Bayesian Model Selection for Hierarchical Copulas and Vines

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Abstract
Copula models provide an effective tool for modeling joint distributions. Model selection allowing to choose an appropriate subclass of copulas remains a critical issue for many applications. The paper suggests an implementation of Bayesian model selection procedure based on ideas of Bretthorst, Huard et al. It allows us to compare several classes of Archimedean copulas (Frank's, Clayton's, and survival Gumbel-Hougaard families) and elliptical copulas (Gaussian and Student t-copulas). For dimensions higher than 2 we consider several types of hierarchical structures including nested Archimedean copulas, hierarchical Kendall copulas and vines. We consider a portfolio based on four national indices. Extreme market co-movements are modeled by the tail behavior of the joint distribution or index returns and currency exchange rates. Estimation of parameters within suggested copula families and hierarchical structures is carried out via empirical Bayes approach using random walk Metropolis algorithm and other Markov chain Monte Carlo techniques.

Key Words: copulas, Bayesian model selection, international markets, stock indices

1. Introduction

1.1 Index Futures, Hedging, and Diversification

The problem of hedging portfolio risks in the international markets is often resolved by taking long or short positions in national stock index futures allowing for the diversification of the risk. This diversification may be achieved if corresponding index futures are negatively associated with the other components of the portfolio.

Traditionally, analyzing correlations between the portfolio components (Markowitz model) provided a reasonable tool of such diversification. Prior to Markowitz approach risk assessment and profitability of portfolio investments was carried out through independent analysis of assets. Introducing the concept of diversification, Markowitz suggested that investors should build a portfolio based on joint performance of risk and return of its constituent assets. This approach became a significant step forward in comparison with a situation when investors formed their portfolios regardless of existing links in returns of included assets. Using historical returns of each asset it is possible to estimate the expected return and variance of the portfolio, which in turn can be viewed as proxy variables for return and risk. Examining different combinations of assets, Markowitz developed a method of forming a portfolio based on the balance of its return and risk.

In practice, many classes of assets are characterized by returns that cannot be described with a Gaussian distribution that comes into contradiction with the basic assumptions of the Markowitz approach. This fact was already discovered by Mandelbrot (1963). Moreover, it has been shown lately that correlations between index futures may change with time and also give a poor assessment of the tail behavior of the joint distributions of index returns.

In recent years, dependence models became a subject of increasing attention and are widely discussed. To update the concept of optimal portfolio researchers are trying to move away from the traditional assumption that asset returns are independent and normally distributed. In the framework of this paper we will not consider dynamic correlation models,

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but rather suggest several models for the entire joint distribution of several index returns. For this purpose we will use the instruments known as copula models. Copula models allow the researchers to analyze the marginal distributions of the random vector separately from their association structure defined by a special copula function, which can be determined for each pair of vector components depending on the strength of their association. Pair copulas are tied together into a multivariate hierarchical structure. Such models provide a wide range of options depending on the class of pair copula functions considered and the hierarchy on the set of variables. The objective of the paper is to compare several copula models and observe to which extent the final hedging decisions may depend on the choice of the model.

1.2 Copula Models for Joint Index Returns

In our study we will concentrate on four national indices: CAC-40 to represent France, HIS for Hong Kong, JSE for South Africa, and Standard and Poors 500 (SPX) for the U.S. market. This portfolio choice is not arbitrary, it includes representatives of geographically diverse clusters, which was demonstrated in Kangina et al. (2016) to provide good diversification benefits. We analyze daily index values for the period 2009-2011 available from a Bloomberg terminal and construct four-dimensional copula models for the joint distribution of the normalized index returns: residual daily logarithmic returns of national indices controlled for autoregression and heteroskedascticity.

Following Hansen (1994) and Gordeev et al. (2012), we will assume asymmetric t-distribution for each of the four variables corresponding to the normalized index returns. Thus we obtain a model for marginals. Elliptical copulas allow for direct four-dimensional construction of the joint distribution based on the marginals and their correlation matrix which can be estimated separately. If we do not assume exchangeability of the variables, Archimedean copulas require an extra step. We will choose a pair copula model and then define a hierarchy structure on the set of variables based on this model.

We will consider three families of Archimedean pair copulas, paying the most attention to Frank's family, which was proven in Gordeev et al. (2012) to provide the best overall fit for the national index data. The most popular methods to define a multivariate copula based on pair copula construction are (i) vine copulas (vines), (ii) hierarchical or nested Archimedean copulas (HAC), and (iii) hierarchical Kendall copulas (HKC). The objective of the paper is to compare these three constructions using same or similar pair copulas and to observe to which extent the tail probabilities may depend on the choice of the multivariate copula model.

