Estimation of Geometric Brownian Motion Model with a t-Distribution Based Particle Filter

Nkemnole, E. B.[1*]. and Abass O[2**].
[*]Department of Mathematics, University of Lagos, Nigeria.
enkemnole@unilag.edu.ng[1]
[**]Department of Computer Sciences, University of Lagos, Nigeria.
olabass@unilag.edu.ng[2]

Abstract
Geometric Brownian motion (GBM) model basically suggests that the distribution of asset returns is normal or lognormal. But, many empirical studies have revealed that return distributions are usually not normal. These studies, time and again, discover evidence of non-normality, such as heavy tails, excess kurtosis, etc. This paper recommends the GBM model based upon the t-distribution to approximate the return distributions of assets, and compares the distribution with normal distribution. In evaluating the recommended GBM level of precision, the model parameters are estimated. A Sequential Monte Carlo (SMC) technique based on t-distribution is developed to estimate the random effects and parameters for the extended model. The SMC or particle filter based upon the t-distribution for the GBM model, which involves randomness, volatility and drift, can precisely capture the aforementioned statistical characteristics of return distributions and can predict the random changes or fluctuation in stock prices. Consequently, it provides an approximate solution to non-Gaussian estimation problem. Through stochastic simulations and the accuracy of the models which was proven by the lower value of the Mean Absolute Percentage Error (MAPE), our analysis shows that the GBM model based on student-t is empirically more successful than the normal distribution.

Keywords: Geometric Brownian motion, Student-t distribution, normal distribution, drift, volatility, particle filter.

1.0 Introduction

Most of the models utilized in the description of financial time series are written in terms of a continuous time diffusion S_t that satisfies the stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where $dB_t \sim N(0, dt)$ is the increment to Brownian motion process , σS_t and μS_t denote the volatility and drift function, respectively. This class of parametric model has been extensively used to portray the dynamics of financial variables, including stock prices, interest rates, and exchange rates. A stochastic process S_t is said to follow a Geometric Brownian Motion (GBM) if it satisfies the above stochastic differential equation.

The GBM is one of the most popular stochastic processes, and without doubt, an effective instrument in modelling and predicting the random changes in stock prices that evolves over time. It is essentially useful for this index price study because the process in question assumes that percentage changes are independent and identically distributed over equal and non-overlapping time length (Luenberger, 1995; Ross, 2000). The GBM assumes that the instantaneously expected rate of return is constant. Hence, the constant instantaneous expected drift assumption of the standard Brownian process is substituted with the constant expected rate of return in the geometric Brownian process (Hull, 2000). The GBM model usually assumes that the distribution of asset returns is normal or lognormal. However, financial data often have heavier tails than can be captured by the standard GBM model. As such, there is need to use non-normal distributions to better model and to deal with the heavy tails (Carol, 2004; Tan 2005, Tan and Tokinaga 2006, Tan 2007a, Tan and Chu 2012). However, intractable likelihood functions for SDEs make inference challenging, necessitating the resort to simulation-based techniques to estimate and maximize the likelihood function. The sequential Monte Carlo methods or the particle filter have allowed for the accurate evaluation of likelihoods at fixed parameter values.

This work is structured thus: Section One introduces the work. Section Two reviews the relevant literature; Section Three gives a synopsis of the standard version of the GBM model, and extends the model by modeling the return distributions of assets using a student-t distribution and gives a brief analysis of the SMC procedure and its implementation. Section Four presents the simulation results and application to the real data that confirms the proposed method based on student-t and normal. Finally, Section Five concludes the work.

2.0 Review of Relevant Literature

A lot of literature has been generated in the area of GBM as a model for stock prices. Some scholars have tried extending and, hence improving the standard GBM model. Duplantier (2005) refers to Louis Bachelier who mentioned in his PhD thesis in 1900 that the stock price dynamics follows Brownian Motion. The process he applied can produce shares that allowed both negative security prices and option prices that exceeded the price of the underlying asset. Osborne (1959) refined the Bachelier model by employing the stochastic exponential of the Brownian motion to model stock price. Samuelson (1965) extended the GBM by using the discount rate in pricing. For him, the return rates, instead of the stock prices, follow Geometric Brownian Motion (Piasecki, 2006). Some scholars represent rare events by jumps and introduce a model of jump diffusion (see Merton (1976) and Kou (2002)). Others presented a more realistic stochastic process for the underlying process (e.g., stock price) by bringing in a stochastic process for the volatility, i.e., with the variance of the stock return as random {for example Hull and White (1987), Stein and Stein (1991) and Heston (1993)}.
Thao (2006) tried replacing the Brownian motions with fractional Brownian motions in the diffusion model. Sattayatham et al. (2007) improved on Thao's results by adding a Poisson jump into the model.

