= -2\frac{\sigma}{\sqrt{n}}\mu_0 \Phi^{-1}\left(\frac{\beta}{2}\right)
\end{aligned}
$$

so that

$$
C\left(\beta, \mu_0\right) = \frac{-2\frac{\sigma}{\sqrt{n}}\mu_0\Phi^{-1}\left(\frac{\beta}{2}\right)}{-2\frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(\frac{\beta}{2}\right)} = \mu_0
$$

which does not depend on β.

5.4 Statistics For One-Sample One-Sided Positive/Negative Location Tests

For the one-sample (positive) one-sided hypothesis test, the power profile is given by

$$
\Pi\left(v; \mu_0\right) = \int_0^1 P\left(Z \le z_\beta + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) d\beta
$$

and under the conditions of the use of the Central Limit Theorem, let

$$
u = \Phi^{-1}\left(\beta\right), \beta = \Phi\left(u\right), d\beta = f\left(u\right) \, du
$$

so that

$$
\begin{aligned}
\Pi\left(v; \mu_0\right) &= \int_0^1 \Phi\left(-\Phi^{-1}\left(\beta\right) + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) d\beta \\
&= \int_{-\infty}^{\infty} \Phi\left(-u + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) f\left(u\right) \, du
\end{aligned}
$$

However,

$$
\begin{aligned}
\int_{-\infty}^{\infty} \Phi\left(-u + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) f\left(u\right) \, du &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{-u+\frac{\nu-\mu_0}{\frac{\sigma}{\sqrt{n}}}} f\left(x\right) \, dx\right) f\left(u\right) \, du \\
&= \int_{-\infty}^{\infty} f\left(x\right) \left(\int_{-x+\frac{\nu-\mu_0}{\frac{\sigma}{\sqrt{n}}}}^{\infty} f\left(u\right) \, du\right) dx
\end{aligned}
$$

$$= \int_{-\infty}^{\infty} f(x) \left(1 - \Phi\left(-x + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right)\right) dx$$

so that

$$\int_{-\infty}^{\infty} \Phi\left(-u + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) f(u)\, du = \frac{1}{2}$$

Therefore,

$$\Pi(v; \mu_0) \equiv \frac{1}{2}$$

Furthermore, the total power is given by

$$\Omega(\beta, \mu_0) = \int_{-\infty}^{\mu_0} P\left(Z \leq -\Phi^{-1}(\beta) + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) d\nu$$

and under the conditions of the use of the Central Limit Theorem, let

$$u = -\Phi^{-1}(\beta) + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}, v = \mu_0 + \frac{\sigma}{\sqrt{n}}\left(u + \Phi^{-1}(\beta)\right), dv = \frac{\sigma}{\sqrt{n}} du$$

so that

$$\begin{aligned}
\Omega(\beta, \mu_0) &= \int_{-\infty}^{\mu_0} \Phi\left(-\Phi^{-1}(\beta) + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) dv \\
&= \frac{\sigma}{\sqrt{n}} \int_{-\infty}^{-\Phi^{-1}(\beta)} \Phi(u)\, du \\
&= \frac{\sigma}{\sqrt{n}} \int_{-\infty}^{-\Phi^{-1}(\beta)} \left(\int_{-\infty}^{u} f(x)\, dx\right) du \\
&= \frac{\sigma}{\sqrt{n}} \int_{-\infty}^{-\Phi^{-1}(\beta)} f(x) \left(\int_{x}^{-\Phi^{-1}(\beta)} du\right) dx \\
&= \frac{\sigma}{\sqrt{n}} \int_{-\infty}^{-\Phi^{-1}(\beta)} f(x) \left(-\Phi^{-1}(\beta) - x\right) dx \\
&= \frac{\sigma}{\sqrt{n}} \left(-\Phi^{-1}(\beta) \int_{-\infty}^{-\Phi^{-1}(\beta)} f(x)\, dx - \int_{-\infty}^{-\Phi^{-1}(\beta)} x f(x)\, dx\right) \\
&= \frac{\sigma}{\sqrt{n}} \left(-\Phi^{-1}(\beta)\, \Phi\left(-\Phi^{-1}(\beta)\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2}\right) \\
&= \frac{\sigma}{\sqrt{n}} \left((\beta - 1)\, \Phi^{-1}(\beta) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2}\right)
\end{aligned}$$

