On bivariate tail non-exchangeability

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Abstract. Non-exchangeable dependence structures exist in the real world, and we are interested in how to identify the existence of non-exchangeability in joint distributional tails and how to quantify the degree of such *tail non-exchangeability*. The results obtained and the approaches proposed benefit bivariate dependence modeling when dependence patterns in the tails are particularly important, as in the fields of quantitative finance, quantitative risk management, and econometrics. We focus on the bivariate case and propose to use conditional expectations as the basis quantities. Then, for random variables X and Y, the departure between tail behavior of $\mathbb{E}[X|Y > t]$ and $\mathbb{E}[Y|X > t]$, or $\mathbb{E}[X|Y = t]$ and $\mathbb{E}[Y|X = t]$, when t is large, becomes useful in detecting tail non-exchangeability. We use a bivariate copula to model the dependence between X and Y. The Khoudraji's device is employed in generating tail non-exchangeability. Three major tail behavior patterns for univariate margins are studied in order to understand the interaction between the strength of dependence together with various types of margins in affecting the measure of tail non-exchangeability. Based on the probabilistic properties of the tail non-exchangeability structures, we develop graphical approaches and statistical tests for analyzing data that may have non-exchangeability in the joint tail. A simulation study is then conducted to demonstrate the usefulness of the proposed approaches.

Key words: conditional expectation, copula, tail dependence, tail behavior.

1 Introduction

The goal of this study is to propose useful measures for quantifying the degree of *non-exchangeability* in the tails of a bivariate joint distribution. The study on tail *non-exchangeability* would be particularly useful for providing more tail-tailored statistical models for modeling *tail non-exchangeability*

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structures observed in risk management, quantitative finance, psychometrics, econometrics, and environmetrics.

There have recently been quite a few papers focusing on the study of quantifying the degree of overall non-exchangeability. For example, Klement and Mesiar (2006) and Nelson (2007) study the extremal cases of bivariate non-exchangeability using copulas, Durante et al. (2010) provides some axioms for measures of bivariate non-exchangeability, Durante and Mesiar (2010) and Durante and Mesiar (2010) study the cases for some specific bivariate copula families, Genest et al. (2012) proposes a test for bivariate non-exchangeability, Harder and Stadtmüller (2014) extends the study of extremal non-exchangeability to multivariate cases. However, all of these studies are for overall non-exchangeability.

In this paper, we are interested in quantifying the degree of tail non-exchangeability for a bivariate random vector that has identical marginal distributions. Therefore, the notion of tail non-exchangeability in this paper relies on limiting properties of bivariate copulas. We first propose a sensible measure for quantifying the strength of tail non-exchangeability, and then details are provided for some non-exchangeable bivariate copula families that are constructed based on some commonly-used approaches. There are some simple (tail) non-exchangeable copulas studied in the literature, such as, Marshall-Olkin copula, the generalized Clayton Copula, mentioned in Furman et al. (2015), and copulas constructed through comonotonic latent variables (see: Hua and Joe (2016)). In this paper, we only consider the Khoudraji's device (see, Khoudraji (1996)) for generating non-exchangeable copulas, and some other approaches such as using non-exchangeable Pickands function for extreme value copulas will be further studied in the future.

One reasonable approach is to consider the difference between certain conditional quantities when we switch X_1 and X_2. Without loss of generality, for identical nonnegative random variables X_1 and X_2, we use the limiting property of $\eta(t) = \mathbb{E}[X_1|X_2 > t]/\mathbb{E}[X_2|X_1 > t]$ as $t \to \infty$ to study the strength of tail non-exchangeability. While in Hua and Joe (2014), the forms $\mathbb{E}[X_1|X_2 > t]$ and $\mathbb{E}[X_1|X_2 = t]$ are used to study the strength of tail dependence as $t \to \infty$, and in Bernard and Czado (2015), conditional quantiles are used to study the strength of tail dependence.

In what follows, we will first introduce the notation system used in the paper. Then in Section 2, we discuss basic concepts including the proposed approach for tail non-exchangeability and the Khoudraji's Device. Section 3 presents the main results for various patterns of tail non-exchangeability for three main different univariate tail heaviness patterns. Section 4 proposes a statistical test for testing the significance of tail non-exchangeability. Finally, Section 5 concludes the article.

1.1 Notation and symbols

In this section we discuss different notations and symbols. We are describing them one by one. Firstly, we define distribution functions as $F(x)$, where in the parenthesis we have the argument. In order to define the survival functions we use $\overline{F}(x)$, where in the parenthesis we have the argument. From the traditional literature we know the first order derivative of the distribution function is itself the density function. Hence, in the next sections when we differentiate $F(x)$ with respect to its argument, we write $f(x)$ as the derivative instead of $F'(x)$. In other words. $f(x) = F'(x) = \partial F(x)/\partial x$.

Secondly, throughout our paper we define the survival copula of an ordinary Copula C^* as $\widehat{C}^*(. \ ,.)$. Important thing in this case is that $\widehat{C}^*(. \ ,.)$ is the Copula before non-exchangeable transformation. After Khoudraji (1996) non-exchangeable transformation we have the survival Copula as $\widehat{C}(. \ ,.)$. Here, throughout our paper by Copula we actually mean Survival Copula where it itself is a function of survival functions [i.e. $\overline{F}(x)$]. In order to calculate the first order derivative we further use $\widehat{C}^*_{1|2}(u|v)$ instead of $\partial \widehat{C}^*(u,v)/\partial v$. From the literature we know that, $\widehat{C}^*_{1|2}(u|v)$ is a Cumulative distribution function(cdf) if $u, v \in [0,1]^2$. In this paper we put $u = \overline{F}(x)$ and $v = \overline{F}(y)$ to make them vary in $[0,1]$. At the tail, when we derive conditional expectation by Laplace approximation, we need to calculate second order derivative of our survival Copula. We use the notation $\widehat{C}^*_{1|2,2}(u|v)$. In other words, we define $\widehat{C}^*_{1|2,2}(u|v) = \partial^2 \widehat{C}^*(u,v)/\partial v^2 = \partial \widehat{C}^*_{1|2}(u|v)/\partial v$.

2 Preliminaries

2.1 Basic concepts and motivations

In dependence modeling one often uses copula functions to account for dependence patterns appearing in the tail part of the joint distribution. This is particularly important when these patterns cannot be well modeled by the commonly-used multivariate models such as the multivariate Normal or Student t distributions. Most of the commonly used bivariate copulas are of the exchangeable structure, meaning that $C(u,v) \equiv C(v,u)$ for any $(u,v) \in [0,1]^2$. As copula modeling often plays an important role in accounting for dependence in the tails, one may be particularly interested in the non-exchangeable structure in the joint tails. Motivated by Hua and Joe (2014), where the tail behavior of $\mathbb{E}[X_1|X_2 > t]$ or $\mathbb{E}[X_1|X_2 = t]$ is studied for capturing the tail dependence strength between the bivariate random vector (X_1, X_2), we introduce the following definition

Definition 1. *Let* (X_1, X_2) *be a bivariate random vector with identically distributed marginals, supported on* $[0,\infty)^2$. *Then the random vector* (X_1, X_2) *is said to be tail exchangeable of Type I if* *the following condition holds:*

$$\textit{Condition I:} \quad \lim_{t\to\infty} \eta_1(t) := \lim_{t\to\infty} \frac{\mathbb{E}[X_1|X_2 > t]}{\mathbb{E}[X_2|X_1 > t]} = 1, \tag{1}$$

and is tail exchangeable of Type II if the following condition holds:

$$\textit{Condition II:} \quad \lim_{t\to\infty} \eta_2(t) := \lim_{t\to\infty} \frac{\mathbb{E}[X_1|X_2 = t]}{\mathbb{E}[X_2|X_1 = t]} = 1. \tag{2}$$

Note here, we define "tail exchangeability" as a limiting property between two random variables when both of them take large values. When either of the conditions (1) and (2) does not hold, the random vector is said to be "tail non-exchangeable". The departure of functions $\eta_1(t)$ and $\eta_2(t)$ to 1 as $t \to \infty$ captures the degree of tail non-exchangeability.

Without loss of generality, assume that the (X_1, X_2) has a unique copula $C(\cdot,\cdot)$, of which the survival copula denoted as $\widehat{C}(\cdot,\cdot)$. Therefore the above conditions can be written as:

$$\text{Condition I':} \quad \lim_{t\to\infty} \eta_1(t) = \lim_{t\to\infty} \frac{\int_0^\infty \widehat{C}(\overline{F}(x), \overline{F}(t))dx}{\int_0^\infty \widehat{C}(\overline{F}(t), \overline{F}(x))dx} = 1, \tag{3}$$

where F is the cdf of the identical univariate marginals. The second condition is then

$$\text{Condition II':} \quad \lim_{t\to\infty} \eta_2(t) = \lim_{t\to\infty} \frac{\int_0^\infty \widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))dx}{\int_0^\infty \widehat{C}_{2|1}(\overline{F}(t)|\overline{F}(x))dx} = 1. \tag{4}$$

It is clear that the tail behavior of $\eta_1(t)$ and $\eta_2(t)$ rely on both the copula C and the marginal F. Our goal is to study the effect of different marginals on the degree of tail non-exchangeability.

As the conditional expectations do not have any closed form solutions, Hua and Joe (2014) suggests to use either Laplace approximation or Watson's lemma for asymptotic approximation when $t \to \infty$. They used these approximations in *exchangeable* copulas. In our paper, we are using the same method after transforming a copula into a *non-exchangeable* structure.

2.2 Khoudraji's device

Assume that X_1 and X_2 have identical marginal distribution functions with the cdf F being continuous on $[0, \infty)$, and density functions and moments exist whenever they are used. Following Khoudraji's device, see, Khoudraji (1996); Genest et al. (1998, 2011, 2012), write the copula in (3) as

$$\widehat{C}(\overline{F}(x), \overline{F}(t)) = \overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2}), \ (\alpha_1, \alpha_2) \in [0, 1]^2, \tag{5}$$

where $\widehat{C}^*$ is the survival copula of C^* that is exchangeable.

3 Tail non-exchangeability

In this section, we focus on obtaining results for tail non-exchangeability with different dependence structures and univariate marginals. There are three main univariate marginals are considered: Pareto, Exponential, and Weibull, that basically represent three different degrees of tail heaviness.