For parametric estimation we use a Bayesian model with informative priors for margins and pair copulas obtained empirically from a broader empirical study of 27 national indexes in Kangina et al. (2016) and Knyazev et al. (2016). Weak or non-informative priors were suggested for the hierarchical copula parameters. Due to complicated expressions for multivariate copula models, Gibbs sampling was hard to implement and a version of random walk Metropolis algorithm was utilized. R environment was used for all computations.

Sections 2 and 3 include a brief description of pair copula models and three hierarchical structures to be compared. Section 4 is dedicated to the results of model selection and parametric estimation for index study, and Section 5 contains the numerical results: calculations of extreme co-movements of index returns, which are defined as probabilities of simultaneous extreme drop of all four indices.

2. Pair Copulas

2.1 Basic Definitions

Copula function is a binary mapping satisfying the following four conditions:

1. $C: [0,1]^2 \to [0,1]$.
2. For any $u, v \in [0;1]$ $C(0,v) = C(u,0) = 0$.
3. For any $u, v \in [0;1]$ $C(1,v) = v,\ C(u,1) = u$.
4. For any $0 \le u_1 \le u_2 \le 1,\ 0 \le v_1 \le v_2 \le 1,\ C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0$.

If $u = F(x)$ and $v = G(y)$ are two marginal distributions of variables X and Y, by virtue of Sklar's theorem [5] any joint distribution of the vector (X,Y) can be represented as a copula $Pr(X \le x, Y \le y) = C(F(x), G(y))$. There exist many copula types, where a particular case of $C(u,v) = uv$ corresponds to independence. In survival analysis and risk management applications paying special attention to the joint tail behavior, a special role is played by the family of Archimedean copulas.

2.2 Elliptical Copulas

The most popular elliptical copula is Gaussian copula which combined with marginal distributions $u_i = F_i(x_i)$ for the components of d-dimensional data vector $X = (X_1, \ldots, X_d)$ defines the joint distribution of vector X as

$$F(x_1, \ldots, x_d) = C_R(u_1, \ldots, u_d) = \Phi_{d,R}(\Phi^{-1}(F_1(x_1)), \ldots, \Phi^{-1}(F_d(x_d))),$$

where $\Phi(x)$ is standard normal distribution and $\Phi_{d,R}$ is d-variate normal with zero mean, unit variances and correlation matrix R. Off-diagonal elements of matrix R describe pairwise associations, so the strength of association may differ for different pairs of components of vector X.

In applications it is often desirable to model heavy-tailed multivariate distributions and tails of the joint distributions. In this situation, the Student t-copula should be used instead of the Gaussian copula. If we use multivariate Student t-distribution with η degrees of freedom and correlation matrix R, we obtain the Student copula or *t-copula*. As we saw for the Gaussian copula, the choice of an elliptical copula model does not prescribe the choice of marginals. They might be chosen separately.

$$F(x_1, \ldots, x_d) = C_{\eta,\eta_1,\ldots,\eta_d,R}(u_1, \ldots, u_d) = T_{d,\eta,R}(T^{-1}_{\eta_1}(F_1(x_1)), \ldots, T^{-1}_{\eta_d}(F_d(x_d))),$$

For inverse transforms we use univariate t-distributions with η_i degrees of freedom. In case of $d = 2$ we obtain pair copulas, but the same construction also works for the case of $d > 2$. In our study, elliptical copulas can be constructed directly for the set of four indices, without a necessity to specify a pair copula model.

2.3 Archimedean Copulas

Let $\phi(t)$ be a continuous, strictly decreasing convex function from $[0;1]$ to $[0;\infty)$ such that $\phi(1) = 0$. We determine *pseudo-inverse* of ϕ as follows:

$$\phi^{[-1]}(t) = \max\{\phi^{-1}(t), 0\}. \tag{1}$$

So $\phi^{[-1]}(t) = \phi^{-1}(t)$ if $0 \le t \le \phi(0)$ and $\phi^{[-1]}(t) = 0$ if $t > \phi(0)$. Pseudo-inverse is additionally defined for the values out of the range of the original function, which makes it different from the regular inverse. If $\phi(t) \to \infty$ when $t \to 0$ than the pseudo-inverse function coincides with the inverse function. Pseudo-inverses serve to extend the inverse transformation to the functions of limited range.

Let $C_\phi :\ I^2 \to I$ be a continuous non-decreasing function

$$C_\phi(u, v) = \phi^{[-1]}(\phi(u) + \phi(v)). \tag{2}$$

A copula $C_\phi(u, v)$ is an *Archimedean copula* if the function $\phi(t)$ is its *generator*.

