Effects of Standard Deviation Estimation on The $\overline{X}$ Control Chart and Adjustments for a Guaranteed In-Control Performance

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Abstract
The $\bar{X}$ control chart is commonly used for monitoring the mean of a process. Its performance when the process parameters are estimated has been widely discussed in the literature. Most of these studies have focused on the unconditional in-control (IC) run length distribution. However, recent works showed that in the face of parameter estimation, the knowledge of the conditional IC average run length ($CARL_0$) distribution or the conditional false alarm rate ($CFAR$) distribution may be more useful. To this end, we study the performance of the $\bar{X}$ chart where the mean is specified but the standard deviation is unknown and is estimated from a set of Phase I reference data, by providing a closed form expression for the cumulative distribution function (c.d.f.) of the $CFAR$ (and the $CARL_0$). Using these expressions, we construct a one-sided prediction interval for the $CFAR$ for several Phase I (reference) sample sizes and show that the minimum number of reference samples that guarantees a desired typical nominal IC performance is large and infeasible in many practical settings. Following up, we propose corrections of the control limits in order to guarantee a desired IC performance for several numbers and sizes of reference samples.

Key Words: $\bar{X}$ Control Chart Performance, Conditional and Unconditional Performance, False Alarm Rate, Control Limits Adjustments, Guaranteed In-Control Performance, Average Run Length

1. Introduction

Control charts, created by Walter A. Shewhart while working for Bell Labs in the 1920s and first published in a book in 1931, are still one of the most used tools for monitoring the quality characteristics of a process (see, for example, Alsyouf et al. (2015)). The control chart to monitor a process mean, namely $\bar{X}$ control chart is widely used in practice in many different areas such as manufacturing industries and medicine (see, for example, Albloushi et al. (2015)). The in-control (IC) process mean (μ_0) and standard deviation (σ_0) are important parameters in the designing of the $\bar{X}$ control chart. Usually these parameters are unknown and they are estimated from m reference samples each of size n when process is in control. This phase or part of SPC is called Phase I. For an overview of Phase I statistical process control the reader is referred to Chakraborti et al. (2009) and Jones-Farmer et al. (2014).

After the Phase I analysis (and hence the estimation of the process parameters and control limits), the $\bar{X}$ control chart is used to start monitoring the process mean, based in incoming or test data, and this part of the SPC is called Phase II. The performance of the Phase II $\bar{X}$ control chart, when their Phase I parameters estimates ($\hat{\mu}_0$ or $\hat{\sigma}_0$) are used, may be severely affected relative to the situation when μ_0 or σ_0 are known. Since Shewhart (1939, p. 76), this problem has been recognized and analyzed by several authors. For example, Quesenberry (1993), Chen (1997) and Chakraborti (2000) have showed that a lot of Phase I data are needed to get accurate control chart limits with unconditional performance comparable to the known parameter case. For reviews of the works on the effect of parameters estimation on the performance of control charts in general until 2006, see Jensen et al. (2006) and, for more recent developments, see Psarakis et al. (2014).

In the present paper we study the conditional performance of the $\bar{X}$ control chart when only the IC process standard deviation (σ_0) is estimated (by $\hat{\sigma}_0$). According to Montgomery (2009), when the mean of the quality characteristic is controlled by adjustments to the machine, the use of the nominal (or target) values of the process mean, μ_0, (instead of estimate the process mean by $\hat{\mu}_0$) and estimate only the process standard deviation, σ_0 (by $\hat{\sigma}_0$) are sometimes helpful in achieving management goals with respect to process performance. Ghosh et al. (1981) also studied the $\bar{X}$ control chart in this case, but differently than us, they focused on the unconditional performance. In the present paper, we argue that the conditional performance approach is more useful to the user.