3.1 Type I

Proposition 1 (Based on Laplace's method). *Suppose that $0 < \alpha_1, \alpha_2 < 1$, $\widehat{C}^*$ is a bivariate copula, X_1 and X_2 are identically distributed positive random variables with univariate cdf F and density function f. Assume that $w := \lim_{x\to 0^+} \log(f(F^{-1}(x)) < \infty$, and write $T := -\log(\overline{F}(t))$ and*

$$g(s,T) = \alpha_2 - s(2-\alpha_1) + \frac{1}{T}\{\log[\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})[f(F^{-1}(1-e^{-sT}))]^{-1}] + w\}. \tag{6}$$

If there exists $T_0 < \infty$ such that, $T > T_0$ implies that $g(0,T) = 0$, $g(\infty,T) = -\infty$, $g'(0,T) > 0$, and $s_0(T) = \arg\max_s g(s,T)$, then,

$$\mathbb{E}[X_1|X_2 > t] \sim Te^{Tg(s_0(T),T)-w}\sqrt{\frac{2\pi}{-Tg''(s_0(T),T)}}, \quad t \to \infty$$

Proof: Following Hua and Joe (2014), together with (5),

$$\begin{aligned}\mathbb{E}[X_1|X_2 > t] &= \int_0^\infty \frac{\widehat{C}(\overline{F}(x), \overline{F}(t))}{\overline{F}(t)}\, dx, \ \forall\ t \\ &= \int_0^\infty \frac{\overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2})}{\overline{F}(t)}\, dx\ \forall\ t \end{aligned} \tag{7}$$

As $y = -\log \overline{F}(x) \implies dy = -\frac{\partial\overline{F}(x)/\partial x}{\overline{F}(x)}dx \implies \overline{F}(x)dy = -\frac{\partial\overline{F}(x)}{\partial x}dx \implies \overline{F}(x)dy = f(F^{-1}(1-\overline{F}(x)))dx$, after changing of variables we get, $e^{-y}dy = f(F^{-1}(1-e^{-y}))dx \implies e^{-y}[f(F^{-1}(1-e^{-y}))]^{-1}dy = dx$. After putting this condition in (7) we get, as $T = -\log(\overline{F}(t))$,

$$\mathbb{E}[X_1|X_2 > t] = \int_0^\infty e^{T\alpha_2 - y(2-\alpha_1)}\widehat{C}^*(e^{-\alpha_1 y}, e^{-\alpha_2 T})[f(F^{-1}(1-e^{-y}))]^{-1}\, dy \tag{8}$$

Let $y = sT$, and thus $dy = Tds$. After putting this condition in (8),

$$\mathbb{E}[X_1|X_2 > t] = Te^{-w}\int_0^\infty e^{Tg(s,T)}ds, \quad \forall\ s \in [0,\infty). \tag{9}$$

Now, based on the Laplace's method,

$$\begin{aligned}\mathbb{E}[X_1|X_2 > t] &\sim Te^{-w}\int_0^\infty \exp\{Tg(s_0(T),T) + \frac{1}{2}(s-s_0(T))^2 g''(s_0(T),T)\}ds \\ &\sim Te^{Tg(s_0(T),T)-w}\sqrt{\frac{2\pi}{-Tg''(s_0(T),T)}},\end{aligned}$$

which completes the proof. □

Remark 1. If $\alpha_1 = \alpha_2 = \alpha$ we have exchangeable survival copula. Furthermore, if we consider $\alpha = 1$ we get the same result with Hua and Joe (2014).

Example 1 (Clayton copula with Pareto marginals)**.** Let $\widehat{C}^*$ be the Clayton copula, that is, $\widehat{C}^*(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}$. Let X_1 and X_2 follow Pareto distributions with cdf $F(x) = 1 - (1+x)^{-\beta}$, and $1 < \beta < \frac{1}{1-\alpha_1}$. Then based on (6), for given $0 < \alpha_1, \alpha_2 < 1$, $T := \beta \log(1+t)$,

$$g(s,T) = \alpha_2 - (2-\alpha_1)s - \log(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)/(T\delta) + (1+\beta^{-1})s \tag{10}$$

It is clear that $g(0,T) = 0$ for any $T \in (0,\infty]$. Moreover, since $\beta > 1$, it can be verified that $g(\infty, T) = -\infty$ for any $T \in [0,\infty)$. Also,

$$g'(s;T) = \frac{1}{\beta} - \frac{\alpha_1}{1 + e^{\alpha_2\delta T - \alpha_1 \delta s T} - e^{-\alpha_1 \delta s T}} - (1-\alpha_1),$$

which implies that for any given $0 < \alpha_1 < 1$ and $1 < \beta$, there exists $T_0 > 0$ such that $T > T_0$ implies that $g'(0,T) > 0$.

For any given $0 < T$, the root $s_0(T)$ of $g'(s,T) = 0$ is

$$s_0(T) = \frac{1}{\alpha_1 \delta T} \log \frac{(e^{\alpha_2 \delta T} - 1)(1 - \beta + \alpha_1 \beta)}{\beta - 1}. \tag{11}$$

Therefore, we require that $1 < \beta < \frac{1}{1-\alpha_1}$ in order to have a well defined root. Moreover, it is clear that $\lim_{T\to\infty} s_0(T) = \alpha_2/\alpha_1$. Now consider

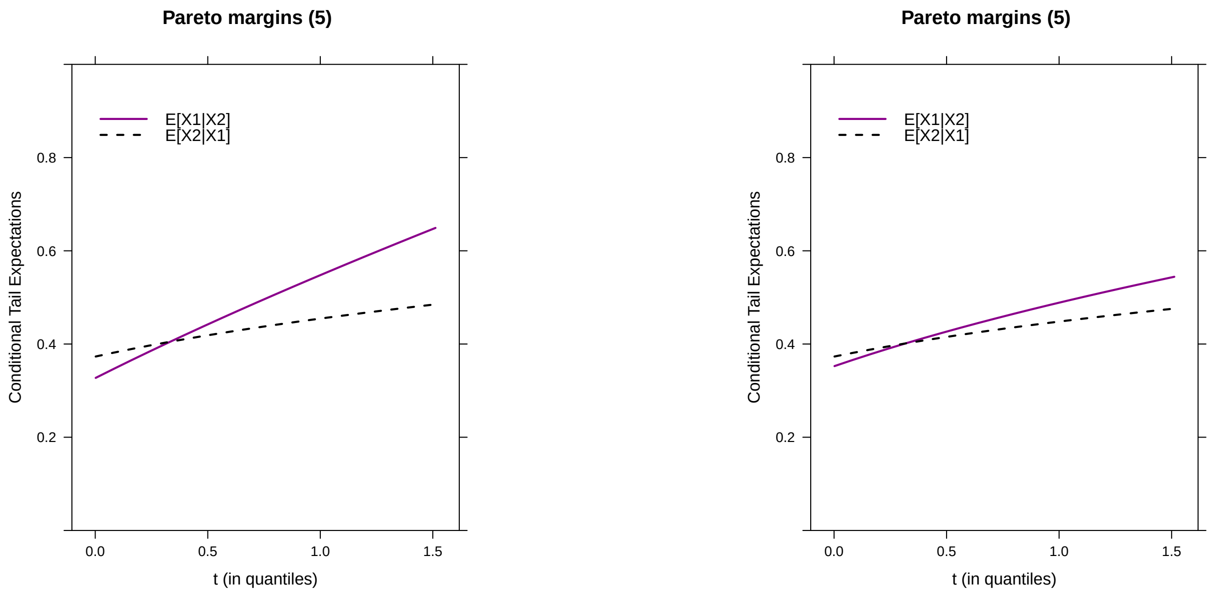
$$-g''(s;T) = \frac{\alpha_1(-\alpha_1 \delta T e^{\alpha_2 \delta T - \alpha_1 \delta s T} + \alpha_2 \delta T e^{-\alpha_2 \delta s T})}{(1 + e^{\alpha_2 \delta T - \alpha_1 \delta s T} - e^{-\alpha_2 \delta s T})^2}, \tag{12}$$

and therefore, with $T = \beta \log(1+t)$,

$$\mathbb{E}[X_1 | X_2 > t] \sim T e^{Tg(s_0(T),T)} \sqrt{\frac{2\pi}{-Tg''(s_0(T),T)}}, \quad t \to \infty, \tag{13}$$

where $g(s,T)$, $s_0(T)$, and $g''(s,T)$ are given in (10), (11), and (12), respectively. □

In Figure 1a and Figure 1b we try to do the simulation using Laplace approximation. In Figure 1a we take α_1 and α_2 0.97 and 0.85 respectively. In Figure 1b we reduce α_1 such that , its value comes closer to α_2. We do this because if α_1 and α_2 are very close to each other, we get *exchangeability* as the result of symmetric copulas. In these two panels we assume $\beta = 5$ and $\delta = 10$ throughout this simulation. The values of α_1 and α_2 are very high. The main reason is that, if we take lower vales, the distance between two conditional expectations are so big that we cannot find any pattern. Apart from that these Laplace approximation simulations look similar to the simulations when we use Pareto margins and use the definitions of conditional expectations.



(a) $\alpha_1 = 0.97$, $\alpha_2 = 0.85$, $\delta = 10$

(b) $\alpha_1 = 0.90$, $\alpha_2 = 0.85$, $\delta = 10$

Figure 1: Comparison of $\mathbb{E}[X_1|X_2 > t]$ and $\mathbb{E}[X_2|X_1 > t]$ when α_1 and α_2 are different, using Laplace Approximation

Proposition 2 (Based on Watson's lemma)**.** *Suppose that* $0 < \alpha_1, \alpha_2 < 1$, $\widehat{C}^*$ *is a bivariate copula,* X_1 *and* X_2 *are identically distributed positive random variables with univariate cdf* F *and density function* f*. Assume that* $\int_0^\infty e^{Tg(s,T)}\, ds < \infty$ *and write* $T := -\log(\overline{F}(t))$ *and*

$$g(s,T) = \alpha_2 - s(2-\alpha_1) + \frac{1}{T}\{\log[\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})[f(F^{-1}(1-e^{-sT}))]^{-1}] + w\}.$$

If there exists $T_0 < \infty$ *such that,* $T > T_0$ *implies that* $g(0,T) = 0$, $g(\infty,T) = -\infty$, $g'(0,T) \not> 0$, *Then,*

$$\mathbb{E}[X_1|X_2 > t] \sim \frac{1}{[(2-\alpha_1) + D_1 + D_2]}, \text{ as } t \to \infty,$$

where $D_1 = \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1, e^{-\alpha_2 T})}$ *and* $D_2 = \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))}$.