2.4 Classes of Archimedean Copulas

Three most popular subclasses of the Archimedean family are:

Clayton's copula

$$\phi_\alpha(t) = t^{-\alpha} - 1.$$

$$P(X \le x, Y \le y) = C_C(u, v|\alpha) = max\{(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}, 0\}.$$

Gumbel-Hougaard's copula

$$\phi_\alpha(t) = (-\log t)^\alpha;$$

$$P(X \le x, Y \le y) = C_{GH}(u, v|\alpha) = \exp\{-[(-\log u)^\alpha + (-\log v)^\alpha]^{1/\alpha}\}.$$

The survival version of Gumbel-Hougaard's copula uses survival functions $1 - u = P(X > x)$ and $1 - v = P(Y > y)$ instead of marginals distributions, so that $P(X > x, Y > y) = C_{GH}(1 - u, 1 - v)|\alpha)$. This copula along with the Clayton's copula is especially convenient for modeling joint lower tails, which play a special role in risk management.

Frank's copula

$$\phi_\alpha(t) = -\log\Big[\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}\Big]$$

$$P(X \le x, Y \le y) = C_F(u, v|\alpha) = -\frac{1}{\alpha}\log\Big[1 + \frac{(e^{(-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\Big].$$

The choice of a specific Archimedean subclass to model pairwise dependence of the variables is a very important part of the model selection. It is certainly most convenient (though not necessary) to select one family of copulas to model all pairs. The most popular approach is to choose the best representative from each parametric subclass based on an estimation procedure for association parameter α such as maximum likelihood (MLE) or Bayes estimators, and then compare these best representatives to each other based on information criteria (AIC, BIC, or DIC) as in Jondeau and Rockinger (2006), or other characteristics of model fit (Kolmogorov-Smirnov or chi-squared distances) as in Fermanian and Scaillet (2003). In the following subsection we discuss a different approach.

2.5 Bayesian Model Selection

Following Bretthorst (1996) and Huard et al. (2006), we suggest to compare the data fit provided by several pair copula models not at a single value of association parameter(s) obtained by MLE, but rather over the entire range of possible association values. This can be accomplished by specifying a prior distribution for association parameter(s) and integrating the likelihood with respect to the prior distribution. The problem is the difference of meaning and ranges of association parameters for different copula classes. If we want to compare several classes of copulas in Bayesian framework, we need to establish the common basis of comparison. For that purpose we need to suggest a universal parameter, which can be evaluated for all classes of copulas under consideration.

One of such universal parameters is Kendall's concordance τ, which can be conveniently expressed in terms of association for many copula families. Sample concordance $\hat{\tau}$ is a reasonable non-parametric estimator of τ. Using formulas expressing concordance through association parameters, see, for example, Genest and Rivest (1993), we can calculate values of τ induced by MLE for parameters of elliptic or Archimedean copulas and compare them to the sample values of $\hat{\tau}$. Proximity of model induced values of τ to the sample value $\hat{\tau}$ may serve as a measure of the model fit and help to compare the model performance, see also Persons et al. (2012). However, this comparison is still using single values representing entire families.

We will assume that the classes of copulas we choose represent exhaustive and mutually exclusive hypotheses $H_1, H_2, \ldots, H_m$. Posterior probabilities of hypotheses $H_k, k = 1, \ldots, m$, for data D may be rewritten as

$$P(H_k \mid D) = \int P(H_k, \tau \mid D) d\tau = \frac{\int P(D \mid H_k, \tau) P(H_k \mid \tau) \pi(\tau) d\tau}{P(D)}, \tag{3}$$

where we will consider all m hypotheses a priori equally likely. If the dependence between variables is positive for all hypotheses, we can assume $\tau \geq 0$. In this case the natural choice of prior for τ is beta distribution, and the choice of parameters for the prior can be subjective or non-informative objective. However, in presence of relevant additional data, similar in nature, the prior might be suggested by sample concordance for this data consistently with empirical Bayes approach: $P(D \mid H_k, \tau) = L_k(D \mid \alpha(\tau))$, $P(H_k \mid \tau) = P(H_k) = \frac{1}{m}$, $\pi(\tau) \sim Beta(\hat{a}, \hat{b})$. If we have multiple pairs of variables included in the study, estimates of parameters of the Beta distribution for empirical Bayes can be obtained from all pairs of components.

