Acceptance Procedure for Process Control Based on Clustering

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Abstract
Periodically an item is inspected during online process control. Inspections are subject to misclassification and thus the item is subjected to repeat classifications. The decision as to whether the item is judged to be conforming is based on whether there is a sufficient cluster of classifications of conforming prior to a pre-specified number of nonconforming judgments.

Key Words: Acceptance Sampling, Process Control, Scan, Clustering

1. Introduction

A number of authors consider on-line process control by attributes in which every h^{th} item produced is inspected [Taguchi, Elsayed, and Hsiang(1989) and Taguchi, Chowdhury, and Wu (2004)]. In this model the process is assumed at the start to have some high fraction of conforming items, close to 100%. That is, an item conforms to specifications with probability p_1, equal to or very close to 1 when the process is in control. The process goes out of control at random and there is a shift to p_2 ($< p_1$) for the fraction conforming, i.e., the probability that the selected item is really conforming. The process is stopped for adjustment if and only if an inspected item is judged to be nonconforming,

Nayebpour and Woodall (1993) assume the random time until the shift from p_1 to p_2 follows a geometric distribution where the items produced are modeled as independent and identically distributed trials with a constant probability π for each item to be the first item produced after the shift of the fraction conforming. Since only every h^{th} item is inspected, the first item produced after the shift might not be inspected resulting in some initial number of items produced before the possibility of the detection even exists.

Borges, Ho, and Turnes (2001) model the inspection process itself as subject to possible errors and thus in a single classification, a conforming item might be mistakenly classified as nonconforming. Let p_{CN} designate this misclassification probability. It is also possible that a nonconforming item might mistakenly be judged as conforming and we let p_{NC} be the probability of this misclassification. We will also define probability p_{CC} (p_{NN}) of the correct classification that a conforming (nonconforming) item is classified as conforming (nonconforming). As a consequence there arises the notion of making repeated classifications of each inspected item before making the final judgment as to whether the item as conforming or nonconforming. If the item is finally judged to be nonconforming, the process is viewed as out of control and is stopped for adjustment. Otherwise, the process is viewed as in control and is not stopped for adjustment. Since errors are possible in the repeated classifications, it is possible that an item is mistakenly judged to be nonconforming and thus that the process is judged out of control, when it actually is not. Nonetheless, it is stopped for adjustment. In this case, no cause can be found and the process is restarted and has not been put out of control by the stopping, searching for a cause, and restarting. It is also possible that the process goes out of control and is not detected. In this case, it stays out of control until this is detected at a later time at which point it will be adjusted and be put back in control.

Trindade, Ho, and Quinino (2007) study a rule in which the final judgment of whether the inspected item is conforming and thus whether the process is in control, is based on a pre-specified number of repeated classifications and uses majority rule. Quinino, Colin, and Ho (2009) consider a rule in which the item is determined to be conforming and the process to be in control if and only if there are k classifications as conforming before f classifications as nonconforming, where k and f are some pre-specified positive integers. The acronym TCTN is used to describe this rule since the decision is based on the total number of classifications as conforming and nonconforming and has been studied by Smith and Griffith (2009) as well. In other papers, Smith and Griffith (2011, 2012) studied some alternative rules including one in which the final determination that an item is conforming, and thus the process is in control, if and only if k consecutive classifications as conforming occur before a total of f classifications as nonconforming (CCTN).

In this paper we will we propose a new rule that is a compromise between the CCTN and TCTN rules. The final determination that an item is conforming if a cluster of classifications of conforming occur prior to a preset number of classifications of nonconforming. We denote this rule by ScanCTN. Hence, we are waiting for k classifications of conforming to occur within a window or a scan of w classifications. This eases the restriction on consecutive classifications from the CCTN rule in the sense that the count for classifications does not necessarily return to zero when a nonconforming classification occurs. That is, for a final judgment, a cluster of conforming classifications is needed rather than the more stringent requirement that a run of consecutive conforming classifications occur. On the other hand, the scan rule requires more consistency in the way conforming classifications are obtained than the TCTN rule. We will perform the probabilistic analysis using the Markov Chain approach.

2. State Space and Transition Probabilities

In order to assist with the readability of this section we will define the following notation for the ScanCTN rule.