Proof. In order to do the approximation we are using Theorem 36 of Breitung (1994) [p. 48]. As $g(s,T)$ is a real valued function on the semi-infinite interval $[0,\infty)$ and in an interval $(0, 0+\epsilon]$ with $\epsilon > 0$, this function is continuously differentiable and $\sup_{0+\epsilon \le s \le \infty} g(s,T) \le g(0,T) - \psi$, with $\psi > 0$.

If $g'(s,T) \not> 0$ and $s \to 0$, following Theorem 36 of Breitung (1994) we can write $g'(s,T) = -as^{r-1} + o(s^{r-1})\ \forall r > 0$. Now if we assume $r = 1$ then

$$g'(s,T) = -a = -\left[(2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT}f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}\right].$$

This particular version of Watson's lemma requires $-a$ to be constant, which is possible only if

$$\lim_{s^+\to 0} g'(s,T) = -[(2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT}f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}],$$

goes to a constant. In other words, if $\lim_{s^+\to 0} \frac{e^{-sT}f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}$ is a constant . Thus,

$$-a = -[(2-\alpha_1) + \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1, e^{-\alpha_2 T})} + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))}]$$

or, $a = (2-\alpha_1) + \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1,e^{-\alpha_2 T})} + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} = 2 + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} > 0$.

Now, let us assume there is another real and continuous function $h(s) \in [0,\infty)$ such that, $h(s) = bs^{m-1} + o(s^{m-1})$ with $m > 0$. More specifically in our case we have, $h(s) = 1$. Thus, $bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1$ when $m = 1$. As we assume $\int_0^\infty e^{g(s,T)}ds < \infty$ then by this version of Watson's lemma we can write ;

$$\mathbb{E}[X_1|X_2 > t] \sim \frac{1}{[(2-\alpha_1) + D_1 + D_2]},\ t \to \infty \tag{14}$$

for all $(\alpha_1, \alpha_2) \in [0,1]^2$, and where $D_1 = \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1,e^{-\alpha_2 T})} = \alpha_1$ and $D_2 = \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))}$. This completes the proof. □

Example 2 (Clayton copula with Weibull marginals)**.** Let $\widehat{C}^*$ be the Clayton copula, that is, $\widehat{C}^*(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}$. Let X_1 and X_2 follow Weibull distributions with cdf $F(x) = 1 - F(x) = e^{-x^\gamma}\ \forall\ x, \gamma > 0$. Then based on (6), for given $0 < \alpha_1, \alpha_2 < 1$, $T := t^\gamma$,

$$g(s,T) = 1 - \frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1 \delta sT} + e^{\alpha_2 \delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T\right] \tag{15}$$

From (15) we clearly see that, $g(0,T) = 0$ and $g(\infty, T) = -\infty$ for any $T \in (0,\infty]$. Also from (15) we can also get,

$$g'(s;T) = -\left[\frac{\alpha_1}{1 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1 \delta sT}} + (1-\alpha_1)\right] < 0 \tag{16}$$

By using *proposition 2* we get,

$$\lim_{s\to 0^+} g(s,T) = -a = -\left[\frac{\alpha_1}{e^{\alpha_2 \delta T}} + (1-\alpha_1)\right] < 0 \tag{17}$$

Furthermore, in our case, $h(s) = s^{\frac{1}{\gamma}-1}$, $b = 1$ and $m = \gamma^{-1}$. Finally before using the result of *proposition 2* we have to show $\int_0^\infty |h(s)|e^{g(s,T)}ds < \infty$. Here we have,

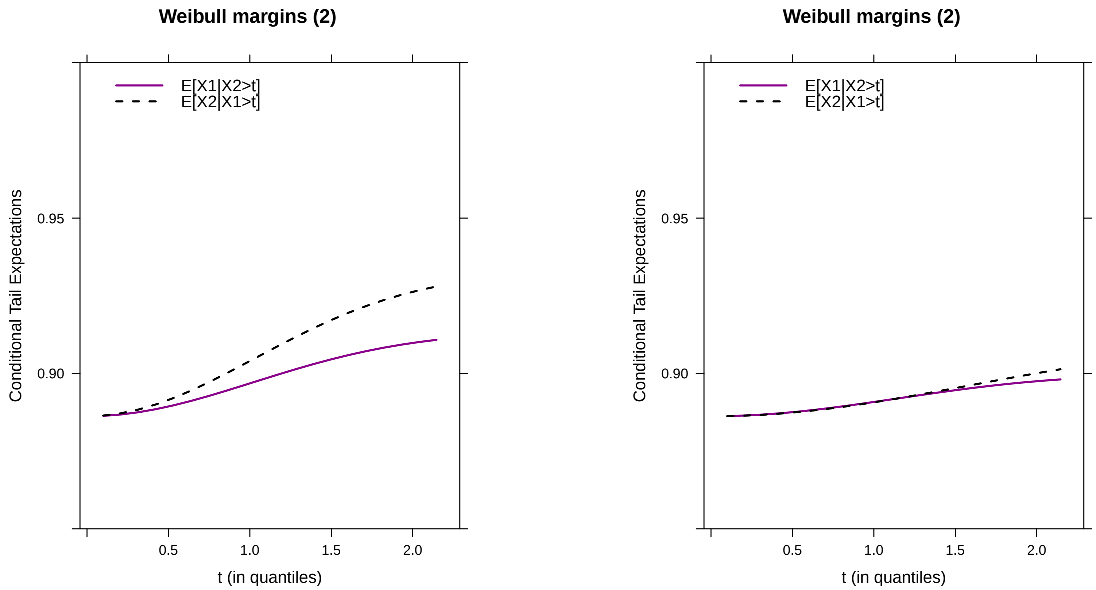
$$\int_0^\infty |h(s)|e^{g(s,T)}\ ds = \int_0^\infty |s^{\frac{1}{\gamma}-1}|\ e^{1-\frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)+(1-\alpha_1)sT+(1-\alpha_2)T]}\ ds$$
$$\leq e^{\alpha_2}\ \left[\int_0^{\frac{\alpha_2}{\alpha_1}}\ s^{\frac{1}{\gamma}-1}\ e^{-\alpha_2-(1-\alpha_1)s}\ ds\ +\int_{\frac{\alpha_2}{\alpha_1}}^\infty\ s^{\frac{1}{\gamma}-1}\ e^{-(\alpha_1+1-\alpha_1)s}\ ds\right],\ at\ T\to\infty$$
$$= e^{\alpha_2}\Bigg[e^{-\alpha_2}\ e^{-(1-\alpha_1)s}\sum_{i=0}^{\frac{1}{\gamma}-1}\ (-1)^{\frac{1}{\gamma}-i-1}\frac{(\frac{1}{\gamma}-1)!}{i!(\alpha_1-1)^{\frac{1}{\gamma}-i}}\ s^i\Bigg|_0^{\frac{\alpha_2}{\alpha_1}}$$
$$+\, e^{-s}\ \sum_{i=0}^{\frac{1}{\gamma}-1}\ (-1)^{\frac{1}{\gamma}-i-1}\frac{(\frac{1}{\gamma}-1)!}{i!}\ s^i\Bigg|_{\frac{\alpha_2}{\alpha_1}}^\infty\Bigg] \tag{18}$$

Both the terms on the right hand side of (18)is always finite. The main reasons are we have e^{-s} and $\gamma > 0$; which leads us three possibilities, $\gamma \in (0, 1)$, $\gamma = 1$ and $\gamma > 1$. Let us discuss each of the cases separately. As we have e^{-s} as the first term, it is always finite as $s \to \infty$. Now, only thing matters is the value of γ. When $\gamma \in (0, 1)$, $\gamma^{-1} - 1$ takes the highest value when $\gamma \to 0$. By assumption $\gamma > 0$. So $\gamma^{-1} - 1 < \infty$. Under this case we still possibility to have $s^i \to \infty$ as $s \to \infty$. Therefore, we need more restriction on γ. In this bound of $(0, 1)$ s^i is not finite. Furthermore, when $\gamma = 1$, $s^i \to \infty$ as $s \to \infty$. Hence, we need $\gamma > 1$ to make $s^i < \infty$ for any large s. using above conditions and *proposition 2* we get;

$$\mathbb{E}[X_1|X_2 > t] \sim \gamma^{-1}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{\frac{\alpha_1}{e^{\alpha_2\delta T}}+(1-\alpha_1)}\ as\ t\to\infty\ and\ T = t^\gamma, \tag{19}$$

where $(\alpha_1, \alpha_2) \in [0, 1]^2$, $\gamma > 0$ and $t \to \infty$.

Figure 2a represents simulation from Watson's lemma and figure 2b is actual integration result. In figure 2a the dotted black line represents $\mathbb{E}[X_2|X_1 > t]$ and the purple line represents $\mathbb{E}[X_1|X_2 > t]$. The pattern of the movement of the two lines are same but the gap between them is more than in figure 2b. Hence, we clearly claim that, Watson's lemma gives more tail *non-exchangeability.*



(a) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$ (b) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$

Figure 2: Comparison between Simulation of Watson's Lemma and Actual Expectations

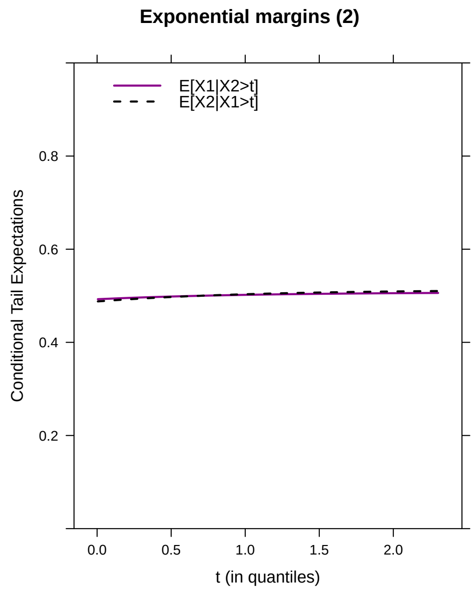
Example 3 (Clayton copula with Exponential marginals)**.** Again like before we have the same Clayton Copula but, we have exponential marginals. Let X_1 and X_2 follow Exponential distributions with cdf $F(x) = 1 - e^{-\lambda x}$, $\forall x \in [0, \infty)$. Then based on (6), for given $0 < \alpha_1, \alpha_2 < 1$, $\delta > 0$ and $T := \lambda t$,

$$g(s,T) = 1 - \frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T\right] \tag{20}$$