We will not need to calculate the denominator of the posterior in (3). It suffices to calculate the weights

$$W_k = \int_0^1 L_k(D \mid \alpha(\tau)) \pi(\tau) d\tau = \int_0^1 \Pi_{i=1}^n c_\alpha(\hat{u}_i, \hat{v}_i \mid \alpha(\tau)) \pi(\tau) d\tau, \tag{4}$$

where $c_\alpha(\hat{u}_i, \hat{v}_i)$ is the corresponding copula density for estimated marginals. Instead of this integral, using Monte-Carlo approach and drawing samples from the Beta prior, evaluate

$$\hat{W}_k = \frac{1}{N} \sum_{j=1}^{N} \Pi_{i=1}^n c_\alpha(\hat{u}_i, \hat{v}_i \mid \alpha(\tau_j)). \tag{5}$$

Then we choose the class with the highest weight and obtain the Bayes estimate of the association parameter using MCMC.

3. Multidimensional Copula Constructions

Elliptical copulas allow for direct multivariate copula construction with no restriction to $d = 2$. However, direct extension of Archimedean copulas to higher dimensions requires exchangeability of the marginals, which seriously restricts the possibility to model different strength of association between the variables. We will consider three different constructions which make it possible to apply Archimedean copulas for $d = 3$ and $d = 4$ without exchangeability assumption.

3.1 Vine Copulas

Let us begin with $d = 3$ and consider the problem of modeling joint distribution $P(X \leq x, Y \leq y, Z \leq z)$ using marginals $u = F(x), v = G(y), w = H(z)$ and pair copulas. Let us designate one out of three variables (say, Y) as the central variable, whose associations with both X and Z are most important. Modeling the association between X and Z will have a lower priority. This "hierarchy" of dependence structure is inevitable, and can be established either from context, or by preliminary estimation of the strength of pairwise associations. Two equivalent ways to graphically illustrate the hierarchy of pairwise associations are suggested in a centered vine diagram in Figure 1. Primary links between the variables are indicated by solid lines, and the secondary links by dashed lines.

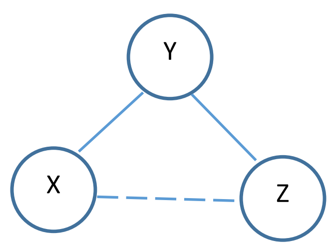

Figure 1: Vine structure, $d = 3$.

We will model primary associations for pairs (X, Y) and (Y, Z) using two pair copulas:

$$C_1(u, v) = C_1(F(x), G(y));\ C_2(v, w) = C_2(G(y), H(z)).$$

We also introduce the conditional copula $C_3(F_{X|Y}, H_{Z|Y})$ for the secondary link.

Therefore, we can express the triple joint density in terms of marginal densities, two pair copulas for (X, Y) and (Y, Z), and one additional copula, which is defined on conditional distributions rather than on marginals. The following diagram describes the two-level tree of associations, where at the first level we need two pair copulas, and at the second just one conditional pair copula which uses conditional distributions instead of marginals as its arguments:

$$(X, Y), (Y, Z); (X, Z|Y).$$

For dimension $d = 4$, we can use the vine structure above but also have to cope with an additional variable W, which should be linked to the dependence diagram for three variables. We will have to add a new link. The new link connects W either with the central variable Y or with a non-central Z (or X) - whatever association is more important. This distinction brings about classification of vine diagrams into two popular types: C-vines and D-vines. This importance can be determined by contextual meaning of the variables or by rough estimation of the strength of their association. The diagram in Figure 2 corresponds to a C-vine, where Y plays the role of the central variable, primary links are shown by solid lines (star symbol indicates the new link), secondary links - by dashed line, and the third-level links by a dotted line. Figure 3 shows a so-called D-vine, where the variables are linked in a straight chain. Three-level trees of the process for a C-vine and D-vine

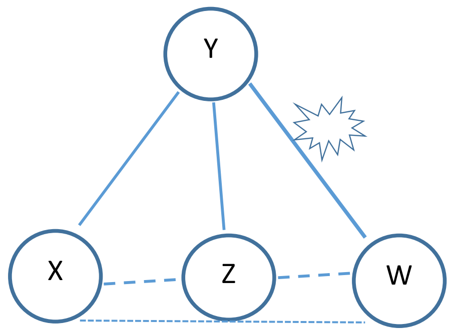

Figure 2: C-vine for $d = 4$.

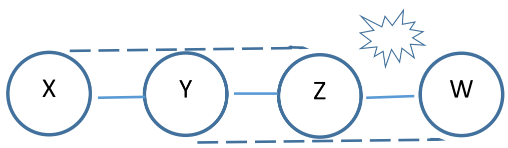

Figure 3: D-vine for $d = 4$.

respectively are:

$$(X, Y), (Y, Z), (\mathbf{Y}, \mathbf{W}); (X, Z|Y), (\mathbf{Z}, \mathbf{W}|\mathbf{Y}); (X, W|Y, Z).$$

and

$$(X, Y), (Y, Z), (\mathbf{Z}, \mathbf{W}); (X, Z|Y), (\mathbf{Y}, \mathbf{W}|\mathbf{Z}); (X, W|Y, Z).$$

with the highlighted differences between C-vine and D-vine caused by the star-designated link in Figure 2 and Figure 3, see Aas et al. (2009).