The most common performance measure of a control chart presented in the literature is the average number of samples until an alarm (or signal) is signaled by the chart. This is also called the average run length (ARL) and it is the expected value of the discrete random variable called the run length (RL), which stands for the number of samples until an alarm. When process parameters are known, the RL follows a geometric distribution in which the parameter is the probability of a signal ($P(Signal)$). Note that, in this case, $E(RL) = ARL = 1/P(Signal)$. When the process is in-control (IC), the $P(Signal|IC)$ is called the False Alarm Rate (FAR) and $ARL = ARL_0 = 1/FAR$. In the case where the IC standard deviation has to be estimated (by $\hat{\sigma}_0$), ARL and $P(Signal)$, or FAR when process is IC, are conditioned on the values of the estimated standard deviation (for example, $ARL = E(RL|\hat{\sigma}_0)$ and $FAR = P(Signal|IC, \hat{\sigma}_0)$), so in this cases, they are random variables (since $\hat{\sigma}_0$ is a random variable) with their own means and standard deviations ($AARL = E\big(E(RL|\hat{\sigma}_0)\big) = E(RL)$ and $SDARL$, respectively for ARL). To make explicit that the values of ARL, $SDRL$ and FAR are conditioned on the values of the estimated standard deviation ($\hat{\sigma}_0$), let´s denote them respectively by $CARL = E(RL|\hat{\sigma}_0)$, $CSDRL$ and $CFAR = P(Signal|IC, \hat{\sigma}_0)$. In this case, RL does not follow a geometric distribution. However, conditioned on $\hat{\sigma}_0$, the conditioned run length distribution of the $\bar{X}$ chart is geometric (with parameter $CFAR$, in the case of IC process). So note that, in this case, $E(RL_0|\hat{\sigma}_0) = CARL_0 = 1/CFAR$. Also note that, according to the Law of Total Expectation, $AARL = E(CARL) = E\big(E(RL|\hat{\sigma}_0)\big) = E(RL)$.

Most researchers studying the effect of the parameter estimation on the performance of control charts have focused on the unconditional IC average run length ($AARL_0 = E(CARL_0) = E(RL_0)$) as the main performance measure. However, in a given application, the user would most likely have a single reference sample to estimate the control chart parameters and thus the performance of the Phase II chart would depend on the estimates obtained from the reference sample. Thus, several authors like Trietsch and Bischak (1998), Chakraborti (2000 and 2006), Bischak and Trietsch (2007), Epprecht et al. (2015) and Saleh et al (2015b) have argued that a study focusing on the distributions of the random variables $CARL_0$ and $CFAR$ is more useful to practitioners. In this context, we argue that the IC performance should not only be measured with the overall average $AARL_0 = E(CARL_0) = E(RL_0)$), but since the $CARL_0$ is a random variable, one should examine the probability that the $CARL_0$ (or the $CFAR$) is higher than a specified large value. This exceedance probability can be a good measure of the in-control performance of the chart with estimated parameters. For example, it is desirable that the probability that the $CARL_0$ exceeds 370 (the most common target for the number of sample until a false alarm in average) should be high, since if it's small, then the $\bar{X}$ chart is not performing well as it is desirable to have a large in-control ARL such as 370. This is called the exceedance probability criterion and it was introduced by Albers et al. (2005).

Focusing on the exceedance probability criteria, one of the main objectives of this work is to better understand the distributions of the $CARL_0$ and $CFAR$ when the standard deviation is estimated. After all, the best scenario is when the $CARL_0$ is large and the $CFAR$ is small, but as noted earlier, these are random variables, so we can study their distributions and calculate some exceedance probability. In the latter sense, we consider a one-sided prediction interval for the $CFAR$. Then, for several sample sizes, we obtain the minimum number of reference samples that guarantees, with a specified high probability (say, 90%), that the conditional false alarm rate does not exceed the nominal value by more than a pre-specified small percentage (say, 10%). This analysis is in the same spirit as in Epprecht et al. (2015), who considered the S chart. We will see that as in Epprecht et al. (2015), the minimum number of the reference samples needed to guarantee an acceptable in-control conditional performance of the $\bar{X}$ chart, is quite large, larger than what has been suggested by earlier authors.

Hence, considering the fact that such a large number of reference samples would be unfeasible in many practical settings, we finally propose some adjustments to the control limits of the $\bar{X}$ chart, so as to limit to a specified small probability, the probability that the $CFAR$ exceeds a pre-specified bound which may be tolerated by the user. Using a similar formulation, Gandy and Kvaloy (2013), Aly et al. (2015) and Faraz et al (2015) used the bootstrap method to make such adjustment for many types of charts, but not for the situation in which we are considering here. Moreover, we obtain exact formulas for the adjustments in the normal distribution case which can be implemented without bootstrapping. It may be noted that using an exceedance probability control criterion, Albers and Kallenberg (2004a, 2004b, 2005) also proposed control limit corrections, bude differently than us, for the X chart (individual data) in the case where the process mean and the process standard deviation are both estimated. To this end, they derived an approximation for the distribution of $CFAR$ instead of deriving exact expressions like us.