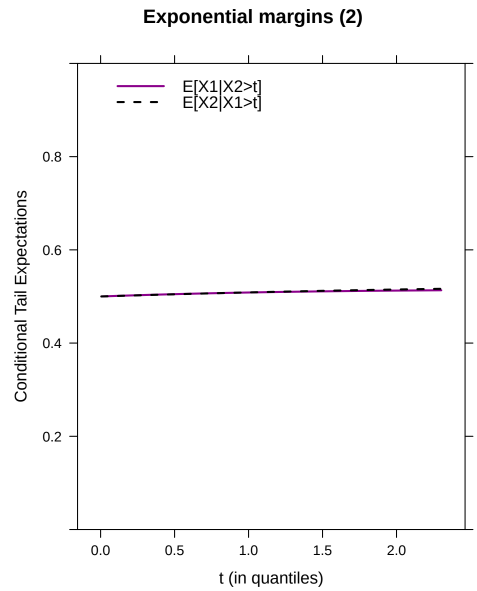
Again, by doing easy calculations we can show that, $g(0,T) = 0$ and $g(\infty, T) = -\infty$ at any $T \in [0, \infty)$. Also we have,

$$g'(s;T) = -\left[\frac{\alpha_1}{1 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1 \delta s T}} + (1-\alpha_1)\right] < 0 \tag{21}$$

In order to satisfy *proposition 2* we need to see the behavior of $g(s,T)$ around zero. In other words, $\lim_{s^+ \to 0} g'(s;T) = -a = -\left[\alpha_1 e^{-\alpha_2 \delta T} + 1 - \alpha_1\right] < 0$ for all $(\alpha_1, \alpha_2) \in (0,1)$ and $\delta > 0$. This implies that the slope of the $g(s,T)$ is negative even in the neighborhood of zero. In this case $b = m = h(s) = 1$. Apart from that like in the Weibull case we can show that, $\int_0^\infty |h(s)| e^{g(s,T)} ds < \infty$. Now using *proposition 2* we can conclude, $\mathbb{E}[X_1|X_2 > t] \sim \lambda^{-1}(\alpha_1 e^{-\alpha_2 \delta T} + 1 - \alpha_1)$ as $t \to \infty$, $\lambda > 0$ and $T = \lambda t$. In figure 3a we did the simulation corresponding to Watson's lemma. On the right hand side in 3b we get the plot of the numerical integration. If we compare these two pictures we clearly see that, Watson's lemma gives almost same simulation result. The important thing is when we use KB4 copula with exponential margins we do not get much *non-exchangeability*.



(a) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$ and $\lambda = 2$, Watson's lemma



(b) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\lambda = 2$

Figure 3: Comparison between Simulation of Watson's Lemma and Actual Expectations

3.2 Type II

Definition 2. *If $\mathbb{E}[X_1|X_2 = t]$ is a conditional expectation for any given t, then it can be expressed in terms of non-exchangeable Copula as; $\mathbb{E}[X_1|X_2 = t] = \int_0^\infty \widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))\,dx,\ \forall\ t$ where*
$\widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t)) = \partial/\partial\overline{F}(t)\left[\overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2})\right]$, with survival functions $\overline{F}(x)$ and $\overline{F}(t)$ and $(\alpha_1, \alpha_2)\ \in [0,1]^2$.

Proposition 3 (Based on Laplace's Method)**.** *Suppose that $0 < \alpha_1, \alpha_2 < 1$, $\widehat{C}^*$ is a bivariate copula, $\widehat{C}^*_{1|2}$ is a conditional bivariate copula, X_1 and X_2 are identically distributed positive random variables with univariate cdf F and density function f. Assume that $w := \lim_{x\to 0^+}\log(f(F^{-1}(x)) < \infty$, and write $T := -\log(\overline{F}(t))$ and*

$$g_1(s,T) = -(\alpha_2 + s\alpha_1) + \frac{1}{T}\left[\log\frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]} + w\right]$$

and

$$g_2(s,T) = -s(2-\alpha_1) + \frac{1}{T}\left[\log\frac{\widehat{C}^*_{1|2}(e^{-\alpha_1 sT}|e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]} + w\right].$$

If there exists $T_0 < \infty$ such that, $T > T_0$ implies that $g_i(0,T) = 0$, , $g_i(\infty,T) = -\infty$, $g_i'(0,T) > 0$, and $s_{0i}(T) = \arg\max_s g_i(s,T)$, for all $i = 1,2$ then,

$$\mathbb{E}[X_1|X_2 = t] \sim -\log \overline{F}(t)(1-\alpha_2)e^{-\log \overline{F}(t) g_1(s_{01}(T), -\log \overline{F}(t))}\sqrt{\frac{2\pi}{\log \overline{F}(t)(1-\alpha_2)g_1''(s_{01}(T), -\log \overline{F}(t))}}$$
$$-\log \overline{F}(t)\alpha_2 e^{-\log \overline{F}(t) g_2(s_{02}(T), -\log \overline{F}(t))}\sqrt{\frac{2\pi}{\log \overline{F}(t)\alpha_2 g_2''(s_{02}(T), T)}},\ t \to \infty$$

Proof. Following Hua and Joe (2014), together with (5),

$$\begin{aligned}\mathbb{E}[X_1|X_2 = t] &= \int_0^\infty \widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))\ dx,\ \forall\ t \\ &= \int_0^\infty \left[(1-\alpha_2)\overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{-\alpha_2}\widehat{C}^*\left(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2}\right) + \alpha_2 \overline{F}(x)^{1-\alpha_1}\widehat{C}^*_{1|2}(\overline{F}(x)^{\alpha_1}|\overline{F}(t)^{\alpha_2})\right] dx\end{aligned} \tag{22}$$

As $y = -\log \overline{F}(x) \implies dy = -\frac{\partial \overline{F}(x)/\partial x}{\overline{F}(x)}dx \implies \overline{F}(x)dy = -\frac{\partial \overline{F}(x)}{\partial x}dx \implies \overline{F}(x)dy = f(F^{-1}(1-\overline{F}(x)))dx$, after changing of variables we get, $e^{-y}dy = f(F^{-1}(1-e^{-y}))dx \implies e^{-y}[f(F^{-1}(1-e^{-y}))]^{-1}dy = dx$. After putting this condition in (22) we get, as $T = -\log(\overline{F}(t))$,

$$\begin{aligned}\mathbb{E}[X_1|X_2 = t] = \int_0^\infty &[(1-\alpha_2)e^{-(1-\alpha_1)y}e^{\alpha_2 T}\widehat{C}^*(e^{-\alpha_1 y}, e^{-\alpha_2 T}) \\ &+ \alpha_2 e^{-(1-\alpha_1)y}\widehat{C}^*_{1|2}(e^{-\alpha_1 y}|e^{-\alpha_2 T})]e^{-y}[f[F^{-1}(1-e^{-y})]]^{-1}\ dy\end{aligned} \tag{23}$$

Let $y = sT$, and thus $dy = Tds$. After putting this condition in (23),

$$\begin{aligned}&\mathbb{E}[X_1|X_2 = t] \\ &= T(1-\alpha_2)e^{-w}\int_0^\infty e^{T\left[-(\alpha_2+s\alpha_1)+\frac{1}{T}\left[\log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{[f[F^{-1}(1-e^{-sT})]]}+w\right]\right]}\ ds \\ &\quad + T\alpha_2 e^{-w}\int_0^\infty e^{T\left[-s(2-\alpha_1)+\frac{1}{T}\left[\log \frac{\widehat{C}^*_{1|2}(e^{-\alpha_1 sT}|e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]}+w\right]\right]}\ ds \\ &= T(1-\alpha_2)e^{-w}\int_0^\infty e^{Tg_1(s,T)}h(s)\ ds + T\alpha_2 e^{-w}\int_0^\infty e^{Tg_2(s,T)}h(s)\ ds \qquad (24) \\ &= Te^{-w}\left[(1-\alpha_2)\Gamma_1 + \alpha_2\Gamma_2\right],\end{aligned}$$

where $\Gamma_1 = \int_0^\infty e^{Tg_1(s,T)}h(s)\ ds$ and $\Gamma_2 = \int_0^\infty e^{Tg_2(s,T)}h(s)\ ds$. Now we have two separate integrations consist of $g_i(s,T)$ (with $i = 1,2$) functions each of which behaves similarly like in *proposition 1*. In this case $h(s) = 1$. Thus, based on Laplace Method we can say that , $\Gamma_i = \int_0^\infty \exp\{Tg_i(s_{0i}(T),T) + \frac{1}{2}(s - s_{0i}(T))^2 g_i''(s_{0i}(T),T)\}ds$ for all $i = 1,2$. Finally, the conditional expectation becomes,

$$\mathbb{E}[X_1|X_2=t] \sim -\log\overline{F}(t)(1-\alpha_2)e^{-\log\overline{F}(t)g_1(s_{01}(T),-\log\overline{F}(t))}\sqrt{\frac{2\pi}{\log\overline{F}(t)(1-\alpha_2)g_1''(s_{01}(T),-\log\overline{F}(t))}}$$
$$-\log\overline{F}(t)\alpha_2 e^{-\log\overline{F}(t)g_2(s_{02}(T),-\log\overline{F}(t))}\sqrt{\frac{2\pi}{\log\overline{F}(t)\alpha_2 g_2''(s_{02}(T),T)}},\ t\to\infty$$

□

Example 4 (Clayton copula with Pareto marginals)**.** Let $\widehat{C}^*$ be the Clayton copula, that is, $\widehat{C}^*(u,v)=(u^{-\delta}+v^{-\delta}-1)^{-1/\delta}$. Let $\widehat{C}_{1|2}$ be the conditional non-exchangeable Clayton copula which has the form, $\widehat{C}_{1|2}(u|v)=(u^{-\alpha_1\delta}+v^{-\alpha_2\delta}-1)^{-1/\delta}u^{1-\alpha_1}v^{-\alpha_2}\left[(1-\alpha_2)+(\alpha_2 v^{-\alpha_2\delta})(u^{-\alpha_1\delta}+v^{-\alpha_2\delta}-1)^{-1}\right]$. Let X_1 and X_2 follow Pareto distributions with cdf $F(x)=1-(1+x)^{-\beta}$, and $1<\beta<\frac{1}{1-\alpha_1}$. Then based on combination of $g_1(s;T)$ and $g_2(s;T)$ in *proposition 3*, for given $0<\alpha_1,\alpha_2<1$, $T:=\beta\log(1+t)$,

$$g(s;T)=\frac{s}{\beta}+\frac{1}{T}\Bigg[\alpha_2 T-(1-\alpha_1)sT-\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)$$
$$+\log\left[(1-\alpha_2)+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]\Bigg] \tag{25}$$

If we carefully look at (25) and combine this result with g_1 and g_2 in *proposition 3*, we can see that one of the g_i's vanishes. As a result, we get only one g function. It is clear that $g(0,T)=0$ for any $T\in[0,\infty)$. Moreover, since $\beta>1$, it can be verified that $g(\infty,T)=-\infty$ for any $T\in[0,\infty)$. Also,

$$g'(s;T)=\frac{1}{\beta}-(1-\alpha_1)-\frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}$$
$$-\frac{\alpha_1\alpha_2\delta e^{\alpha_1\delta sT+\alpha_2\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]}$$

which implies that for any given $0<\alpha_1<1$ and $1<\beta$, there exists $T_0>0$ such that $T>T_0$ implies that $g'(0,T)>0$.