3.2 Hierarchical (Nested) Archimedean Copulas (HAC)

Let us suppose that $u = F(x), v = G(y), w = H(z)$ are three marginal distributions in case of $d = 3$. Hierarchical copula construction

$$C((u, v), w) = C_2[C_1(u, v), w] \tag{6}$$

uses two different generators ϕ_1 for the inner (*cluster*) copula C_1 and ϕ_2 for the outer (*nesting*) copula C_2. Not only their association values α_1 and α_2, but also the subclasses

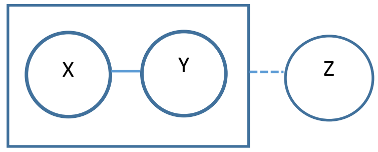

Figure 4: Hierarchical Archimedean copulas, $d = 3$.

of Archimedean copulas (e.g., Clayton, Frank, Gumbel-Hougaard's families) could be different for C_1 and C_2. The suggested hierarchical clustering/nesting diagram is shown in Figure 4. Using generator representation, since $C_1(u, v) = \phi_1^{[-1]}(\phi_1(u) + \phi_1(v))$ and

$$C_2(C_1, w) = \phi_2^{[-1]}(\phi_2(C_1) + \phi_2(w)),$$

$$C((u, v), w) = \phi_2^{[-1]}[\phi_2 \circ \phi_1^{[-1]}(\phi_1(u) + \phi_1(v)) + \phi_2(w)]. \tag{7}$$

Additional conditions are required for the key element of this construction, the superposition $\psi(t) = \phi_1 \circ \phi_2^{[-1]}(t)$ to guarantee that $C((u, v), w)$ is a legitimate copula function.

In case of four variables one may consider two following possibilities of initial clustering depending on the importance of pairwise associations. In Figure 5 the hierarchy diagram corresponds to a fully nested model

$$C(((u, v), w), t) = \phi_3^{[-1]}\Big(\phi_3 \circ \phi_2^{[-1]}[\phi_2 \circ \phi_1^{[-1]}(\phi_1(u) + \phi_1(v)) + \phi_2(w)] + \phi_3(t)\Big), \tag{8}$$

and in Figure 6 - to a non-nested model

$$C((u, v), (w, t)) = \phi_3^{[-1]}\Big(\phi_3 \circ \phi_1^{[-1]}[\phi_1(u) + \phi_1(v)] + \phi_3 \circ \phi_2^{[-1]}[\phi_2(w) + \phi_2(t)]\Big). \tag{9}$$

If pair copula in a HAC construction belong to the same class (e.g., all Clayton's or all Frank's), then generators in all links may be parametrized as $\phi_i = \phi_{\alpha_i}$ and conditions for (7), (8), and (9) being legitimate copulas can be formulated in terms of α_i, see Hofert and Maechler (2011). For applications in higher dimensions see also Hofert and Scherer (2011), Okhrin and Ristig (2014), and Puzanova (2011).

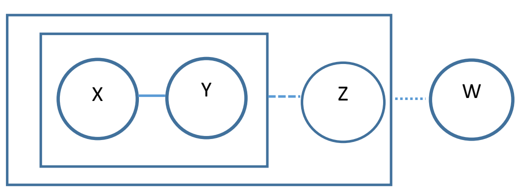

Figure 5: Hierarchical Archimedean copulas, fully nested.

3.3 Hierarchical Kendall Copulas (HKC)

This construction introduced in Brechmann (2014) utilizes Kendall's distribution function $K_C(t) = P(C(U, V) \le t)$ defined for Archimedean copulas as $K_C(t) = 1 - \frac{\phi(t)}{\phi'(t)}$. For a simple illustration let us suppose that a triple copula $C(u, v, w)$ is built in two steps:

1. Choose a cluster (inner) copula $C_1(u, v)$, estimate its association parameter, and calculate $K_{C_1}(t)$.

2. Choose a nesting (outer) copula C_2 and use $C(u, v, w) = C_2(K_{C_1}(t), w)$.

This construction can be used not only for Archimedean copulas, but in the latter case the calculation of Kendall's distribution can be done straightforwardly through its generator.

Comparison of all three constructions on index data for $d = 4$ is provided in Section 5. Differences between these methods become more dramatic in higher dimensions. Vine copulas offer much flexibility in combining various types of copulas in one model and providing a huge variety of dependence hierarchies (compare C-vines and D-vines for different component ordering).