Finally, the centroid power is given by

$$C(\beta, \mu_0) = \frac{\int_{-\infty}^{\mu_0} \nu P\left(Z \leq -\Phi^{-1}(\beta) + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) d\nu}{\int_{-\infty}^{\mu_0} P\left(Z \leq -\Phi^{-1}(\beta) + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) d\nu}$$

and under the conditions of the use of the Central Limit Theorem, let

$$u = -\Phi^{-1}(\beta) + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}, v = \mu_0 + \frac{\sigma}{\sqrt{n}}\left(u + \Phi^{-1}(\beta)\right), dv = \frac{\sigma}{\sqrt{n}} du$$

so that

$$\begin{aligned}\int_{-\infty}^{\mu_0} \nu\pi_+ (\nu; \beta, \mu_0)\ d\nu &= \int_{-\infty}^{\mu_0} v\Phi\left(-\Phi^{-1}(\beta) + \frac{\nu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) dv \\ &= \frac{\sigma}{\sqrt{n}} \int_{-\infty}^{-\Phi^{-1}(\beta)} \left(\mu_0 + \frac{\sigma}{\sqrt{n}}\left(u + \Phi^{-1}(\beta)\right)\right) \Phi(u)\ du \\ &= \frac{\sigma}{\sqrt{n}} \int_{-\infty}^{-\Phi^{-1}(\beta)} \left(\left(\mu_0 + \frac{\sigma}{\sqrt{n}}\Phi^{-1}(\beta)\right) + \frac{\sigma}{\sqrt{n}}u\right) \Phi(u)\ du \\ &= \frac{\sigma}{\sqrt{n}} \left(\left(\mu_0 + \frac{\sigma}{\sqrt{n}}\Phi^{-1}(\beta)\right) \int_{-\infty}^{-\Phi^{-1}(\beta)} \Phi(u)\ du + \frac{\sigma}{\sqrt{n}} \left(\int_{-\infty}^{-\Phi^{-1}(\beta)} u\Phi(u)\ du\right)\right)\end{aligned}$$

However,

$$\begin{aligned}\int_{-\infty}^{-\Phi^{-1}(\beta)} u\Phi(u)\ du &= \int_{-\infty}^{-\Phi^{-1}(\beta)} u \left(\int_{-\infty}^{u} f(x)\ dx\right) du \\ &= \int_{-\infty}^{-\Phi^{-1}(\beta)} \int_{x}^{-\Phi^{-1}(\beta)} uf(x)\ dudx \\ &= \frac{1}{2}\int_{-\infty}^{-\Phi^{-1}(\beta)} f(x) \left(\left(\Phi^{-1}(\beta)\right)^2 - x^2\right) dx \\ &= \frac{1}{2}\left(\left(\Phi^{-1}(\beta)\right)^2 \int_{-\infty}^{-\Phi^{-1}(\beta)} f(x)\ dx - \int_{-\infty}^{-\Phi^{-1}(\beta)} x^2 f(x)\ dx\right) \\ &= \frac{1}{2}\left(\begin{array}{c}\left(\Phi^{-1}(\beta)\right)^2 \Phi\left(-\Phi^{-1}(\beta)\right) \\ -\left(\Phi^{-1}(\beta) f\left(-\Phi^{-1}(\beta)\right) + \int_{-\infty}^{-\Phi^{-1}(\beta)} f(x)\ dx\right)\end{array}\right), \ \left(\begin{array}{c} w = x, dq = xf(x)\ dx \\ dw = dx, q = -f(x)\end{array}\right) \\ &= \frac{1}{2}\left(\Phi^{-1}(\beta)\left(\Phi^{-1}(\beta)(1-\beta) - \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2}\right) - (1-\beta)\right)\end{aligned}$$