For any given $0<T$, the root $s_0(T)$ of $g'(s,T)=0$ is

$$s_0(T)=\frac{1}{\alpha_1\delta T}\Big[\log\left\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta-2[1-\beta(1-\alpha_1)])\right.$$
$$\left.+2[1-\beta(1-\alpha_1)]\right\}\Big]-\frac{1}{\alpha_1\delta T}\Big[\log\left\{2[1-\beta(1-\alpha_1)]\right\}\Big] \tag{26}$$

Therefore, we require that $1<\beta<\frac{1}{1-\alpha_1}$ in order to have a well defined root. Moreover, it is clear that $\lim_{T\to\infty}s_0(T)=\alpha_2/\alpha_1$. Now consider

$$-g''(s;T)=-\frac{\partial}{\partial s}\left[\frac{1}{\beta}-(1-\alpha_1)+A+B\right] \tag{27}$$

where $A = -(\alpha_1 e^{\alpha_1\delta sT})(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-1}$ and, $B = -(\alpha_1\alpha_2\delta e^{\alpha_1\delta sT+\alpha_2\delta T})((e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2[1-\alpha_2 + (\alpha_2 e^{\alpha_2\delta T})(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-1}])^{-1}$. Furthermore, $\partial A/\partial s$ is $-(\alpha_1^2\delta T e^{\alpha_1\delta sT})(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-1}\left[1 - (e^{\alpha_1\delta sT})(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-1}\right]$ which converges to zero as $t \to \infty$. Again,

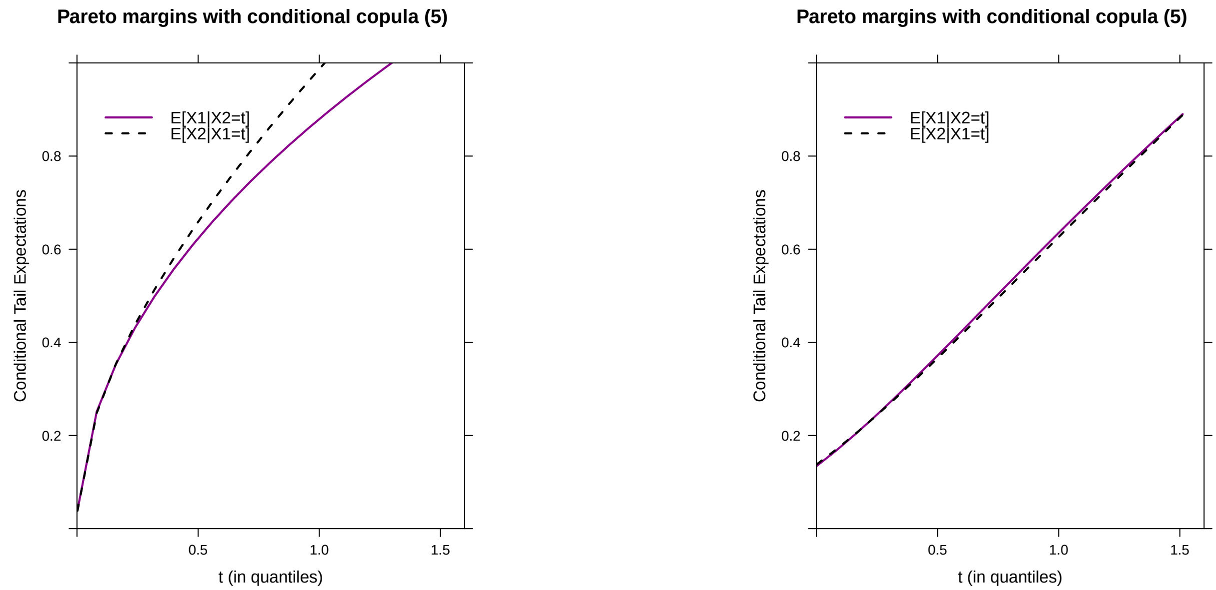
$$\begin{aligned}\frac{\partial B}{\partial s} = -\Bigg[&\frac{\alpha_1^2\alpha_2\delta^2 e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]} \\ &- \frac{\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^4\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]^2} \\ &- \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}\left[2\alpha_1\delta T(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]\right]}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^4\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]^2}\Bigg]\end{aligned} \tag{28}$$

Furthermore, $\lim_{T\to\infty} -g''(s;T) = (\alpha_2\alpha_1^2\delta^2)2^{-1}$ when $\alpha_2 > \alpha_1$. and , with $T = \beta\log(1+t)$,

$$\begin{aligned}\mathbb{E}[X_1|X_2>t] &\sim Te^{Tg(s_0(T),T)}\sqrt{\frac{2\pi}{-Tg''(s_0(T),T)}}, \quad t\to\infty, \\ &\sim \frac{1}{\beta}(1+t)^{\frac{\alpha_2}{\alpha_1}-(1-\alpha_1)\frac{\alpha_2\beta}{\alpha_1}}\sqrt{\frac{4\pi\beta\log(1+t)}{\alpha_2(\alpha_1\delta)^2}},\ as\ t\to\infty.\end{aligned} \tag{29}$$

where $g(s,T)$, $s_0(T)$, and $g''(s,T)$ are given in (25), (26) and (27) respectively. ☐

In Figure 4a and 4b we try to compare the simulation using Laplace approximation with the actual conditional tail expectations. In Figure 4a we are using the simulation results obtained in (29) . Throughout our simulations we assume $\alpha_1 = 0.85$, $\alpha_2 = 0.90$, $\beta = 5$ and $\delta = 1$. We take $\delta = 1$ because for the higher and lower values we can see uneven fluctuations. We are not able to find any pattern in these cases. After fixing the values of the parameters we see in Figure 4a there is no significant *non-exchangeability* at around 0, but this *non-exchangeability* increases as we come closer to 90 th percentile. On the other hand, in Figure 4b we do not find that much *non-exchangeability* throughout the plot. We can say Laplace approximation might overestimate the small changes between two tail order conditional expectations at higher quantiles. Again, if we carefully look at α_1 and α_2, we find they are not significant different from each other. Hence, even these two parameters are very close to each other we can find higher tail *non-exchangeability*.



(a) $\alpha_1 = 0.85$, $\alpha_2 = 0.90$, $\beta = 5$ and $\delta = 1$

(b) $\alpha_1 = 0.85$, $\alpha_2 = 0.90$, $\beta = 5$ and $\delta = 1$

Figure 4: Comparison between Laplace Approximation and the Actual Conditional Tail Expectations when α_1 and α_2 are different

Proposition 4 (Based on Watson's Lemma)**.** *Suppose that* $0 < \alpha_1, \alpha_2 < 1$, $\widehat{C}^*$ *is a bivariate copula,* $\widehat{C}^*_{1|2}$ *is a conditional bivariate copula,* X_1 *and* X_2 *are identically distributed positive random variables with univariate cdf* F *and density function* f*. Assume that* $\int_0^\infty e^{Tg_i(s,T)}\,ds < \infty$ *for all* $i = 1, 2$ *and write* $w := \lim_{x \to 0^+} \log(f(F^{-1}(x)) < \infty$*, and write* $T := -\log(\overline{F}(t))$ *and*

$$g_1(s,T) = -(\alpha_2 + s\alpha_1) + \frac{1}{T}\left[\log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]} + w\right]$$

and

$$g_2(s,T) = -s(2-\alpha_1) + \frac{1}{T}\left[\log \frac{\widehat{C}^*_{1|2}(e^{-\alpha_1 sT}|e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]} + w\right].$$

For $i = 1, 2$ *if there exists* $T_0 < \infty$ *such that,* $T > T_0$ *implies that* $g_i(0,T) = 0$, $g_i(\infty, T) = -\infty$, $g_i'(0,T) \not> 0$*, Then,*

$$E[X_1|X_2 = t] \sim \frac{1-\alpha_2}{\Upsilon_1} + \frac{\alpha_2}{\Upsilon_2},\ \text{as } t \to \infty, \tag{30}$$

where $\Upsilon_1 = \alpha_1 + \{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})\}[\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})]^{-1} + \{e^{-sT} f'[F^{-1}(1-e^{-sT})]\}$ $f^{-2}[F^{-1}(1-e^{-sT})]$ *and* $\Upsilon_2 = (2-\alpha_1) + \{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})\}[\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})]^{-1} +$ $\{e^{-sT} f'[F^{-1}(1-e^{-sT})]\} f^{-2}[F^{-1}(1-e^{-sT})]$ *for all* $\Upsilon_1,\ \Upsilon_2 \in \mathbb{R} \setminus \{0\}$*.*

Proof. We are using Theorem 36 of Breitung (1994) [p. 48]. As $g_1(s,T)$ and $g_2(s,T)$ are real functions on the semi-infinite interval $[0,\infty)^2$ and in an interval $(0, 0+\epsilon_i]^2$,where $i = 1,2$, with $\epsilon_1, \epsilon_2 > 0$, these functions are continuously differentiable and

$$\sup_{0+\epsilon_1 \le s \le \infty} g_1(s,T) \le g_1(0,T) - \psi_1, \ and$$
$$\sup_{0+\epsilon_2 \le s \le \infty} g_2(s,T) \le g_2(0,T) - \psi_2 \tag{31}$$

with $\psi_1, \psi_2 > 0$.