- f = total number nonconforming classifications for final judgment of nonconforming
- w = number of classifications in the window or scan
- k = number of conforming classifications within the window, w, for final judgment of conforming
- p = probability that the classification of an item is judged "conforming"
- $q = 1-p$
- $\{X_n\}$ = Markov-Chain where $X_n = (x_1,x_2,x_3,x_4,\ldots,x_w,s,h)$
- x_i = classification result (1 = conforming, 0 = nonconforming) for $i = 1$ to w-1
- x_w = the outcome of the w^{th} classification within the window of w
- s = total number of conforming classifications in the entries x_1 to x_w
- h = total number of nonconforming classifications

In the case of the ScanCTN, the item is judged conforming if k conforming classifications are achieved within w consecutive classifications prior to observing f total nonconforming classifications among all classifications. Likewise, the item is judged nonconforming if f total nonconforming classifications among all classifications occur prior to k total conforming classifications within w consecutive classifications. Consider the Markov Chain $\{X_n\}$ where $X_n = (x_1,x_2,x_3,x_4,\ldots,x_w,s,h)$ means that after the n^{th} classification the first w-1 entries, x_1 to x_{w-1}, contain the classification results (1 = conforming classification, 0 = nonconforming classification) of the previous w-1 classifications, the w^{th} entry, x_w, is the classification result of the w^{th} trial, s counts the total number of conforming classifications in the entries x_1 to x_w, and h is the total number of nonconforming classifications among all the classifications. When $n < w$, the n^{th} entry will contain the classification of the n^{th} trial, we will let NA be a placeholder in the entries x_{n+1} to x_w. Obviously, the index s is not necessarily needed but it does aid in the computation. The probability of conforming classification is given by p and probability of nonconforming classification is given by $q = 1 - p$.

It is a rather difficult task to write out the state space in set notation and the transition probabilities and state space in a simple diagram even for small values of k, w, and f. Therefore, to aid in understanding the states involved in this Markov Chain, we have listed the absorbing and transient states in Table 1 for a ScanCTN rule where $k = 3$, $w = 4$ and $f = 3$.

For $n = 1$: $P(X_1 = (1,\text{NA},\text{NA},\ldots\text{NA},1,0) \mid X_0 = (\text{NA},\text{NA},\text{NA},\ldots\text{NA},0,0)) = p$
$P(X_1 = (0,\text{NA},\text{NA},\ldots\text{NA},0,1) \mid X_0 = (\text{NA},\text{NA},\text{NA},\ldots\text{NA},0,0)) = q$

$n = 2$: $P(X_2 = (1,1,\text{NA},\ldots\text{NA},2,0) \mid X_1 = (1,\text{NA},\text{NA},\ldots\text{NA},1,0)) = p$
$P(X_2 = (1,0,\text{NA},\ldots\text{NA},1,1) \mid X_1 = (1,\text{NA},\text{NA},\ldots\text{NA},1,0)) = q$
$P(X_2 = (0,1,\text{NA},\ldots\text{NA},1,1) \mid X_1 = (0,\text{NA},\text{NA},\ldots\text{NA},0,1)) = p$
$P(X_2 = (0,0,\text{NA},\ldots\text{NA},0,2) \mid X_1 = (0,\text{NA},\text{NA},\ldots\text{NA},0,1)) = q$

Etc. for $n \leq w$

For $n > w$, $P(X_n = (b,c,d,e,\ldots,1,s+1,h) \mid X_{n-1} = (a,b,c,d,\ldots,g,s,h)) = p$ if $a = 0$
$P(X_n = (b,c,d,e,\ldots,1,s,h) \mid X_{n-1} = (a,b,c,d,\ldots,g,s,h)) = p$ if $a = 1$
$P(X_n = (b,c,d,e,\ldots,0,s,h+1) \mid X_{n-1} = (a,b,c,d,\ldots,g,s,h)) = q$ if $a = 0$
$P(X_n = (b,c,d,e,\ldots,0,s-1,h+1) \mid X_{n-1} = (a,b,c,d,\ldots,g,s,h)) = q$ if $a = 1$

For the Markov chain there are absorbing (recurrent) states, which correspond to the termination of the rule. Let A denote the set of absorbing states and a denote the number of absorbing states. In fact, the singleton sets consisting of each of these absorbing states are recurrent classes. The remaining states are transient which we will denote by T and likewise the number of transient states by t. Written in canonical form, the one-step transition probability matrix $\mathbf{P}$ for the Markov chain is $\begin{bmatrix} \mathbf{P_I} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}$, where $\mathbf{P_I}$ is the $a \times a$ identity matrix for the absorbing states, $\mathbf{R}$ is a $t \times a$ matrix containing the one-step probabilities of the transient states to the recurrent (absorbing) states, $\mathbf{Q}$ is a $t \times t$ matrix containing the one-step probabilities among the transient states, and $\mathbf{0}$ is the $a \times t$ zero matrix. The one-step probabilities of $\mathbf{R}$ and $\mathbf{Q}$ are determined by the transition probabilities given for each test. The first row of $\mathbf{Q}$ contains the one step transition probabilities from state (NA,NA,NA,…NA,0,0).