so that

$$\int_{-\infty}^{\mu_0} \nu\pi_+ (\nu; \beta, \mu_0)\ d\nu = \frac{\sigma}{\sqrt{n}}\left(\begin{array}{c}\left(\mu_0 + \frac{\sigma}{\sqrt{n}}\Phi^{-1}(\beta)\right)\left((\beta - 1)\Phi^{-1}(\beta) + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2}\right) \\ +\frac{\sigma}{\sqrt{n}}\left(\frac{1}{2}\left(\Phi^{-1}(\beta)\left(\Phi^{-1}(\beta)(1-\beta) - \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2}\right) - (1-\beta)\right)\right)\end{array}\right)$$

Therefore,

$$C(\beta, \mu_0) = \mu_0 + \frac{\sigma}{2\sqrt{n}}\left(\Phi^{-1}(\beta) + \left(\frac{1-\beta}{(1-\beta)\Phi^{-1}(\beta) - \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2}}\right)\right)$$

which does depend on β.

Note that when $\beta = \frac{1}{2}$, we have

$$(1-\beta)\Phi^{-1}(\beta) - \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2} = -\frac{1}{\sqrt{2\pi}} < 0$$

and

$$\begin{aligned}\frac{d}{dx}\left(\left(1 - \int_{-\infty}^{x} \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}\, du\right)x - \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right) &= \left(1 - \int_{-\infty}^{x} \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}\, du\right) + \left(-\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\right)x + \frac{x}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \\ &= \int_{x}^{\infty} \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}\, du \\ &> 0\end{aligned}$$

which means $(1-\beta)\,\Phi^{-1}(\beta)-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2}$ is a strictly increasing function of $\Phi^{-1}(\beta)$, and since $\Phi^{-1}(\beta)$ is a strictly increasing function of β, then $(1-\beta)\,\Phi^{-1}(\beta)-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2}$ is a strictly increasing function of β, and we have

$$\begin{aligned}\lim_{x\to\infty}\left(\left(1-\int_{-\infty}^{x}\frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}\,du\right)x-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right)&=\lim_{x\to\infty}\frac{\int_{x}^{\infty}\frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}\,du}{\frac{1}{x}}\\&=\lim_{x\to\infty}\frac{-\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}}{-\frac{1}{x^2}}\\&=\frac{1}{\sqrt{2\pi}}\lim_{x\to\infty}x^2e^{-\frac{1}{2}x^2}\\&=0\end{aligned}$$

which means that $(1-\beta)\,\Phi^{-1}(\beta)-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\Phi^{-1}(\beta)\right)^2}$ is never 0 for any $0<\beta<1$.

Similar results are available for the one-sample (negative) one-sided location hypothesis test following these same analytical methods.

6. Required Sample Sizes

Under the conditions of the Central Limit Theorem, the sample size n_0 needed to obtain $100\,(1-\gamma)\,\%$ power given $(\nu,\beta,\mu_0,\sigma,n)$ is implicitly determined by

$$\pi(\nu;\beta,\mu_0)=\Phi\left(-\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n_0}}}\right)-\Phi\left(\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n_0}}}\right)=1-\gamma$$

The functions

$$G(n|\nu,\beta,\mu_0)=\Phi\left(-\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)-\Phi\left(\Phi^{-1}\left(\frac{\beta}{2}\right)-\frac{\mu_0-\nu}{\frac{\sigma}{\sqrt{n}}}\right)+\gamma-1$$

and

$$n_0|\nu,\beta,\mu_0,\sigma,n=\min_{n>1}\left\{\operatorname{sgn}\left(G(n-1|\nu,\beta,\mu_0)\right)+\operatorname{sgn}\left(G(n|\nu,\beta,\mu_0)\right)=0\right\}$$

are implicitly solved to find the required sample sizes, with corresponding functions for the one-sample one-sided positive/negative cases.

Tables 1-5 of required sample sizes demonstrate the sensitivity of each parameter under the conditions of the Central Limit Theorem.