The key to our results is the cumulative distribution function (c.d.f.) of the $CFAR$ (and the $CARL_0$). We derive exact expressions for the c.d.f.'s under normality and do not consider any asymptotic methods or approximations (or simulations) unlike many authors studying the problem of parameter estimation for the $\bar{X}$ control chart.

Finally, it should be noted that the present paper is a part of a bigger work in which we also study other cases of parameter estimation, including the case when the process mean is estimated and we also analyse the out-of-control performance after the adjustment of the limits.

The paper is organized as follows: In Section 2, we consider the $\bar{X}$ control chart when the process standard deviation is estimated. We derive exact expressions for the $CFAR$ and the c.d.f. of the $CFAR$. Prediction bounds for the $CFAR$ is considered in Section 3. In Sections 4 and 5 we consider the minimum number of reference samples and the adjustment to the control limits in order to achieve a specified in-control performance. Finally, conclusions are presented in Section 6.

2. The $\bar{X}$ Control Chart model when Process Standard Deviation is Estimated

First note that in the ideal (but unreliable) case, the process mean and standard deviation are both known. The general expressions for the upper and low control limits (UCL and LCL) of the $\bar{X}$ chart are

$$UCL = \mu_0 + L\frac{\sigma_0}{\sqrt{n}}, \tag{1}$$

$$LCL = \mu_0 - L\frac{\sigma_0}{\sqrt{n}} \tag{2}$$

where μ_0 is the in-control process mean and σ_0 is the in-control process standard deviation. This situation is referred to as Case KK.

In many cases, however, just the standard deviation σ_0 is unknown but the mean is specified or known. This is referred to as Case KU. In this case, the process standard deviation is typically estimated from m historical (reference) samples each of size n when process is in control from a Phase I analysis. Let $\hat{\sigma}_0$ denote this estimator. Mahmoud et al. (2010), after having analyzed several estimators of the standard deviation (σ_0) in terms of the mean squared error, recommended that $\hat{\sigma}_0 = S_p = \sqrt{\frac{1}{m}\sum_{i=1}^{m} S_i^2}$ where $S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{i,j} - \bar{X}_i\right)^2$ and $\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ (where $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ and X_{ij} is the j-th observation of the i-th Phase I sample of size n For this reason, in the present work, we considered $\hat{\sigma}_0 = S_p$. We also assume that X_{ij} is normally distributed and independent ($X_{ij} \sim N(\mu_0, \sigma_0)$).
If the operator of the chart wishes to set limits in function of a nominal false alarm rate, namely α_{nom}, then $L = z_{\frac{\alpha_{nom}}{2}} = \Phi^{-1}\left(1 - \frac{\alpha_{nom}}{2}\right)$ where Φ^{-1} is the inverse of the standard normal cumulative distribution function. The usual 3-sigma limits correspond, thus, to a nominal false alarm $\alpha_{nom} = 0.0027$, be it explicitly desired by him/her or just ignored.

2.1 The Conditional False Alarm Rate

In Case KU, with $\boldsymbol{\hat{\sigma}_0 = S_p}$, the conditional probability of a signaling event ($\boldsymbol{S}$), during Phase II, is expressed as below.

$$P\left(Signal \middle| S_p\right) = 1 - P\left(\mu_0 - L\frac{S_p}{\sqrt{n}} \le \bar{X}_k \le \mu_0 + L\frac{S_{pooled}}{\sqrt{n}}\right) \tag{3}$$

$\bar{X}_k$ is the mean of kth sample collected during Phase II, so $\bar{X}_k = \frac{1}{n}\sum_{j=1}^{n} X_{kj}$ and $k = m+1, m+2, m+3, \ldots$ To maintain generality, let μ denote the process mean during Phase II, so $\bar{X}_k \sim N(\mu, \frac{\sigma_0}{\sqrt{n}})$. Note that if $\mu = \mu_0$, the process mean is in control. However, when $\mu = \mu_1 \neq \mu_0$, the process mean is out of control. Also let $W = \frac{S_{pooled}}{\sigma_0}$ denote the *error factor* of the estimate of the in-control process standard deviation σ_0. In the situation when process is in control, the conditional false alarm rate ($CFAR$), which is the probability of a signal in the in-control case, can be expressed as below.