GBM has been expansively used as a model for the stock prices, commodity prices and growth in demand for products and services and real options analysis (Nembhard et al., 2002; Thorsen, 1998; Benninga and Tolkowsky, 2000). It has also been used for representing future demand in capacity studies (Whitt, 1981; Lieberman, 1989; Ryan, 2006). On the whole, its acceptance was motivated from the assumption that random changes over time follow a GBM process (Marathe and Ryan, 2005). On the other hand, some scholars have raised relevant questions concerning the accuracy of the GBM (for example Watteel-Sprague (2000), Ross (1999) Thorsen (1998), Marathe and Ryan (2005)).

Works on modeling return distributions of financial assets also exist. The most used are the normal, the lognormal and the non-Gaussian stable distributions. Other types of distributions, such as the Student t, the skewed Student t, the generalized t, the Generalized Error Distribution (GED), the skewed GED, and mixture distribution of Gaussian distributions have been applied. The normal distribution is one of the most usually applied distributions. It was extensively used in the 1700's; in 1800, Karl Gauss successfully applied it to astronomical data analysis. It became known as the Gaussian distribution. Empirical analyses, from the late 1960s, were not successful in supporting the normal assumption on estimating the return distribution of real financial data. Mandelbrot (1963) affirmed that while financial prices or its logarithm following a Brownian motion is mathematically convenient; it is hard to fit the real financial data with this assumption. Fama (1965) analyzed equilibrium asset pricing and noted that the daily return distribution follows a non-Gaussian distribution. Both Mandelbrot (1963) and Fama (1965) pointed out that excess kurtosis and heavy tails exist in real financial data.

Hsu, et al. (1974) and Hagerman (1978) showed from their studies that return distributions are non-normal. Bollerslev (1987) found leptokurtosis in monthly Standard & Poor's 500 Index returns. Kariya, et al. (1995) and Nagahara (1996) revealed that the return distributions of Japanese stocks are fat-tailed and skewed. Kitagawa, Sato and Nagahara (1999) found that daily or weekly return distributions are not normal but fat-tailed and skewed according to observed financial data. Harvey and Siddique (2000), as

well as Premaratne and Bera (2000) confirmed the asymmetry of return distribution exists in real business data. Gerig, Vicente and Fuentes (2009) presented a model that explained the shape and scaling of the distribution of intraday stock price fluctuations and verified the model by using a large database made up of several stocks traded on the London Stock Exchange. Their findings showed that the return distributions for these stocks are non-Gaussian, similar in shape and appear to be stable over intraday time scales.
Theodossiou (1998) advocated using a skewed generalized t distribution, which embraces the Student t and skewed Student t, to model return distributions. Furthermore, Theodossiou (2000) pointed out that a skewed GED fits the financial data well, while the asymmetry and excess kurtosis are observed in the financial data.

In this paper, we extended our investigations by the introduction of a GBM model based upon the t-distribution based particle filter to approximate the return distributions of assets and compare the distribution with normal distribution. In evaluating the proposed GBMion level of precision, the model parameters are estimated. A Sequential Monte Carlo or particle filter technique based on student-t distribution is developed to estimate the parameters for the extended model. The ensuing models are applied to modeling the closing stock price of 5 firms of the Nigerian Stock Exchange.

3.0 Methodology

3.1 The Geometric Brownian Motion (GBM) Model

GBM is the stochastic process used in the Black-Scholes methodology to model the evolution of prices in time. As in a typical structural model, let us consider a firm with its value of the asset V_t following a geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma_v V_t dB_t \tag{1}$$

μ and σ are drift and volatility parameters to be estimated. The drift informs us on the average rate at which a value increases in a stochastic process while the volatility is the constant characteristic of the stock prices that tells us the measure of the fluctuations of the stock prices. Relatively high volatility means that the stock price varies continuously within relatively large interval. dt is an infinitely approaching 0 time difference between time points t and $t-1$ and the last term involves random $dB_t \sim N(0, dt)$ increment to Brownian motion process. The right hand side term $\mu V_t d_t$ controls the "trend" of this trajectory and the term $\sigma V_t dB_t$ controls the random noise in the trajectory. Nevertheless, one of the foremost challenges in applying this model to financial market data is the fact that the underlying asset value process is unobservable.