Each table has the same format. The value in the far-left column is the power function value (for the given hypothesis test) with increments given across the columns. The values in the body of the table are the minimum sample size n required under the parameter values for that table for the power function to reach the specified values.

The tables are grouped by values of ν, β, μ_0, and σ. The title of each table contains the ordered values of these parameters. Only results for the two-sided tests are shown in these tables, however, similar values may be calculated for one-sided positive and negative tests.

A value of *NaN* in the table signifies that there is no unique sample size corresponding to the given combination of parameter values (since the power function in this situation has no inverse function). A value of 0 in the table signifies that there is no sample size available to produce the specified power function value. Rows with all 0 values are not shown in the tables.

Table 1: Required Sample Size For ($\nu = \frac{1}{10}$, $\beta = \frac{1}{1000}$, $\mu_0 = 0$, $\sigma = 1$)
One-Sample Two-Sided Location Hypothesis Test To Achieve π () Power

π ()	**-0.00**	**-0.01**	**-0.02**	**-0.03**	**-0.04**	**-0.05**	**-0.06**	**-0.07**	**-0.08**	**-0.09**
1.0	NaN	3154	2856	2674	2541	2435	2347	2271	2204	2144
0.9	2090	2040	1994	1950	1910	1872	1836	1801	1768	1737
0.8	1707	1678	1650	1623	1597	1572	1547	1523	1500	1477
0.7	1455	1433	1412	1391	1371	1351	1331	1312	1293	1274
0.6	1255	1237	1219	1201	1184	1167	1149	1132	1116	1099
0.5	1082	1066	1050	1033	1017	1001	985	969	953	938
0.4	922	906	891	875	859	844	828	812	796	781
0.3	765	749	733	717	700	684	667	651	634	617
0.2	599	582	564	545	527	508	488	468	447	426
0.1	403	380	355	329	301	270	237	198	152	92

Table 2: Required Sample Size For ($\nu = \frac{1}{10}$, $\beta = \frac{1}{100}$, $\mu_0 = 0$, $\sigma = 1$)
One-Sample Two-Sided Location Hypothesis Test To Achieve π () Power

π ()	**-0.00**	**-0.01**	**-0.02**	**-0.03**	**-0.04**	**-0.05**	**-0.06**	**-0.07**	**-0.08**	**-0.09**
1.0	NaN	2403	2143	1986	1871	1781	1706	1641	1584	1533
0.9	1487	1445	1406	1370	1336	1304	1274	1246	1218	1192
0.8	1167	1143	1120	1098	1077	1056	1036	1016	997	979
0.7	961	943	926	909	892	876	861	845	830	815
0.6	800	785	771	757	743	729	716	702	689	676
0.5	663	650	637	625	612	600	587	575	563	551
0.4	539	527	515	503	491	479	468	456	444	432
0.3	420	409	397	385	373	361	349	337	325	313
0.2	300	288	275	262	250	236	223	210	196	181
0.1	167	152	136	120	103	86	67	47	24	0

Table 3: Required Sample Size For ($\nu = \frac{1}{10}$, $\beta = \frac{1}{20}$, $\mu_0 = 0$, $\sigma = 1$)
One-Sample Two-Sided Location Hypothesis Test To Achieve π () Power

π ()	**-0.00**	**-0.01**	**-0.02**	**-0.03**	**-0.04**	**-0.05**	**-0.06**	**-0.07**	**-0.08**	**-0.09**
1.0	NaN	1837	1611	1475	1376	1299	1235	1180	1132	1089
0.9	1050	1015	982	952	924	897	872	849	826	805
0.8	784	765	746	728	710	694	677	661	646	631
0.7	617	603	589	575	562	550	537	525	513	501
0.6	489	478	467	456	445	434	424	414	404	393
0.5	384	374	364	355	345	336	327	318	308	300
0.4	291	282	273	264	256	247	239	230	222	214
0.3	205	197	189	181	172	164	156	148	140	132
0.2	124	116	108	99	91	83	75	67	59	50
0.1	42	34	25	17	8	0	0	0	0	0