$$CFAR = 1 - \big(\Phi(LW) - \Phi(-LW)\big) = 2\Phi(-LW) \quad (4)$$

Using the fact that $Y = m(n-1)W^2 \sim \chi^2_{m(n-1)}$ and the probability integral transformation, the c.d.f. of Y, namely $F_Y(Y)$ (or $F_{\chi^2_{m(n-1)}}(Y)$), has the same distribution of a random variable U uniformly distributed between 0 and 1, we can write $Y = F^{-1}_{\chi^2_{m(n-1)}}(U)$. So, it is clear that the $CFAR$ in Case KU depends on one random variable Y (or, equivalently, on U or on W) and hence is itself a random variable. The $CFAR$, using (4), can be written as:

$$CFAR = 2\Phi\left(-L\sqrt{\frac{F^{-1}_{\chi^2_{m(n-1)}}(U)}{m(n-1)}}\right) \quad (5)$$

It is interesting to examine the $CFAR$ as a function m, n and U. For example, Figure 1 shows the curves of $CFAR$ (in function of $u \in (0,1)$), for $n = 5,$, $L = 3$ which corresponds to $\alpha_{nom} = 0.0027$ for varying $m = 10, 20, 50,100,500$. The effect of the standard estimation on $CFAR$ for different number of Phase I samples m is clearly seen. The difference between the $CFAR$ and the α_{nom} (the horizontal line) is larger for small values m comparing to the same difference when m is large for all values of U. It is interesting to note that the $CFAR = 0.0027$ for all m when u is close to 0.53. This means that if one is “lucky” to obtain an estimate which is close to the 0.53 quantile of the $\chi^2_{m(n-1)}$ distribution, then the $CFAR$ will be equal to $\alpha_{nom} = 0.0027$. If not, different estimates from different sets of in-control data from the same/different practitioners can produce vastly different $CFAR$ values some much larger and some smaller than the nominal value. This is evidence of what has been called “practitioner-to-practitioner” variation. Also, note that the $CFAR$ curves are not equi-distant from 0.0027 on the upper and the lower sides of u close to 0.53 which means that getting an estimate in the lower tail, further from roughly the median is more problematic (much higher false alarm rate than the nominal) than the one on the upper tail (lower than nominal false alarm rate). In some cases, the difference is relatively very big. This behavior is because, as we will see in this paper, the distribution of $CFAR$ is skewed and with less samples to estimate the standard devotion, less precise the estimation is.

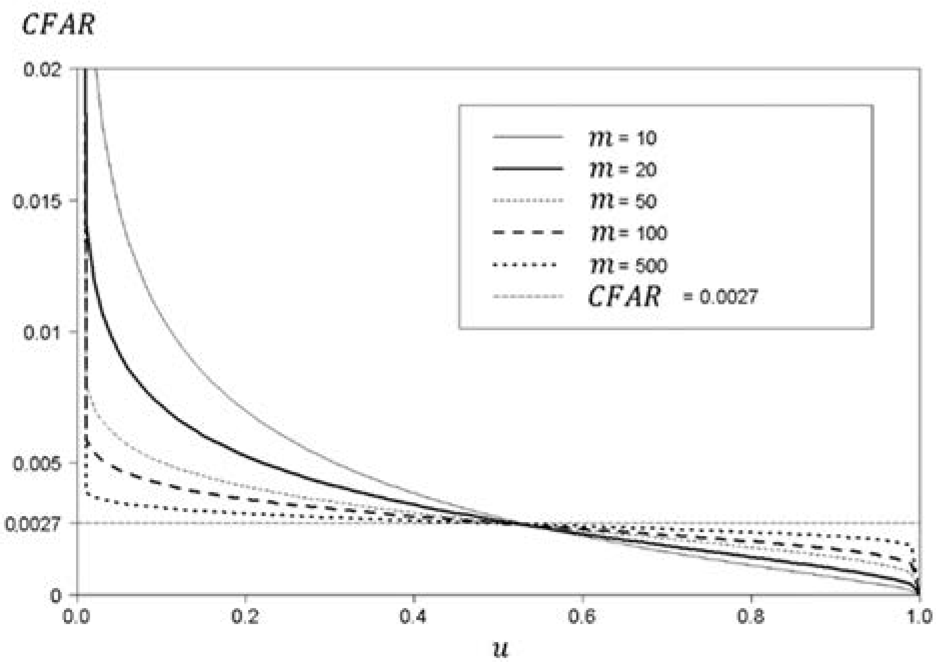

Figure 1: $CFAR$ as function of u for $n = 5$, $m = 10,20,50,100,500$ and $\alpha_{nom} = 0.0027$ (L = 3)