Applying the Ito's formula (see Lamberton and Lapeyre 1997) on equation (1) with $F(S) = \log S$, we obtain:

$$\ln S_t = \left[\mu - \frac{1}{2}\sigma^2\right] t + \sigma B_t \tag{2}$$

The stochastic process, as characterized by equation (2), indicates that $\ln S$ is normally distributed. Equivalently, S is lognormally distributed.

Taking the exponential of both side and inserting the initial condition S_0, we obtain the solution. The analytical solution of this GBM is given by:

$$S_t = S_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \quad (3)$$

This stochastic differential equation is principally significant in modeling of many asset classes. Equation (3) is the asset price model that is able to predict an asset price at specific time t. we can represent GBM solution in the form:

$$S_t = S_0 e^{X_t} \quad (4)$$

where $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$

3.2 GBM Model Maximal Likelihood Estimation

The parameters μ and σ can be estimated using historical data for stock price, bearing in mind also that the time difference for data with monthly frequency is $\Delta t = \dfrac{1}{12}$.

As Damiano Brigo et al., (2007) noted, the parameters that need to be optimised are $\theta = (\mu, \sigma)$ for the GBM.

The likelihood function is denoted as:

$$\begin{aligned} L(\theta) &= f_\theta(y_{t_i}, y_{t_2}, \cdots, y_{t_n}) \\ &= \prod_{i=1}^{n} f_\theta(y_{t_i}) \\ &= \prod_{i=1}^{n} f(y_{t_i} | \theta). \end{aligned}$$

Here f_θ is the probability density function, $y_{t_i}, y_{t_2}, \cdots, y_{t_n}$ are the log returns.

Let $\theta = (\mu, \sigma)$, then the probability density function f_θ is:

$$f_\theta(y_{t_i}) = \frac{1}{x_{t_i}\sigma\sqrt{2\pi t}} \exp \frac{\left(\left(\dfrac{y_{t_i}}{y_{t_o}}\right) - \left(\mu - \dfrac{1}{2}\sigma^2\right)t\right)^2}{2\sigma^2 t}$$

The likelihood function is maximised to get the optimal estimators $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$. The natural logarithm of the likelihood function is differentiated in terms of μ and σ then equated to zero to give equations :

$$\hat{y} = \left(\hat{\mu} - \frac{1}{2}\hat{\sigma}^2\right)\Delta t \quad (5)$$

$$\hat{s} = \hat{\sigma}^2 \Delta t \quad (6)$$

where

$$\hat{y} = \frac{\sum_{i=1}^{n} y_{t_i}}{n}$$

$$\hat{v} = \frac{\sum_{i=1}^{n} (y_{t_i} - \hat{y})^2}{n}$$

Determining $\hat{y}$ and $\hat{s}$, the corresponding MLE of μ and σ are: $\hat{\sigma}^2 = \frac{\hat{s}}{\Delta t}$ and $\hat{\mu} = \frac{1}{2}\hat{\sigma}^2 + \frac{\hat{y}}{\Delta t}$.

3.3 The GBM with t-distribution

Asset return distributions are frequently presumed to follow a normal or lognormal distribution. It also can follow GBM based upon the Gaussian process. However, many empirical studies have shown that return distributions are usually not normal. They often find evidence of non-normality, such as heavy tails, excess kurtosis, finite moments, etc. One class of fat-tailed distributions with the potential to give a better approximation to the distribution of stock returns is the t-distribution.

An extension of the version of the GBM model, wherein it is assumed that the random noise process, dB_t is a student-t distribution is considered. The proposed student-t distribution with degrees of freedom, v, for the last term, dB_t , effects a change in the equation:

$$dV_t = \mu V_t dt + \sigma V_t dB_t \qquad dB_t \sim t_v,\ t = 1,\cdots,n. \tag{7}$$

The distribution of the error term for this specification according to Shimada and Tsukuda (2005) takes the form:

$$f_\theta(y_t) = \frac{1}{\sqrt{\pi(v-2)}} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} e^{-\frac{y_t}{2}} \left(1 + \frac{y_t^2 e^{-x_t}}{v-2}\right)^{-\frac{v+1}{2}}$$

3.4 Sequential Monte Carlo (SMC) Algorithm Analysis

The SMC, otherwise known as particle filter algorithm (Gordon, Salmond, and Smith 1993) in sequential estimation on hidden asset value and model parameters estimation are applied under the GBM model. This method applies the concept of sampling-important-resampling (SIR) (Rubin 1987). One of the key challenges in applying structural models to financial market data is the fact that the underlying asset value process is unobservable. Furthermore, at each time *t*, market values of stock are known only up to the time *t*, which means that the information needs to be updated sequentially. In this section, with known model parameters, we apply the particle filter algorithm to update the information about the underlying asset value process recursively from the observed times series. By running the filtering algorithm, the conditional distribution of the underlying asset value is approximated and recursively updated, given observed prices.