Table 4: Required Sample Size For ($\nu = \frac{1}{10}, \beta = \frac{1}{10}, \mu_0 = 0, \sigma = 1$)
One-Sample Two-Sided Location Hypothesis Test To Achieve $\pi\,()$ Power

$\pi\,()$	**-0.00**	**-0.01**	**-0.02**	**-0.03**	**-0.04**	**-0.05**	**-0.06**	**-0.07**	**-0.08**	**-0.09**
1.0	NaN	1577	1367	1243	1152	1082	1023	973	930	891
0.9	856	824	795	767	742	718	696	675	655	636
0.8	618	600	584	568	552	537	523	509	496	483
0.7	470	458	446	434	423	412	401	390	380	370
0.6	360	350	340	331	322	313	304	295	286	278
0.5	270	261	253	245	237	230	222	214	207	200
0.4	192	185	178	171	164	157	150	143	136	130
0.3	123	117	110	104	97	91	84	78	72	66
0.2	60	53	47	41	35	29	23	17	11	5

Table 5: Required Sample Size For ($\nu = \frac{1}{10}, \beta = \frac{3}{20}, \mu_0 = 0, \sigma = 1$)
One-Sample Two-Sided Location Hypothesis Test To Achieve $\pi\,()$ Power

$\pi\,()$	**-0.00**	**-0.01**	**-0.02**	**-0.03**	**-0.04**	**-0.05**	**-0.06**	**-0.07**	**-0.08**	**-0.09**
1.0	NaN	1418	1220	1102	1017	951	896	849	809	772
0.9	740	710	683	658	634	612	592	572	554	536
0.8	520	504	488	474	460	446	433	420	408	396
0.7	385	374	363	352	342	332	322	313	303	294
0.6	285	277	268	260	251	243	236	228	220	213
0.5	205	198	191	184	177	170	163	157	150	144
0.4	137	131	125	119	113	107	101	95	89	84
0.3	78	72	67	61	56	51	45	40	35	30
0.2	25	19	14	9	4	0	0	0	0	0

7. MAPLE Implementations

The following MAPLE[2] code implements the required sample size calculations (given all other parameters to achieve a given power level in the one-sample location hypothesis tests).

```
with(Statistics):
Phi:=RandomVariable(Normal(0,1)):
```

```
PowerOneSampleTwoSidedLocationTest:=proc(v,b,m,s,n)
   local c,cc,X;
   description "Power Function For A One-Sample Two-Sided
                Location Hypothesis Test";
   options `Copyright 2016 PQI Consulting All Rights Reserved`;
   X:=RandomVariable(Normal(0,1));
   c:=Quantile(X,b/2);
```

[2]MAPLE is a registered trademark of Maplesoft (a division of Waterloo Maple Inc.), 615 Kumpf Drive, Waterloo, Ontario, Canada, N2V 1K8. The MAPLE version used to produce the results was 16.02, November 18, 2012, Maple Build ID 788210.

```
   cc:=(m-v)/(s/surd(n,2));
   return evalf(CDF(X,-c-cc)-CDF(X,c-cc));
end proc;
```

```
PowerOneSampleOneSidedPositiveLocationTest:=proc(v,b,m,s,n)
   local c,cc,X;
   description "Power Function For A One-Sample One-Sided
                Positive Location Hypothesis Test";
   options ‘Copyright 2016 PQI Consulting All Rights Reserved‘;
   X:=RandomVariable(Normal(0,1));
   c:=Quantile(X,b);
   cc:=(v-m)/(s/surd(n,2));
   return evalf(1-CDF(X,-c+cc));
end proc;
```

```
PowerOneSampleOneSidedNegativeLocationTest:=proc(v,b,m,s,n)
   local c,cc,X;
   description "Power Function For A One-Sample One-Sided
                Negative Location Hypothesis Test";
   options ‘Copyright 2016 PQI Consulting All Rights Reserved‘;
   X:=RandomVariable(Normal(0,1));
   c:=Quantile(X,b);
   cc:=(m-v)/(s/surd(n,2));
   return evalf(CDF(X,c-cc));
end proc;
```