Given that the in control conditional run length (RL_0) distribution of the $\bar{X}$ chart is geometric with parameter $CFAR$ (see, for example, Chakraborti (2000)), then its expected value, the conditional in-control average run length $CARL_0$, is:

$$E(RL_0|\hat{\sigma}_0) = CARL_0 = \frac{1}{CFAR} \tag{6}$$

Note that $E(CARL_0) = E\big(E(RL|\hat{\sigma}_0)\big) = E(RL_0)$ is the unconditional in control average run length and it has been the most commonly used metric to measure the in control performance of a control chart (see, for example, Faraz et al. (2015)).

2.2 The cumulative distribution function of $CFAR$ and $CARL_0$

As shown in Equation (6), the $\boldsymbol{CARL_0}$ is a monotonic decreasing function of $\boldsymbol{CFAR}$, so the cumulative distribution function (c.d.f.) of $\boldsymbol{CFAR}$ ($\boldsymbol{F_{CFAR}(t)}$) is related with the c.d.f of $\boldsymbol{CARL_0}$ ($\boldsymbol{F_{CARL_0}(t)}$) as shown below:

$$F_{CFAR}(t) = P(CFAR \le t) = P(CARL_0 \ge t^{-1}) = 1 - F_{CARL_0}(t^{-1}) \tag{7}$$

From Figure 1 it can be seen that $CFAR$ is a monotonically decreasing function of u. Also the $CFAR$ is an invertible function and so it is possible to find the expression of its c.d.f. from Equation (5) as shown below.

$$F_{CFAR}(t) = P(CFAR \le t) = P\left(2\Phi\left(-L\sqrt{\frac{F^{-1}_{\chi^2_{m(n-1)}}(U)}{m(n-1)}}\right) \le t\right)$$

$$= P\left(\Phi\left(-L\sqrt{\frac{Y}{m(n-1)}}\right) \leq \frac{t}{2}\right) = P\left(-L\sqrt{\frac{Y}{m(n-1)}} \leq \Phi^{-1}\left(\frac{t}{2}\right)\right)$$

$$= P\left(Y \geq m(n-1)\left(-\frac{\Phi^{-1}\left(\frac{t}{2}\right)}{L}\right)^2\right) = 1 - P\left(Y \leq m(n-1)\left(-\frac{\Phi^{-1}\left(\frac{t}{2}\right)}{L}\right)^2\right)$$

$$= 1 - F_{\chi^2_{m(n-1)}}\left(m(n-1)\left(-\frac{\Phi^{-1}\left(\frac{t}{2}\right)}{L}\right)^2\right) \tag{8}$$

Where t is any possible value for $CFAR$ $(0 \leq t \leq 1)$.

Figure 2 shows the c.d.f. of $CFAR$ given by Equation (8) for values of $n = 5$, $\alpha_{nom} = 0.0027$ ($L = 3$ and for varying $m = 10, 20, 50, 100,$ and 500 The effect of estimation (and the impact of the amount of Phase I data) on the $CFAR$ is again clearly seen in Figure 2. It can be seen that the distribution of $CFAR$ is skewed to the right and when the number of reference sample, m, grows, more skewed the distribution is.