3.4.1 Particle filter Algorithm

Assuming that we have at time t weighted particles $\{f_t^{(i)}, w_t^{(i)}\}$ drawn from $f(x_t \mid y_t)$, $f_t^{(i)}$ is a set of particle filter with associated weight $w_t^{(i)}$. This is seen as an empirical approximation for the density made up of point masses,

$$f(x_t \mid y_t) \approx \sum_{i=1}^{M} w_t^{(i)} \delta(x_t - f_t^{(i)}). \tag{8}$$

Kitagawa and Sato (2001) and Kitagawa (1996) offer an algorithm for filtering thus:

1. For $i = 1,\cdots,N$, generate a random number $f_0^{(i)} \sim p(x_0)$
2. Repeat the following steps for $t = 1,\cdots,T$.
   a. For $i = 1,\cdots,N$, generate a random number $w_t^{(i)} \sim q(w)$.

b. For $i=1,\cdots,N$, Compute $p_t^{(i)} = F(f_{t-1}^{(i)}, w_t^{(i)})$

c. For $i=1,\cdots,N$, Compute $w_t^{(i)} = p(y_t \mid p_t^{(i)})$

d. Generate $f_t^{(i)}, i=1,\cdots N$ by resampling $p_t^{(i)},\cdots,p_t^{(N)}$

3. This Monte Carlo filer returns

$\{f_t^{(i)}, i=1,\cdots,N, t=1,\cdots,m\}$ so that $\sum_{i=1}^{N}\frac{1}{N}\delta(x_t - f_t^{(i)}) \approx f(x_t \mid Y_t)$.

3.5 Estimation Procedure

In this section, with known model parameters, we apply the particle filter algorithm based on t-distribution to update the information about the underlying asset value process recursively from the observed times series of stock prices.

With known parameters $\Theta = \{\mu, \sigma\}$, we observe the time series of stock prices $S = \{S_t; t=1,\cdots T\}$ and have the hidden asset process to be estimated $V = \{V_t; t=1,\cdots T\}$. The algorithm is as follows:

The algorithm for the filtering shows an extension of Godsill et al. (2004) and Kim and Stoffer (2008). From here M samples from $f(V_t, | S_t)$ for each t were obtained.

i) Given $\{V_t^{(i)}\}_{i=1}^{m}$, draw $\{V_{t+1}^{(*i)}\}$ from $p_{t+1}(V_{t+1} \,|\, V_t^{(i)}) \sim t_v$ for $i=1,\cdots,M$

Generate $f_0^{(i)} \sim t_v$

ii) Generate a random number $w_t^{(i)} \sim t_v$,

iii) Compute $p_t^{(i)} = \mu f_{t-1}^{(i)} + \sigma w_t^{(i)}$

a.Compute $w_t^{(i)} = p(s_t \mid p_t^{(i)},) \propto e^{-\frac{s_t}{2}}\left(1+\frac{s_t^2 e^{-s_t}}{v-2}\right)^{-\frac{v+1}{2}}$

b. Generate $f_t^{(i)}$ by resampling with weights, $w_t^{(j)}$

Resample from $\{(V_{t+1}^{(*1)}), (V_{t+1}^{(*2)}),\cdots,(V_{t+1}^{(*M)})$ with probability proportional to $w_t^{(j)}$