To compute the moments of the rule length, we will define the following notation. Since elements of T appear as subscripts, we will use i and j as typical elements of T. However, it should be noted that when we do so, each of i and j refer to an ordered $(w + 2)$ tuple. Let,

- $\mathbf{I}_{t \times t}$ = identity matrix of dimension $t \times t$
- $\mathbf{M}_{t \times t} = (\mathbf{I}_{t \times t} - \mathbf{Q}_{t \times t})^{-1}$ - the fundamental matrix of dimension $t \times t$
- $\mathbf{e}_m$ = column vector of length t where the m^{th} element is one and the remaining elements are zero.
- $\mathbf{e}_m'$ is defined to be the transpose of $\mathbf{e}_m$
- $\mathbf{u}_{\{NS\}}$ = column vector where all the elements corresponding to the final judgment of nonconforming states are one, and the remainder of the elements are zero.
- $\mathbf{1}_z$ = column vector of ones of length z
- N_{ij} = random variable that represents the number of times the process visits state j before it eventually enters a recurrent state, having initially started from state i $(i,j \in \text{T})$.
- $\mu_{ij} = E(N_{ij})$ for $i,j \in$ T.
- $\mathbf{M}_\rho = \left[\sum_{j \in T} \mu_{ij} \right] = \mathbf{M}\,\mathbf{1}_t$ = column vector such that the m^{th} element is the sum of the m^{th} row of $\mathbf{M}$
- $\mathbf{M}_{\rho^2} = \left[\left(\sum_{j \in T} \mu_{ij} \right)^2 \right]$ = diag $(\mathbf{M}_\rho)\,\mathbf{M}_\rho$ - column vector such that the m^{th} element is the square of the sum of the m^{th} row of $\mathbf{M}$. Note: diag $(\mathbf{M}_\rho)$ is a diagonal matrix whose entries are the corresponding entries of $\mathbf{M}_\rho$.

Using the notation of the preceding section, the geometric distribution as a waiting time distribution, and basic probability results such as the law of total probability, we can obtain a number of results. These results are based on formulas in Bhat[18].

Proposition 1

An item is inspected by being subjected to repeated classifications.

A) Given that the item being inspected is conforming, the probability that it is judged to be conforming is $P(judged\ conforming|\ conforming) = ScanCTN(p_{CC}) = 1 - \mathbf{e_1}'\mathbf{MR}\,\mathbf{u}_{\{NS\}}$ where $p = p_{CC}$.

B) Given that the item being inspected is nonconforming, the probability that it is judged to be conforming is $P(judged\ conforming|\ nonconforming) = ScanCTN(p_{NC}) = 1 - \mathbf{e_1}'\mathbf{MR}\,\mathbf{u}_{\{NS\}}$ where $p = p_{NC}$.

C) Given that the item being inspected is conforming, the probability that it is judged to be nonconforming is $P(judged\ nonconforming|\ conforming) = \mathbf{e_1}'\mathbf{MR}\,\mathbf{u}_{\{NS\}}$ where $p = p_{CC}$.

D) Given that the item being inspected is nonconforming, the probability that it is judged to be nonconforming is $P(judged\ nonconforming|\ nonconforming) = \mathbf{e_1}'\mathbf{MR}\,\mathbf{u}_{\{NS\}}$ where $p = p_{NC}$.

Proposition 2

A) Given that the process is in control and the inspected item is conforming with probability p_1 and nonconforming with probability 1-p_1, then the probability that the process is judged to be in control

$$P_{II} = P(judged\ in\ control|\ in\ control)$$
$$= p_1 ScanCTN(p_{CC}) + (1 - p_1) ScanCTN(p_{NC})$$

B) Given that the process is out of control and the inspected item conforms with probability p_2 and fails to conform with probability $1 - p_2$, then the probability that the process judged to be in control

$$P_{OI} = P(judged\ in\ control|\ out\ of\ control)$$
$$= p_2 ScanCTN(p_{CC}) + (1 - p_2) ScanCTN(p_{NC})$$

Proposition 3

Once it goes out of control, the distribution of the number of inspections needed to determine it is out of control is the geometric distribution with parameter $1 - P_{OI}$.

Proposition 4

Let Y = time measured in decision time until the process actually goes out of control (or would go out of control if no inspections were being completed or ignored) and π is the probability of a shift on any item produced then the

$$P(Y = y) = [(1-\pi)^h]^{y-1}[1-(1-\pi)^h] = \theta(1-\theta)^{y-1}, \quad y = 1, 2, 3, \ldots \quad .$$

So Y has a geometric distribution with parameter $\theta = 1 - (1-\pi)^h$.