4. Stock Index Study

Our goal is to build a four-dimensional copula model for four indices (CAC, HIS, JSE, SPX). We will first select a model for the margins, then decide on the model for pair copulas. Finally, for Archimedean copulas we will choose the best way to link pairs into a hierarchical structure. We will separately do it for vines, HAC, and HKC based on the data for the period from 2009 to 2011. Then in Section 5 we will compare the results of applying these models to estimation of the tails of joint distributions. We also perform out-of-sample model validation using new data for the period from 2012 to 2015.

4.1 Model for Margins

Independent of the choice of the hierarchical structure, all pair copula parameters are estimated using empirical Bayes approach. The first step is to transform raw data for each of the national indices CAC, HIS, JSE, and SPX, converting time series of daily index prices $S_i, i = 1, \ldots, n$ into stationary matched samples of normalized residual logarithmic returns t_i, filtering out effects of autocorrelation and heteroskedasticity, as in Gordeev et al. (2012) and Knyazev et al. (2016).

$$t_i = \frac{\varepsilon_i}{\sqrt{h_i}},\ \varepsilon_i = R_i - \beta_0 - \beta_1 R_{i-1},\ h_i = \beta_2 + \beta_3 h_{i-1} + \beta_4 \varepsilon_i^2,\ R_i = \log\frac{S_i}{S_{i-1}},$$

where β_k are coefficients of ARIMA/GARCH model estimated for each index time series independently. As it was shown in Gordeev et al. (2012), assumption of normality usually does not hold for such samples. A better model for the margins is provided by an asymmetric t-distribution with degrees of freedom η and parameter of asymmetry λ: $t_i \sim T_{\eta,\lambda}$ defined in Hansen (1994).

We will use the results of a broader study of 27 indices from Knyazev et al. (2016). The empirical Bayes estimates in Table 1 for four indices considered in the present paper were

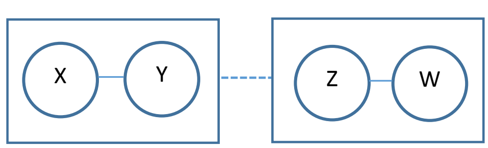

Figure 6: Hierarchical Archimedean copulas, non-nested.

obtained in Kangina et al. (2016) and the resulting models were tested via Kolmogorov-Smirnov goodness-of-fit test.

Table 1: Parameters of t- Margins

Index	η	λ	p-value
CAC (France)	24.50	0.03	0.17
HIS (Hong Kong)	23.95	-0.04	0.12
JSE (South Africa)	18.00	-0.08	0.58
SPX (USA)	7.40	-0.13	0.08

4.2 Pair Copulas

For pair copula selection we use the same study of 27 indices (378 pairs). Applying Bayesian model selection from Subsection 2.5 with empirical Beta priors on Kendall's τ, we obtain results in Table 2, showing for how many pairs out of 378 a certain pair copula model demonstrated the best fit from Bayesian model selection point of view.

Table 2: Number of Pairs with the Best Fit

Pair copula model	Number of pairs
Clayton	0
Gumbel-Hougaard (survival)	5
Frank	13
Gaussian	160
Student t	200

These results confirm the finding of Persons et al. (2012), suggesting that elliptical models, especially Student t-copulas, provide a better overall fit for national index data. This is not necessary true when we use different criteria of model selection such as AIC or BIC. Therefore we will use t-copulas in our final comparison with Archimedean models. The choice of Archimedean model suggested in Kangina et al. (2016) and supported by Table 2 is Frank's copula. Therefore for every pair of indices out of four (total of six pairs) the parameter of association α is estimated under assumption of Frank's copula model. Similar estimation was done for Clayton's and survival Gumbel-Hougaard's models, which are known to better model the joint distribution tails, but it goes out of the scope of this paper. Table 3 shows the results of Frank's pair copula parametric estimation in the decreasing order of association.

4.3 Vines: Conditional Copula Parameters

In order to build a vine copula model, one has to determine the best possible vine structure for dimension $d = 4$. We will use information approach and judge the fit of the model by the lower value of AIC and also by the error of the generalized method of moments: comparing the discrepance $\delta = |\hat{\tau} - \tau(\alpha)|$ between the empirical Kendall's concordance $\hat{\tau}$ and the model induced value of $\tau(\alpha)$. These two measures yield rather consistent results.