As averred by Lawrence et al., (2009), we have three measurement of forecasting model which involve time period, t. The measurements are number of period forecast, n, actual value in time period at time, t, Y_t and forecast value at time period t, F_t. The mean absolute percentage error (MAPE) happens to be the most widely used to evaluate the forecasting method that considers the effect of the magnitude of the actual values. It can be calculated as follows:

$$MAPE = \frac{1}{n}\sum\left|\frac{Y_t - F_t}{Y_t}\right|$$

4. Empirical Results and Discussion

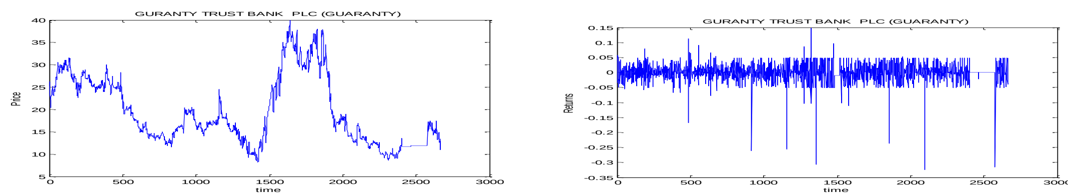
4.1 Data

We apply the above-described methodology to model the stock prices of 5 firms of the NSE. Each from five different sectors; namely the banking sector (GTB), from Oil & Gas sector (OANDO), from

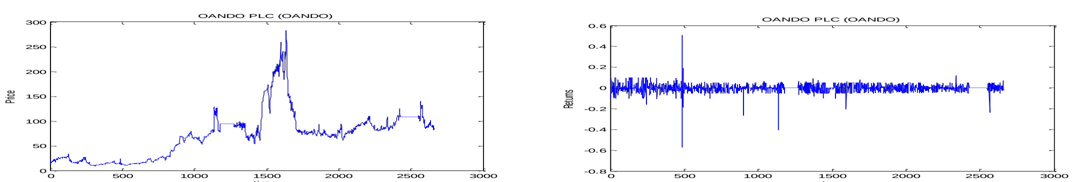
Construction sector (Juius Berger), from Health care sector (Glaxo Smith) and from Industrial goods sector {Chemical & Apllied Product (CAP)} over the period January 2, 2010 to December 31, 2014.

The data series is transformed into daily log returns series so that we obtain stationary series. Descriptive statistical summary is obtained to view the data for the daily stock prices and returns of all the indices.
Table 1 reported the descriptive statistics for all the five selected indices. The value of the kurtosis for returns is high and greater than three. This shows that the distribution is leptokurtic, that is, it is fat tailed and that the returns display financial characteristics of volatility clustering and leptokurtosis. The skewness for both prices and returns is positive, showing that the distribution has a long right tail. The high values of kurtosis for the returns indicate that extreme price changes occurred frequently during the sampling period. The positive skewness and kurtosis indicated non-normal series. With reference to Jarque -Bera statistics, the stock index series is non-normal at the confidence interval of 99% since probability is 0.000000 which is less than 0.01.

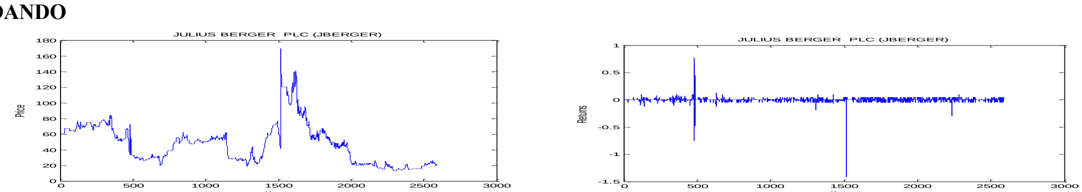
Figs. 1 - 5 shows the plot of each of the five firms stock prices