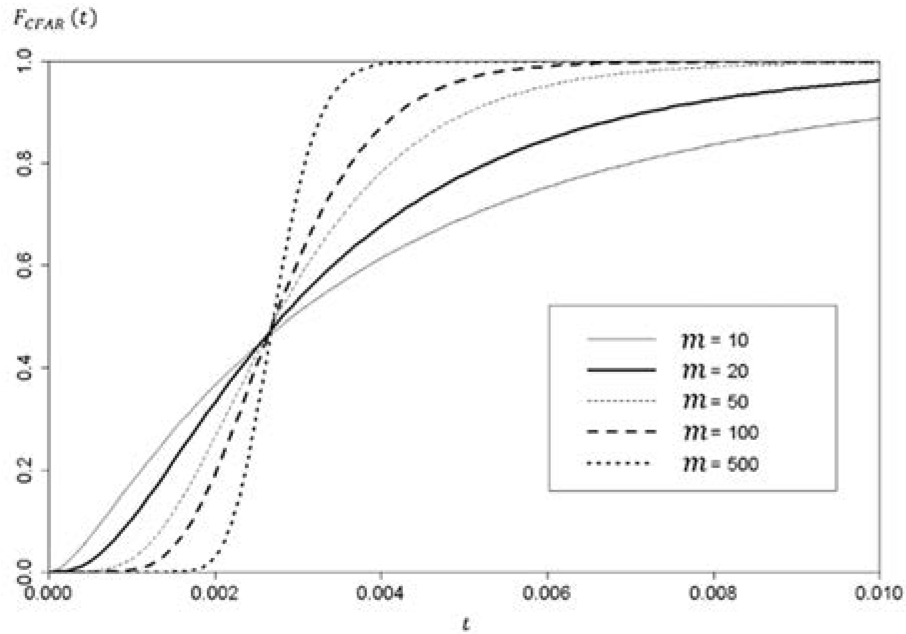

Figure 2: c.d.f. of $CFAR$ for different values of m (the number of initial samples), $\alpha_{nom} = 0.0027 (L = 3)$

3. Prediction Intervals

We next consider the problem of prediction of $CFAR$ since it is a random variable. For example, it may be of interest to know, in practice, in a given Phase II application, what the highest $CFAR$ (an upper bound to the $CFAR$) can be, with a certain (high) probability. Put in another way, for a given m and n, that is some Phase I reference data, it may be of practical interest to know the value of $CFAR$ that has only a specified high probability $1-p$ (say, 90%) of not being exceeded. This means that we need to find the value α_p such that when the process is in control:

$$F_{CFAR}(\alpha_p) = P(CFAR \leq \alpha_p) = 1 - p \tag{9}$$

Using Equation (8), it is possible to derive an exact expression of α_p as shown below

$$F_{\chi^2_{m(n-1)}}\left(m(n-1)\left(-\frac{\Phi^{-1}\left(\frac{\alpha_p}{2}\right)}{L}\right)^2\right) = p \therefore \alpha_p = 2\Phi\left(-L\sqrt{\frac{F^{-1}_{\chi^2_{m(n-1)}}(p)}{m(n-1)}}\right) \tag{10}$$

Tables 1 presents the values of α_p for $p = 5\%$ (the 0.95 quantile), $p = 10\%$ (the 0.9 quantile) and for some values of m and n. As it can be seen, when m and/or n are small, the values of $CFAR$ that are exceeded only with a probability of 5% or 10% are much higher than the desired α_{nom}. For example, for $m = 10$, $n = 2$ and $p = 5\%$, $CFAR = 0.05968$, more than 22 timer larger than 0.0027.

This means that, if one would like only a 5% or a 10% probability that the $CFAR$ exceeds α_{nom}, for small values of m and n, this is mostly not possible for typical values of α_{nom}. For example, the lowest entries in the table, that occur when $m = 1000$ and $n = 50$ are still greater than a typical $\alpha_{nom} = 0.0027$ So, id the user desire a $p = 5\%$ or $p = 10\%$, with an $\alpha_{nom} = 0.0027$, the user should actually focus in the $CFAR$ values in Table 1 (α_p).

Table 1 - 0.95-quantiles ($p = 5\%$) and 0.9-quantiles ($p = 10\%$) of $CFAR$

		m						
	n	5	10	20	50	100	500	1000
p = 5%	2	0.15103	0.05968	0.02713	0.01237	0.00809	0.00446	0.00386
	5	0.02713	0.0146	0.00915	0.00593	0.00473	0.00348	0.00323
	10	0.01335	0.00856	0.00618	0.00459	0.00393	0.0032	0.00304
	20	0.00831	0.00605	0.0048	0.0039	0.0035	0.00304	0.00293
	50	0.00551	0.00449	0.00387	0.0034	0.00318	0.0029	0.00284
p = 10%	2	0.08866	0.03639	0.01797	0.0092	0.00648	0.00401	0.00357
	5	0.01797	0.01057	0.00716	0.00503	0.0042	0.00329	0.00311
	10	0.00981	0.00679	0.0052	0.0041	0.00363	0.00308	0.00297
	20	0.00662	0.00511	0.00425	0.0036	0.00331	0.00296	0.00288
	50	0.00474	0.00403	0.00358	0.00323	0.00307	0.00286	0.00281

4. Finding m for a Guaranteed In-Control Performance

Another relevant question for the practitioner may be the minimum number m of Phase I reference samples that guarantees, with a specified high probability $1-p$ (say, 0.9), that $CFAR$ does not exceed a desired α_{nom} by more than a given small percentage ε (e.g.