Proposition 5

Let X = be the time measured in decision time until the process is judged out of control.

$$P(X = x) = \sum_{y=1}^{\infty} P(X = x|Y = y)P(Y = y) \quad \text{where}$$

$$P(X = x|Y = y) = \begin{cases} [P_{II}]^{x-1}[1 - P_{II}], & x < y \\ [P_{II}]^{x-1}[1 - P_{OI}], & x = y \\ [P_{II}]^{y}[P_{OI}]^{x-1-y}[1 - P_{OI}], & x > y \end{cases}$$

$$P(X = x) = \sum_{y=1}^{x-1} [P_{II}]^{y}[P_{OI}]^{x-1-y}[1 - P_{OI}]\ (\theta(1-\theta)^{y-1}) +$$

$$[P_{II}]^{x-1}[1 - P_{OI}](\theta(1-\theta)^{x-1}) + \sum_{y=x+1}^{\infty} [P_{II}]^{x-1}[1 - P_{II}](\theta(1-\theta)^{y-1})$$

Proposition 6

Consider the decision time for a single item for i.i.d. Bernoulli classifications with as stated in section 1. Then the mean, variance, and probability mass function for a decision time is given below based on whether or not the item is conforming or nonconforming.

A) Expected decision time

E(Decision time |conforming) = $\mathbf{e}_1{}'\mathbf{M1}_t$ where $p = p_{CC}$

E(Decision time |nonconforming) = $\mathbf{e}_1'\mathbf{M}\mathbf{1}_t$ where $p = p_{NC}$

E(Decision time|in control) = p_1 E(Decision time |conforming) + (1-p_1) E(Decision time |nonconforming)

E(Decision time|out of control) = p_2 E(Decision time |conforming) + (1-p_2) E(Decision time |nonconforming)

B) The variance of decision time
Var(Decision time |conforming) =$e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}}-\mathbf{M}_{\rho^2}\right]$ where $p = p_{CC}$
Var(Decision time |nonconforming) = $e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}}-\mathbf{M}_{\rho^2}\right]$ where $p = p_{NC}$

C) The probability mass function of the decision time
P(decision time = m | conforming) = $\mathbf{e}_1'\,\mathbf{Q}^{m-1}\mathbf{R}\,\mathbf{1}_a$ where $p = p_{CC}$

P(decision time = m | nonconforming) = $\mathbf{e}_1'\,\mathbf{Q}^{m-1}\mathbf{R}\,\mathbf{1}_a$ where $p = p_{NC}$

References

Bhat, U.N., 1984. Elements of Applied Stochastic Processes, 2nd ed., John Wiley & Sons, Inc., New York, N.Y.

Borges, W., Ho, L.L., and Turnes, O. (2001). An analysis of Taguchi's on-line quality monitoring procedure for attributes with diagnosis errors. Applied Stochastic Models in Business and Industry, 17, 271-276.

Griffith, W. S., Smith, M. L, (2011). Process Control When Items Are Subject to Misclassification, Proceedings of the American Statistical Association, Section on Quality and Productivity, 1179 – 1184.

Nayebpour, M.R., and Woodall, W.H. (1993). An analysis of Taguchi's on-line quality monitoring procedure for attributes. Technometrics, 35, 53-60.

Quinino, R., Colin, E.C., and Ho, L.L. (2009). Diagnostic Errors and Repetitive sequential classifications in on-line process control by attributes. European Journal of Operational Research, doi:10.1016/j.ejor.2009.02.017.

Smith, M. L, Griffith, W. S. (2009). Repeated Classification Subject to Errors in Process Control, Proceedings of the American Statistical Association, Section on Quality and Productivity, pp 1529-1534.

Smith, M. L, Griffith, W. S. (2012). Procedure for Process Control with Inspection Errors, *Proceedings of the American Statistical Association, Section on Quality and Productivity, pp 1585-1592.*

Taguchi, G., Chowdhury, S., and Wu, Y. (2004). Taguchi's Quality Engineering Handbook. John Wiley & Sons, Inc. New Jersey.

Taguchi, G., Elsayed, E.A., and Hsiang, T. (1989). Quality Engineering in Production in Systems. McGraw-Hill, New York.

Trindade, A., Ho, L.L., and Quinino, R. (2007). Monitoring process for attributes with quality deterioration and diagnosis errors. Applied Stochastic Models in Business and Industry, 23 (4), 339-358.