Now for $g_1'(s,T)$ and $g_2'(s,T)$ we have $g_1'(s,T) < 0$ and $g_2'(s,T) < 0$ for all $s \in (0, 0+\max\{\epsilon_1, \epsilon_2\}]$. We can also write $g_1'(s,T) = -as^{r_1-1} + o(s^{r-1}) \ \forall r_1 > 0$ and $g_2'(s,T) = -as^{r_2-1} + o(s^{r-1}) \ \forall r_2 > 0$. Now if we assume $r = 1$ then $r_1 = r_2 = 1$ and, $g_1'(s,T) = -a_1$ and $g_2'(s,T) = -a_2$. We also know that,

$$g_1'(s,T) = -\alpha_1 - \frac{\alpha_1 e^{-\alpha_1 sT} \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} - \frac{e^{-sT} f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]} = -\Upsilon_1$$

with $\lim_{s^+ \to 0} \ g_1'(s,T) = -2\alpha_1 - f'[F^{-1}(0)]f^{-2}[F^{-1}(0)]$ and

$$g_2'(s,T) = -(2-\alpha_1) - \frac{\alpha_1 e^{-\alpha_1 sT} \widehat{C}^*_{2|1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})} - \frac{e^{-sT} f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]} = -\Upsilon_2$$

with $\lim_{s^+ \to 0} \ g_2'(s,T) = -2 - f'[F^{-1}(0)]f^{-2}[F^{-1}(0)]$; which are constants at $s^+ \to 0$ and $t \to \infty$. Thus, $-a_1 = -\Upsilon_1$ or, $a_1 = \Upsilon_1 > 0$. Similarly, we can say that, $a_2 = \Upsilon_2$.

Let us assume there is another real and continuous function $h(s,T) \in [0,\infty)$ such that, $h_i(s,T) = bs^{m_i-1} + o(s^{m_i-1})$ with $m_i > 0$ and $i = 1, \ 2$. More specifically we assume $h_i(s,T) = 1 \ \forall \ i = 1,2$ in our case. Thus, $b_i s^{m_i-1} + o(s^{m_i-1}) = 1 \implies b_i = 1$, where $m_i = 1$ for all $i = 1,2$.

Now, after using this theorem we get;

$$E[X_1|X_2 = t] \sim \frac{1-\alpha_2}{\Upsilon_1} + \frac{\alpha_2}{\Upsilon_2}, \ as \ t \to \infty, \ with \ s^+ \to 0 \tag{32}$$

where $T = -\log \overline{F}(t)$, $(\alpha_1, \alpha_2) \in [0,1]^2$, $\Upsilon_1 = \alpha_1 + \{\alpha_1 e^{-\alpha_1 sT} \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})\}[\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})]^{-1}$ $+ \{e^{-sT} f'[F^{-1}(1-e^{-sT})]\} f^{-2}[F^{-1}(1-e^{-sT})]$ and $\Upsilon_2 = (2-\alpha_1) + \{\alpha_1 e^{-\alpha_1 sT} \widehat{C}^*_{2|1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})\}$ $\left[\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})\right]^{-1} + \{e^{-sT} f'[F^{-1}(1-e^{-sT})]\} f^{-2}[F^{-1}(1-e^{-sT})]$ for all $\Upsilon_1, \ \Upsilon_2 \in \mathbb{R} \setminus \{0\}$. □

Example 5 (Clayton copula with Weibull marginals)**.** Let $\widehat{C}^*$ be the Clayton copula, that is, $\widehat{C}^*(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}$. Let $\widehat{C}_{1|2}$ be the conditional non-exchangeable Clayton copula which has the form, $\widehat{C}_{1|2}(u|v) = (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta} u^{1-\alpha_1} v^{-\alpha_2} \left[(1-\alpha_2) + (\alpha_2 v^{-\alpha_2\delta})(u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1}\right]$.

Suppose, X_1 and X_2 follow Weibull distributions with cdf $F(x) = 1 - F(x) = e^{-x^\gamma}\ \forall\ x, \gamma > 0$. Then based on (31), for given $0 < \alpha_1, \alpha_2 < 1$, $T := t^\gamma$,

$$g(s;T) = \frac{1}{T}\Bigg[\alpha_2 T - (1-\alpha_1)sT - \frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + \log\left[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\right]\Bigg]. \quad (33)$$

From (33) we clearly see that, $g(0,T) = 0$ and $g(\infty, T) = -\infty$ for any $T \in (0, \infty]$. Also from (33) we can also get,

$$g'(s;T) = -\left[(1-\alpha_1) + \frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1} + \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2\left[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\right]}\right] < 0 \quad (34)$$

By using *proposition 4* we get;

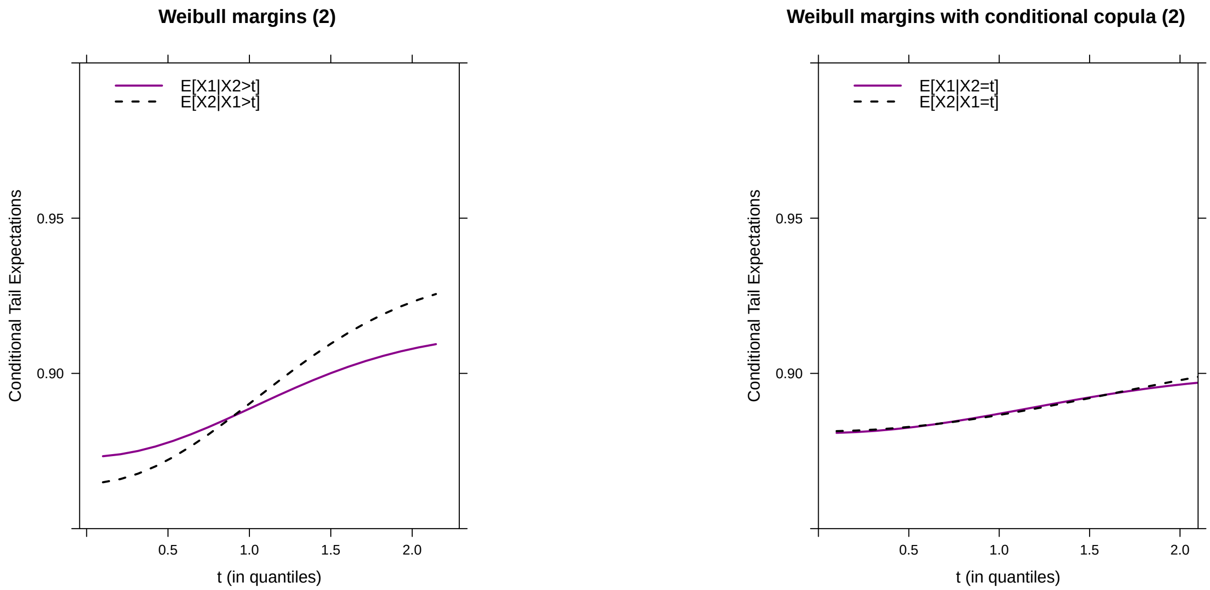
$$\lim_{s^+\to 0} g(s,T) = -\left[(1-\alpha_1) + \frac{\alpha_1}{e^{\alpha_2\delta T}} + \frac{\alpha_1\alpha_2\delta}{e^{\alpha_2\delta T}}\right] < 0 \quad (35)$$

Furthermore, in our case, $h(s) = s^{\frac{1}{\gamma}-1}$, $b = 1$ and $m = \gamma^{-1}$. In a similar calculation like in *Example 2* we can easily show that $\int_0^\infty |h(s)|e^{g(s,T)}ds < \infty$. Now, by using *proposition 4* we get;

$$\mathbb{E}[X_1|X_2 > t] \sim \gamma^{-1}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{(1-\alpha_1) + \frac{\alpha_1}{e^{\alpha_2\delta T}} + \frac{\alpha_1\alpha_2\delta}{e^{\alpha_2\delta T}}}\ as\ t \to \infty\ and\ T = t^\gamma, \quad (36)$$

where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta > 0$, $\gamma > 0$ and $t \to \infty$.

In figure 5a and figure 5b we are comparing between the simulation result based on Watson's lemma and result corresponding to numerical integration. If we look carefully figure 5a we can see that, there is some non-exchangeability at around zero but when the two lines corresponding to two conditional expectations come close to 1, they take the same value. At that point there is exchangeability of some extent exists. Now, if we go further to the 90th quartile, we experience less exchangeability. Furthermore, it shows more *non-exchangeability* than in figure 5b. Apart from that, in figure 5a $\mathbb{E}[X_1|X_2 = t] > \mathbb{E}[X_2|X_1 = t]$ before taking the value of 1 in x-axis but, after hitting 1 it is $\mathbb{E}[X_1|X_2 = t] < \mathbb{E}[X_2|X_1 = t]$.