We have two choices: C-vine or D-vine, and we have to determine the relative placement of the nodes in the vine. To help with initial determination of the vine structure, we

Table 3: Frank's Pair Copula Parameter α

Pair	Frank's α
CAC, SPX	6.36
CAC, JSE	5.49
JSE, SPX	3.53
HIS, JSE	2.98
CAC, HIS	2.65
HIS, SPX	1.84

can draw a diagram of association between four indexes Figure 7 using empirical Kendall's concordance $\hat{\tau}$ calculated for each matched pair of samples t_i. The best choice according

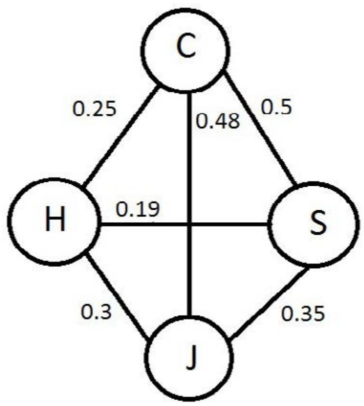

Figure 7: Diagram of Association.

to AIC is provided in Figure 8, while its conditional copulas and their concordances are shown in Figure 9. A D-vine in Figure 10 gives a very close value of AIC.

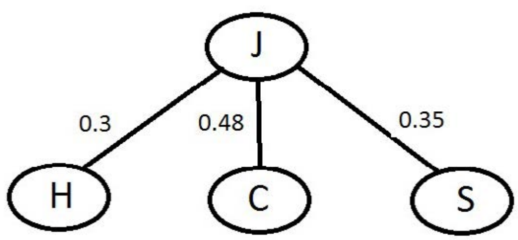

Figure 8: C1. Best Choice of C-vine.

4.4 HAC and HKC: Cluster and Nesting Association Parameters

Assuming that copulas on all hierarchy levels belong to the same class and vary only by the value of parameter α, we take into the account that for (7) to represent a copula, for any class from Subsection 2.3 it is sufficient that $\alpha_1 > \alpha_2 > \alpha_3$ for the nested model (8) and $\alpha_1 > \alpha_3; \alpha_2 > \alpha_3$ for the non-nested model (9) (see Hofert and Maechler (2011)). Therefore we first estimate the inner associations α_i, and then the outer associations α_j, for which we suggest a weak prior $\alpha_j \sim Unif[0, \alpha_i]$. The choice of the hierarchy is based on

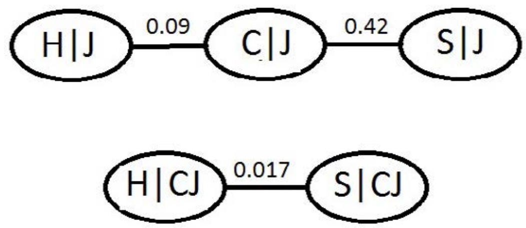

Figure 9: Conditional Copulas for the Best C-vine.

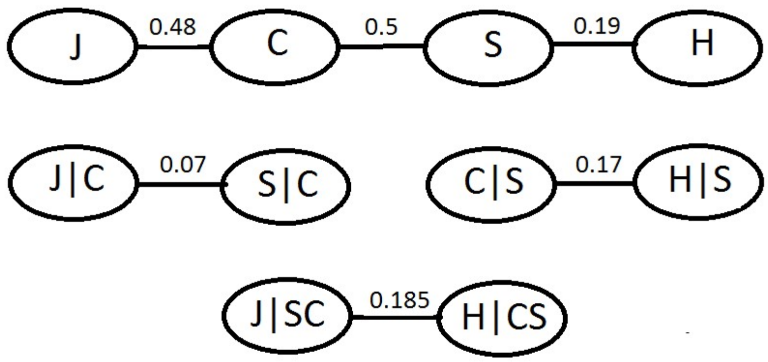

Figure 10: D1. Best Choice of D-vine.

a simple rule applied to the diagram in Figure 7: for any of the four triplets possible, the pair with the highest concordance value forms the inner link.

For HKC construction, Kendall's distribution function for the Frank's copula may be represented as

$$K(t) = t + \frac{(1 - e^{\alpha t})}{\alpha} \log\Big(\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}\Big).$$

For quadruplets the hierarchy of concordance dictates the choices summarized in Table 4. The first choice is a non-nested model with two cluster parameters α_1 and α_2, and one nesting parameter α_3. The second is a completely nested model with three levels of hierarchy.

Table 4: Nesting Copula Parameters in HAC and HKC

Triplet	α_1	$\alpha_2(HAC)$	$\alpha_3(HAC)$	$\alpha_2(HKC)$	$\alpha_3(HKC)$
((CAC, SPX), (HIS, JSE))	6.36	2.98	2.97	2.98	4.58
(((CAC, SPX), JSE),HIC)	6.36	4.27	1.90	4.99	3.19

From AIC standpoint, the non-nested model provides a slightly better fit for HAC and the nested model fits HKC better.