10%). This can be formulated as follows: Given the values of n, α_{nom}, ε and p find the minimum number of in-control Phase I samples, m, such that

$$P(CFAR \leq (1+\varepsilon)\alpha_{nom}) = 1 - p \tag{11}$$

This problem is similar to the one in the previous section, with the difference that now α_p is given and is equal to a tolerated upper bound to the false alarm rate, (that is, $\alpha_{TOL} = (1+\varepsilon)\alpha_{nom}$) and m is the unknown parameter that needs to be found. Note that, since m is an integer, a perfect match is generally not possible, so, reformulating, m should be the smallest integer such that $P(CFAR \leq (1+\varepsilon)\alpha_{nom}) \geq 1 - p$.

In order to find the values of m, we can try to find a formula for m manipulating Equation (10) like it is shown in Equation (12). But this equation shows that finding m is more involved because it is a function of a quantile of a chi-squared variable whose number of degrees of freedom, in turn, is a function of m. Hence finding the required solution requires a search process.

$$\Phi^{-1}\left(\frac{\alpha_p}{2}\right) = L\sqrt{\frac{F^{-1}_{\chi^2_{m(n-1)}}(p)}{m(n-1)}} \therefore F^{-1}_{\chi^2_{m(n-1)}}(p) = m(n-1)\left(\frac{\Phi^{-1}\left(\frac{\alpha_p}{2}\right)}{L}\right)^2$$

$$m = \left\lceil \frac{F^{-1}_{\chi^2_{m(n-1)}}(p)}{(n-1)\left(\frac{\Phi^{-1}\left(\frac{\alpha_p}{2}\right)}{L}\right)^2} \right\rceil \tag{12}$$

where $\lceil a \rceil$ denotes the smallest integer greater or equal to a.

Table 2 shows the minimum number of in-control Phase I samples, m. We considered $\alpha_{nom} = 0.0027$, $p = 5\%$, 10% and some values of n and ε. As can be seen in Tables 2 for small values of n, one needs a large number of Phase I samples to guarantee such performance. These values are larger than the values proposed by other authors like Saleh et al. (2015). For example, for $n = 5$ and the case in which the process mean is also estimated, Saleh et al. (2015) proposed $m = 1200$. It can be seen in Table 2 that one will need much more than 1200 reference samples for $\varepsilon = 5\%$ and $\varepsilon = 10\%$ when $n = 5$.

Table 2 - Minimum number of in control Phase I samples, m, required for $P(CFAR \leq (1+\varepsilon)\alpha_{nom}) = 1 - p$

	ε (p = 5% and α_{nom} = 0.0027)				ε (p = 10% and α_{nom} = 0.0027)			
n	5%	10%	15%	20%	5%	10%	15%	20%
2	54938	14349	6652	3897	33402	8738	4057	2380
5	13735	3588	1663	975	8351	2185	1015	595
10	6105	1595	740	433	3712	971	451	265
20	2892	756	351	206	1758	460	214	126
30	1895	495	230	135	1152	302	140	83
50	1122	293	136	80	682	179	83	49

5. Adjustment of the Limits for a Guaranteed In-Control Performance

Since the required amount of Phase I data is very large and often impractical to guarantee a desirable in-control chart performance, in this section we present the adjustment of the control limits of the $\bar{X}$ chart in order to ensure a low probability, p, that the conditional false alarm rate ($CFAR$) exceeds a tolerated false alarm rate (α_{tol}) even for small values of m and n. Quite usually α_{nom} is 0.0027, leading to the standard 3-sigma limits. Note that α_{tol} is greater than α_{nom} by a percentage ε. This formulation allows the user the flexibility during implementation of the control chart.

When $L = 3$ (the commonly used 3-sigma limits) is used, $P(CFAR \geq (1+\varepsilon)\alpha_{nom})$ is large, especially when ε, m and n are small. One solution for this problem is to make an adjustment on the control limits replacing L by $L(p, \alpha_{tol})$, where $L(p, \alpha_{tol})$ represent the value for the control limit factor that guarantees that $P(CFAR \geq \alpha_{tol}) = p$ or $P\left(CARL_0 \geq \frac{1}{\alpha_{tol}}\right) = 1 - p$ (according to Equation (7)) for a given values of $\alpha_{tol} = (1+\varepsilon)\alpha_{nom}$, m and n. It is relevant for the practitioners to know the control limit factor, for a given value of m and n, that guarantee a low p probability of a conditional false alarm rate exceed.