(a) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$, $\delta = 10$

(b) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$ and $\delta = 10$

Figure 5: Comparison between Watson's Lemma and the Actual Conditional Tail Expectations with Weibull Margins when α_1 and α_2 are different

Example 6 (Clayton copula with Exponential marginals)**.** Let $\widehat{C}^*$ be the Clayton copula, that is, $\widehat{C}^*(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}$. Let $\widehat{C}_{1|2}$ be the conditional non-exchangeable Clayton copula which has the form, $\widehat{C}_{1|2}(u|v) = (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta} u^{1-\alpha_1} v^{-\alpha_2} \left[(1-\alpha_2) + (\alpha_2 v^{-\alpha_2\delta})(u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1}\right]$. Let X_1 and X_2 follow Exponential distributions with cdf $F(x) = 1 - e^{-\lambda x}$, $\forall x \in [0,\infty)$. Then based on (31), for given $0 < \alpha_1, \alpha_2 < 1$, $\delta > 0$ and $T := \lambda t$,

$$g(s;T) = \frac{1}{T}\Bigg[\alpha_2 T - (1-\alpha_1)sT - \frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + \log\left[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\right]\Bigg]. \tag{37}$$

From (37) we clearly see that, $g(0,T) = 0$ and $g(\infty,T) = -\infty$ for any $T \in (0,\infty]$. Also from (37) we can also get,

$$g'(s;T) = -\left[(1-\alpha_1) + \frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1} + \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2\left[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\right]}\right] < 0 \tag{38}$$

By using *proposition 4* we get;

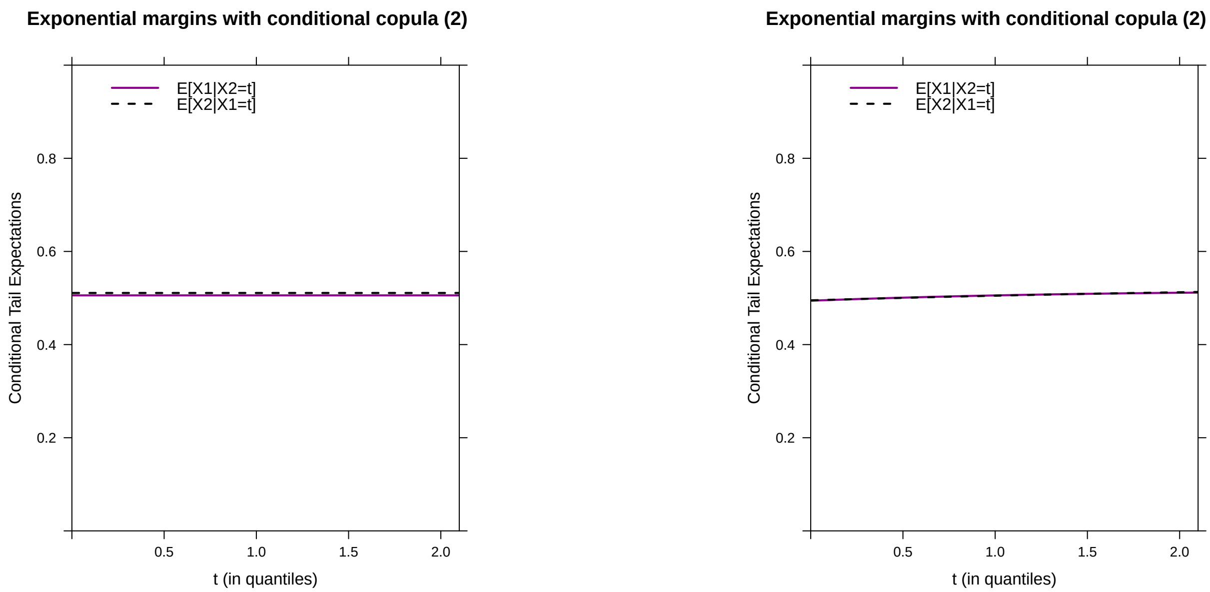
$$\lim_{s^+\to 0} g(s,T) = -\left[(1-\alpha_1) + \frac{\alpha_1}{e^{\alpha_2\delta T}} + \frac{\alpha_1\alpha_2\delta}{e^{\alpha_2\delta T}}\right] < 0 \tag{39}$$

Furthermore, in our case, $h(s) = b = m = 1$. In a similar calculation like in *Example 3* we can easily show that $\int_0^\infty |h(s)|e^{g(s,T)}ds < \infty$. Now, by using *proposition 4* we get;

$$\mathbb{E}[X_1|X_2 > t] \sim \left(\frac{1}{\lambda}\right) \frac{1}{(1-\alpha_1) + \frac{\alpha_1}{e^{\alpha_2 \delta T}} + \frac{\alpha_1 \alpha_2 \delta}{e^{\alpha_2 \delta T}}} \; as\ t \to \infty\ and\ T = \lambda t, \tag{40}$$

where $(\alpha_1, \alpha_2) \in [0, 1]^2$, $\delta > 0$, $\gamma > 0$ and $t \to \infty$.

In figure 6a and figure 6b we are comparing the simulation results corresponding to Watson's lemma with the numerical integration respectively. In both cases we are not able to find much non-exchangeability. Throughout our paper we only consider KB4 copula which is non-exchangeable by nature. Probably in this case the marginal distribuion dominates the copula. As a result we can not find less non-exchangeability.



(a) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\lambda = 2$, $\delta = 10$

(b) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\lambda = 2$ and $\delta = 10$

Figure 6: Comparison between Watson's Lemma and the Actual Conditional Tail Expectations with Exponential Margins when α_1 and α_2 are different

Remark 2. *If we compare the g functions in example 4,5 and 6 with propositions 3 and 4 we can see that, in the propositions we have g functions in the form of g_1 and g_2 simultaneously but, in the examples we do not. The main reason behind it is one g_i (for $i = 1, 2$) function dominates the other. As a result, another g_i function vanishes.*

4 Test of tail non-exchangeability

We have done simulations by using either Laplace method or Watson's lemma with different non-exchangeable survival Copulas with Pareto, Weibull or Exponential margins respectively. In this chapter we are going to test *non-exchangeability* empirically. There has been a lot of research in this field and different kinds of estimators have been used in hypothesis testing of this literature. Fermanian et al. (2004) defines that any empirical Copula process converges to a weak Gaussian process. A detailed discussion in weak Gaussian process takes place in Van Der Vaart and Wellner (1996). Furthermore, Fermanian et al. (2004) mention Theorem 3.9.4 of Van Der Vaart and Wellner (1996) in order to prove Theorem 3 in their paper. In this section we are trying to do a statistical test of *tail non-exchangeability* of Condition I.

Definition 3. *We define empirical version of the survival copula* $\widehat{C}^*(\overline{F}(x), \overline{F}(t))$ *as*

$$\widehat{C}^*_n(\overline{F}(x), \overline{F}(t)) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}(U_{1j} \le \overline{F}(x), U_{2j} \le \overline{F}(t))$$

where, $\overline{F}(x)$ *and* $\overline{F}(t)$ *are associated any continuous marginal distributions such that* $[\overline{F}(x), \overline{F}(t)] \in [0,1]^2$*;* $\forall\ j = 1, 2, 3, ..., n,$ *and at* $t \to \infty$*.*

Proposition 5. *If* $\mathbf{\Sigma}$ *is positive definite,* $\boldsymbol{d(\theta)} \neq \mathbf{0}$ *and* $\boldsymbol{d(\theta)}$ *is continuous in a neighborhood of* $\boldsymbol{\theta}$*, then*

$$n^{1/2}\left[h(\bar{\boldsymbol{V}}_n) - h(\boldsymbol{\theta})\right] = n^{1/2}\left(\frac{I_n}{J_n} - \frac{\boldsymbol{\theta}_1}{\boldsymbol{\theta}_2}\right) \overset{w}{\to} \boldsymbol{Z},$$

where $\boldsymbol{Z} \sim N[0, \boldsymbol{d'(\theta)}\ \mathbf{\Sigma}\ \boldsymbol{d(\theta)}]$ *with* $\boldsymbol{\theta} = E[\boldsymbol{V}_j]$*,* $\mathbf{\Sigma} = Var[\boldsymbol{V}_j]$*,* $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$*,* $\boldsymbol{d(\theta)} = \partial h(\boldsymbol{\theta})/\partial(\boldsymbol{\theta})$ *and*

$$\bar{\boldsymbol{V}}_n = n^{-1}\sum_{j=1}^{n} \boldsymbol{V}_j = \begin{bmatrix} I_n \\ J_n \end{bmatrix},$$

where I_n *and* J_n *are empirical versions of* $\mathbb{E}[X_1|X_2 > t]$ *and* $\mathbb{E}[X_2|X_1 > t]$ *respectively.*

Proof. From above we know that, the conditional expectation can be defined as, $\mathbb{E}[X_1|X_2 > t] = \int_0^\infty \widehat{C}^*_n(\overline{F}(x), \overline{F}(t))\overline{F}(t)^{-1}dx$. Furthermore as we are assuming $\overline{F}(t)$ is constant the behavior of this conditional expectation depends only on the integration part of the expectation or $\int_0^\infty n^{-1}\sum_{j=1}^{n} \mathbb{I}[U_{1j} \le \overline{F}(x), U_{2j} \le \overline{F}(t)]\ dx$

Let us define,

$$\begin{aligned}
I_n &= \int_0^\infty n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{1j} \le \overline{F}(x), U_{2j} \le \overline{F}(t)]\ dx \\
&= n^{-1} \int_0^\infty \sum_{j=1}^{n} \mathbb{I}[U_{1j} \le \overline{F}(x), U_{2j} \le \overline{F}(t)]\ dx \\
&= n^{-1} \sum_{j=1}^{n} \int_0^\infty \mathbb{I}[U_{1j} \le \overline{F}(x)]\ \mathbb{I}[U_{2j} \le \overline{F}(t)]\ dx \\
&= n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{2j} \le \overline{F}(t)] \int_0^\infty \mathbb{I}[U_{1j} \le \overline{F}(x)]\ dx,\ \forall\ t, \qquad (41)
\end{aligned}$$

where $[\overline{F}(x), \overline{F}(t)] \in [0,1]^2$.

Furthermore, Serfling (2009) [pg.3] implies that $F(x) \ge y$ if and only if $x \ge F^{-1}(y)$ where

$$F^{-1}(y) = inf_{x \in \mathbb{R}}\ \{x : F(x) \ge y\}.$$

Therefore,

$$\begin{aligned}
U_{1j} \le \overline{F}(x) &\iff U_{1j} \le\ 1 - F(x) \\
&\iff 1 - U_{1j} \ge F(x) \\
&\iff x \le F^{-1}(1 - U_{1j})
\end{aligned}$$

and

$$\begin{aligned}
I_n &= n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{2j} \le \overline{F}(t)] \int_0^\infty \mathbb{I}[x \le F^{-1}(1 - U_{1j})]\ dx \\
&= n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{2j} \le \overline{F}(t)] \int_0^{F^{-1}(1-U_{1j})} dx \\
&= n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{2j} \le \overline{F}(t)]\ F^{-1}(1 - U_{1j}) \\
&= n^{-1} \sum_{j=1}^{n} B_j Y_j, \qquad (42)
\end{aligned}$$

where $B_j = \mathbb{I}[U_{2j} \le \overline{F}(t)]$ and $Y_j = F^{-1}(1 - U_{1j})$. In this case, $1 - U_{1j} \sim \textsc{Uniform}(0,1)$ so that $Y_j = F^{-1}(1 - U_{1j}) \sim\ F$ and that $B_j = \mathbb{I}[U_{2j} \le \overline{F}(t)] \sim \textsc{Bernoulli}(\pi)$ where $\pi = P[U_{2j} \le \overline{F}(t)]$.