5. Tail Probabilities

Figure 11 demonstrates a diagonal cross-section of joint CDF of normalized index log-return residuals. Black curve corresponds to the empirical CDF, green line to the best vine model, blue line to best HAC, and red to the best HKC (all with Frank's pair copula). Magenta line corresponds to four-dimensional Student t-copula with correlations R estimated by the method of moments and degrees of freedom $\eta = 7$ (the best integer approximation to MLE). Figure 12 zooms in at the joint left tail (simultaneous drops of four indices from 1 to 3 standard deviations).

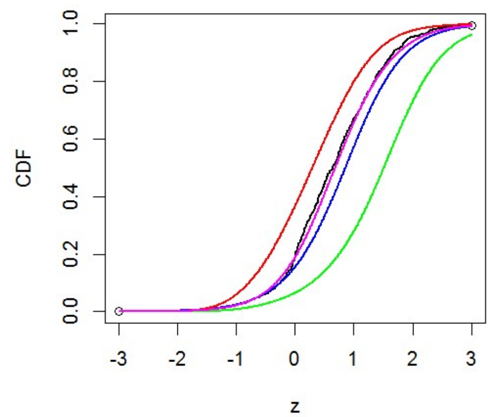

Figure 11: Estimates (2009-2011). Diagonal cross-section of joint CDF.

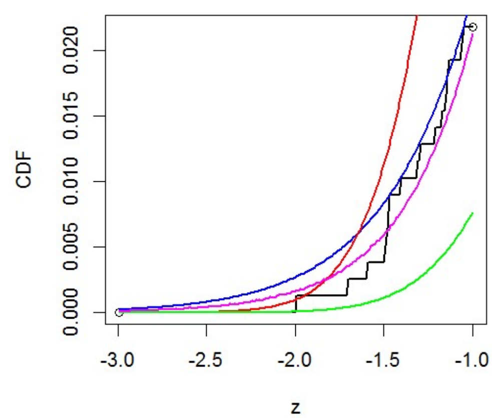

Figure 12: Estimates (2009-2011). Left tails.

The last two figures demonstrate the results of out-of-sample model vaildation. Figure 13 demonstrates a diagonal cross-section of joint CDF of normalized residuals for 2012-2015. Black curve corresponds to the actual empirical CDF for this period, and the model lines with parameters estimates from 2009-2011 demonstrate the predictions. Figure **??** zooms in at the joint left tail.

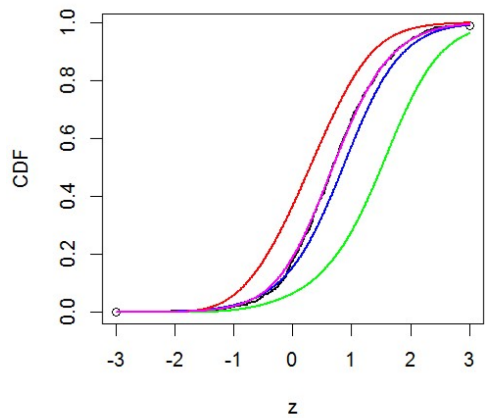

Figure 13: Prediction (2012-2015). Diagonal cross-section of joint CDF.

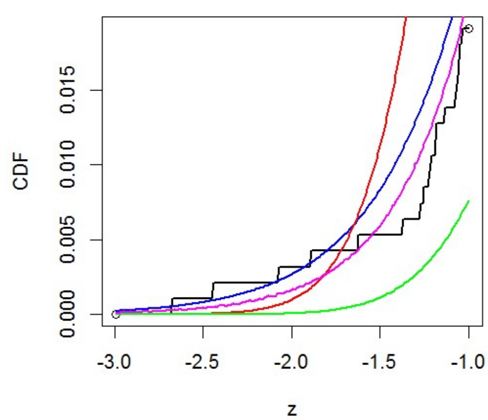

Figure 14: Prediction (2012-2015). Left tails.

6. Conclusions

Results presented in this paper suggest that studying non-linear effects and extreme co-movements of financial variables can explain the behavior of complex multinational investment portfolios in different ways than the traditional correlation analysis. Modeling national stock indexes with the help of copula models provide valuable insights on their dependence structure beyond correlation used in Markowitz model. The effects of thisn dependence structure may be observed for relatively low dimension of four.

Numerical results also suggest that:

- Elliptical copulas provide more robust choices for bivariate models.
- Tail probabilities depend on the hierarchical structure: best choice is provided t-copulas. For Archimedean copulas HAC seem to provide the best structure.
- Models have surprisingly good predictive quality.

Finally, Bayesian approach has several advantages:

- Incorporation of prior information obtained for geographically and temporally diverse financial environments.
- Stable results resolving some issue with MLE in high parametric dimensions.
- Carefully chosen priors promise certain advantages.

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