Since we derived a closed-form expression for the c.d.f of $CFAR$ given by Equation (8), we can develop a closed-form expression for $L(p, \alpha_{tol})$ replacing L by $L(p, \alpha_{tol})$ in Equation 12 rearranging the terms.

$$L(p, \alpha_{tol}) = \frac{\Phi^{-1}\left(\frac{\alpha_{tol}}{2}\right)}{\sqrt{\frac{F^{-1}_{\chi^2_{m(n-1)}}(p)}{m(n-1)}}} \quad (13)$$

Tables 3 and 4 show the exact values of $L(p, \alpha_{tol})$ for same values of m and n (for $p = 10\%$, $\varepsilon = 0\%$ and $\alpha_{nom} = 0.0027$). Note the values in Table 3 leads to $P(CARL_0 \geq 370.4) = 90\%$ for a given values of m and n.

According to Table 4 for $m = 30$ and $n = 5$, we have $L(10\%, 0\%) = 3.28$. In the case of $m = 30$ and $n = 5$, using 3-sigma limits ($L = 3$), one has $P(CARL_0 \geq 370.4) = 48.28\%$. In other words, there is a high probability (almost 50%) that the attained average in-control run length be smaller than 370.4. This is not the case for 3.28-sigma limits, with which this probability is only of 10%.

Table 3– Values of $L(10\%, 0\%)$ for $P(CARL_0 > 370.4) = 90\%$

	m					
n	3	5	10	15	20	25
2	6.80	5.29	4.30	3.97	3.80	3.70
3	4.95	4.30	3.80	3.62	3.52	3.46
4	4.41	3.97	3.62	3.48	3.41	3.36
5	4.14	3.80	3.52	3.41	3.35	3.31
10	3.66	3.48	3.32	3.26	3.22	3.20
15	3.51	3.37	3.25	3.20	3.17	3.15
20	3.42	3.31	3.21	3.17	3.15	3.13
25	3.37	3.28	3.19	3.15	3.13	3.12
30	3.33	3.25	3.17	3.14	3.12	3.11
50	3.25	3.19	3.13	3.10	3.09	3.08

Table 4– More values of $L(10\%, 0\%)$ for $P(CARL_0 > 370.4) = 90\%$

	m					
n	30	50	100	300	500	1000
2	3.62	3.46	3.31	3.17	3.13	3.09
3	3.41	3.31	3.21	3.12	3.09	3.06
4	3.32	3.24	3.17	3.09	3.07	3.05
5	3.28	3.21	3.14	3.08	3.06	3.04
10	3.18	3.14	3.09	3.05	3.04	3.03
15	3.14	3.11	3.07	3.04	3.03	3.02
20	3.12	3.09	3.06	3.04	3.03	3.02
25	3.11	3.08	3.06	3.03	3.03	3.02
30	3.10	3.07	3.05	3.03	3.02	3.02
50	3.07	3.06	3.04	3.02	3.02	3.01

6. Conclusions

In the present article, we studied the impact of the process standard deviation estimation on the in-control performance of the $\bar{X}$ Control Chart. We verified that if the user uses the standard 3 sigma limits and estimates the process standard deviation with a small amount of Phase I reference data, chances are high that the chart will not have the nominal in-control performance. We conclude that when the m (the number of reference samples) and/or the n (size of each sample) are small, the values of the conditional false alarm rate that are exceeded only with a small probability of 5% or 10% are much higher than the typical desired nominal false alarm rate. Also we checked that for small values of n, one needs a large number of reference samples, m, in order to guarantee some specified in-control conditional performance. So we provided some corrections on the control limits in order to achieve a desired in-control conditional performance given a fix value of m and n.

The main contribution of the present work is that the majority of the existing papers studying the effects of parameter estimation on the $\overline{X}$ control chart have not focused on the case that is when just the process standard deviation is unknown (Case KU) and thus is estimated, like we did. Also most authors have focused on the unconditional average run length as a performance criterion. We, on the other hand, have focused on the conditioned average run length, since, once the process standard deviation is estimated, the performance of the chart will be conditioned to the value of the estimator. Finally, we provide exact formulas under normality which do not require the use of any approximations, asymptotic methods, bootstrapping or simulations (although we checked our results with simulations) to obtain the distribution of the false alarm rate of the $\overline{X}$ chart.

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