In the similar fashion we can define,

$$\begin{aligned} J_n &= \int_0^\infty n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{2j} \le \overline{F}(x), U_{1j} \le \overline{F}(t)]\ dx \\ &= n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{1j} \le \overline{F}(t)]\ F^{-1}(1 - U_{2j}) \\ &= n^{-1} \sum_{j=1}^{n} C_j Z_j, \end{aligned} \tag{43}$$

where $Z_j = F^{-1}(1 - U_{2j}) \sim\ F$ and $C_j = \mathbb{I}[U_{1j} \le \overline{F}(t)] \sim \text{BERNOULLI}(\rho)$ where $\rho = P[U_{1j} \le \overline{F}(t)]$.

Now, let us define,

$$\boldsymbol{V}_j = \begin{bmatrix} B_j Y_j \\ C_j Z_j \end{bmatrix}$$

for $j \in \{1, 2, 3, .., n\}$ and it is important to note that,

$$\bar{\boldsymbol{V}}_n = n^{-1} \sum_{j=1}^{n} \boldsymbol{V}_j = \begin{bmatrix} I_n \\ J_n \end{bmatrix}$$

Now, let us define $\boldsymbol{\theta} = E[\boldsymbol{V}_j]$ and $\boldsymbol{\Sigma} = Var\ [\boldsymbol{V}_j]$. Suppose that $\boldsymbol{\Sigma}$ has all finite elements, then MULTIVARIATE CENTRAL LIMIT THEOREM [i.e Theorem 4.22 of Polansky (2011)] implies that, $n^{1/2}(\bar{\boldsymbol{V}}_n - \boldsymbol{\theta}) \xrightarrow{w} \boldsymbol{Z}$, where $\boldsymbol{Z} \sim \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$. Now, consider the function $h(\boldsymbol{u}) = h(u_1, u_2) = u_2^{-1} u_1$ and define

$$\boldsymbol{d}(\boldsymbol{u}) = \frac{\partial}{\partial \boldsymbol{u}} h(\boldsymbol{u}) = \begin{bmatrix} u_2^{-1} \\ -u_2^{-1} u_1 \end{bmatrix}$$

Therefore, If $\boldsymbol{\Sigma}$ is positive definite, $\boldsymbol{d(\theta)} \neq \boldsymbol{0}$ and $\boldsymbol{d(\theta)}$ is continuous in a neighborhood of $\boldsymbol{\theta}$, then by Theorem 6.5 of Polansky (2011) implies that,

$$n^{1/2} \left[h(\bar{\boldsymbol{V}}_n) - h(\boldsymbol{\theta}) \right] = n^{1/2} \left(\frac{I_n}{J_n} - \frac{\boldsymbol{\theta}_1}{\boldsymbol{\theta}_2} \right) \xrightarrow{w} \boldsymbol{Z},$$

where $\boldsymbol{Z} \sim N[0, \boldsymbol{d'(\theta)}\ \boldsymbol{\Sigma}\ \boldsymbol{d(\theta)}]$. This completes the proof. □

Remark 3. *In order to get the above result we have be more careful as U_{1j} and U_{2j} are not independent. Their dependence is defined through the Copula structure. In this case if we do not know the structure of $\boldsymbol{\Sigma}$, we can not get above proposition. That is why we need to make a heuristic assumption that we do know the structure of $\boldsymbol{\Sigma}$ and we can get this by using Monte Carlo method. In our case $\boldsymbol{\theta}_1 = \mathbb{E}[X_1 | X_2 > t]$ and $\boldsymbol{\theta}_2 = \mathbb{E}[X_2 | X_1 > t]$. In the previous chapters we know the exact values of these two parameters for Clayton and Gumbel Copulas. We can use those expressions as the representatives of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$.*

As from above we see the ratio of two empirical representations of conditional expectations weakly follows normal distribution with mean zero and some variance covariance matrix, we can create a hypothesis testing environment on it. In our case, testing null hypothesis (i.e $\mathscr{H}_0$, say) amounts to check that the variables X_1 and X_2 are dependent and the dependence structure of two conditional expectations $\mathbb{E}[X_1|X_2 > t]$ and $\mathbb{E}[X_2|X_1 > t]$ is *exchangeable*. Furthermore, we are more interested in $\eta_1(t)$.

The main objective of this section is to propose a test of hypothesis for a given t sufficiently large,

$$\mathscr{H}_0: \ \eta_1(t) = \frac{\mathbb{E}[X_1|X_2 > t]}{\mathbb{E}[X_1|X_2 > t]} = 1,$$

against the general alternative

$$\mathscr{H}_1: \ \eta_1(t) = \frac{\mathbb{E}[X_1|X_2 > t]}{\mathbb{E}[X_1|X_2 > t]} \neq 1,$$

From the previous proposition we know that, the proportion of empirical forms of conditional expectations weakly follows normal distribution.

Thus, the test statistic should be,

$$\mathscr{Z}_n = \left(\frac{n^{1/2}\left[\frac{I_n}{J_n} - 1\right]}{(\boldsymbol{d'(\theta)} \ \boldsymbol{\Sigma} \ \boldsymbol{d(\theta)})^{1/2}} \right) \overset{w}{\sim} N[0,1]$$

as $n \to \infty$ where $\boldsymbol{\theta} = E[\boldsymbol{V}_j]$, $\boldsymbol{\Sigma} = Var[\boldsymbol{V}_j]$, $\boldsymbol{d(\theta)} = \partial h(\boldsymbol{\theta})/\partial(\boldsymbol{\theta})$ and

$$\bar{\boldsymbol{V}}_n = n^{-1}\sum_{j=1}^{n} \boldsymbol{V}_j = \begin{bmatrix} I_n \\ J_n \end{bmatrix},$$

with I_n and J_n are empirical versions of $\mathbb{E}[X_1|X_2 > t]$ and $\mathbb{E}[X_2|X_1 > t]$ respectively.

5 Concluding remark

Our primary objective of study throughout this paper is tail *non-exchangeability*. In order to do so firstly, we take an *exchangeable* Copula. Then we do Khoudraji (1996) *non-exchangeable* transformation of our *exchangeable* Copula. Then we construct conditional tail expectation. As this integration in conditional expectation does not have any closed form solution, following Hua and Joe (2014) we use numerical approximation either by *Laplace Method* or *Watson's Lemma*. First we theoretically develop tail *non-exchangeability* by above two methods and derive the general conditions under which we are able to get some forms of approximation at the tail. Then we take Clayton Copula with *Pareto, Weibull* and *Exponential* margins. We use *Laplace Method* or

Watson's lemma based on the conditions satisfied by different margins. *Laplace Method* works only for Khoudraji (1996) *non-exchangeable* Clayton survival Copula with *Pareto* margin $1 < \beta < (1-\alpha_1)^{-1}$ otherwise, we are using *Watson's Lemma.* In the case of *non-exchangeable* survival Clayton Copula with *Weibull* and *Exponential* margins we can *only* use *Watson's Lemma.* From the simulation results we conclude that, KB4 with pareto and weibull margins give some non-exchangeability but, in the case of exponential margins the value of exchangeability is close to unity.

While deriving *conditional tail expectation* for $\mathbb{E}[X_1|X_2 = t]$ with *non-exchangeable* Copula in *propositions 3* and *4*, we do find two separate terms which are equally powerful at the tail. Thus, we have to do *Laplace Approximation* or *Watson's Lemma* two times for each expression. In survival Clayton Copula we do not see these kind of two separate terms because, one term is dominated by the other. In this paper we just do the *non-exchangeable* transformation of only survival Clayton Copula. If we do this numerical approximations with different Copulas, we might get some other results.

Finally, in the chapter of *Testing of Hypothesis* we develop a test statistic based on the empirical survival Copulas. As we know empirically we cannot show the limiting properties because, when we consider extreme values, we concentrate on fewer data points, as a result, we are losing information of the whole data set. Therefore, we only consider whole set of data while testing *non-exchangeability* empirically. We show that, our test statistic weakly follows standard normal distribution under certain conditions. As the data have dependence structure through empirical Copula, our proposition regarding test statistic works if we know the structure of variance-covariance matrix [i.e., $\mathbf{\Sigma}$] in *Proposition 5.*

Firstly, as in our paper we did not find the structure of variance-covariance matrix [i.e.$\mathbf{\Sigma}$], in future we can do further study on the structure of variance-covariance matrix. If we assume the structure is known, we can do *Monte Carlo* simulation in order to get this. If we assume this structure is unknown we cannot use our proposition and the proof is not valid at all. In this case we can use *boot-strap* method in order to estimate $\mathbf{\Sigma}$. Again, in the chapter of *Testing of Hypothesis* we are only able to estimate the empirical test for $\mathbb{E}[X_1|X_2 > t]$ not $\mathbb{E}[X_1|X_2 = t]$ because, for later finding out an empirical test statistics is way more harder. Secondly, in this paper we only try to get a mathematical derivation of *non-exchangeability* but we do not check how the degree of *dependence* in the tail affects the measure of *tail non-exchangeability.* In this paper, we only consider *non-exchangeability* in the presence of *positive* dependence, we never consider *non-exchangeability* with *negative* dependence. We do not know how this looks like. Probably, this is going to be a very good future research. Finally, in our paper we define, the random vector (X_1, X_2) is said to be *tail exchangeable* of Type I if the following condition holds:

$$\text{Condition I:} \quad \lim_{t\to\infty} \eta_1(t) := \lim_{t\to\infty} \frac{\mathbb{E}[X_1|X_2 > t]}{\mathbb{E}[X_2|X_1 > t]} = 1,$$

and is *tail exchangeable* of Type II if the following condition holds:

$$\text{Condition II:} \quad \lim_{t\to\infty} \eta_2(t) := \lim_{t\to\infty} \frac{\mathbb{E}[X_1|X_2 = t]}{\mathbb{E}[X_2|X_1 = t]} = 1.$$

But, we do not know anything regarding the relationship between these two conditions. We cannot say, if one system is *non-exchangeable* under *Condition I* then, it is *non-exchangeable* under *Condition II* or the other way. To verify the relationship between them, some *regularity* conditions such as *stochastically increasing* may be needed.

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