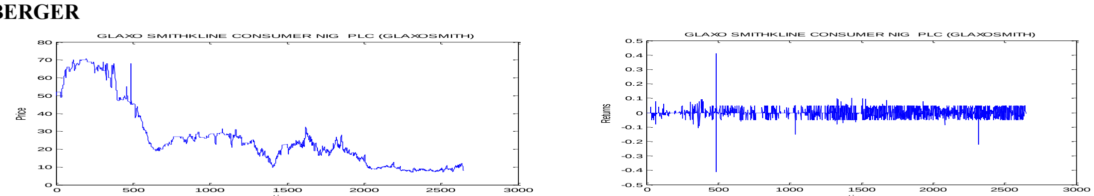
GTB



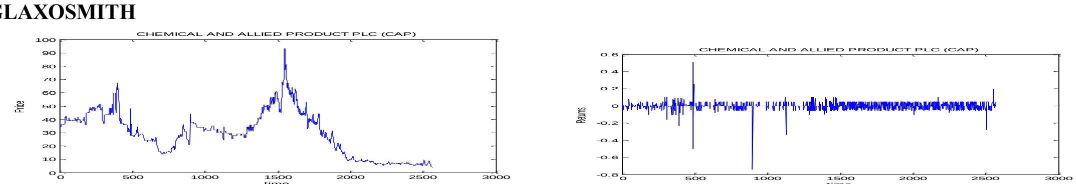
OANDO



JBERGER



GLAXOSMITH



CAP

Table 1 Descriptive statistical summary for the daily stock prices

	Prices					Returns				
Index	**Mean**	**Std. Dev.**	**Skewness**	**Kurtosis**	**Jarque-Bera**	**Mean**	**Std. Dev.**	**Skewness**	**Kurtosis**	**Jarque-Bera**
GTB	40.3754	18.4573	0.6718	3.3487	194.387 (0.0000)	0.0030950	0.0278	-2.4739	29.4686	684.195 (0.0000)
OANDO	70.9741	47.4833	1.0297	4.8445	202.394 (0.0000)	-0.00062810	0.0352	-1.3034	54.4961	312.573 (0.0000)
JBERGER	46.8880	25.5413	0.9931	4.2569	207.281 (0.0000)	0.0041886	0.0458	-1.5541	42.6513	497.263 (0.0000)
GLAXOSMITH	76.9873	48.2532	1.1542	4.2342	213.237 (0.0000)	0.0070399	0.0300	-0.1911	30.2326	513.240 (0.0000)
CAP	49.3441	27.3775	0.2783	3.6845	195.142 (0.0000)	0.000875734	0.0367	-3.6781	20.7944	795.142 (0.0000)

The stock prices of each of the five firms of the NSE of the year 2010 to 2014 were used to derive the drift and volatility. Table 2 shows the observed values:

Table 2: Drift and Volatility values of stock prices

Index	Drift (μ)	Volatility (σ)
GTB	0.072	0.2816
OANDO	0.0485	0.2794
JBERGER	0.0514	0.2723
GLAXOSMITH	0.0354	0.2837
CAP	0.0624	0.2808

These two parameters were then used to create the Geometric Brownian path for both the GBM normal and student t distribution of each of the five firms of the NSE. We compute the MLEs and the corresponding log-likelihood for each stock. The results presented on Table 3 summarize the estimated parameters for both the GBM normal and student t distribution.

Table 3: Estimated parameters of the GBM normal and student t distribution

	GBM's-normal				GBM'student t			
Index	μ	σ	**Log-lik.**	**AIC**	μ	σ	**Log-lik.**	**AIC**
GTB	0.45	0.37	-2797	4359	0.34	1.09	-2740	4248
OANDO	0.12	0.33	-2341	4686	0.30	1.27	-2331	4568
JBERGER	0.10	0.40	-2149	4302	0.37	1.03	-2135	4176
GLAXOSMITH	0.23	0.36	-2344	4255	0.41	1.04	-1234	4234
CAP	0.34	0.44	-2783	4684	0.32	1.23	-1345	4221

The log-likelihood for the GBM student-t distribution model is higher than the model for the GBM normal for each of the five stock series. Appraising the two models based on the Akaike Information Criterion (AIC), the GBM t distribution model outperforms the GBM normal model for each of these five stocks.

The GBM normal and student t based particle filter method are then run on the simulated prices process, and the average mean absolute percentage error (MAPE) is calculated. Both models estimate the volatility process of each of the five firms' stock prices using known parameters. Table 4 shows the observed values.

Table 4. Evaluation Statistic-Distribution comparison on technique based on the Normal and Student's-t

Models	Mean Absolute Percentage Deviation
GBM'S Normal	0.0967 = 9.67%
GBM'Student t	0.0652 = 6.52%

Graphically, for a single run, the estimation results obtained from running these two models are shown below: Each of the figures presented below contains a GBM's normal paths (in blue), GBM's student-t paths (in red) and the actual volatility price curve of each of the five firms stock prices (in black).

Figure 6 – 10 shows the plot of volatility estimation for each of the five firms stock prices

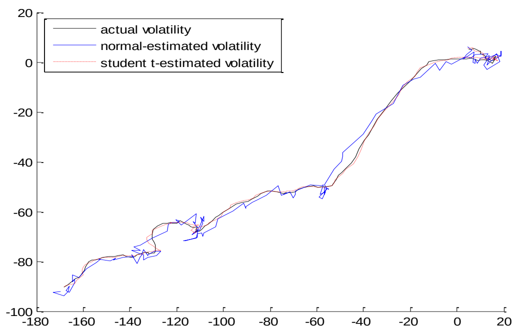


Fig. 6: Brownian Path for GTB

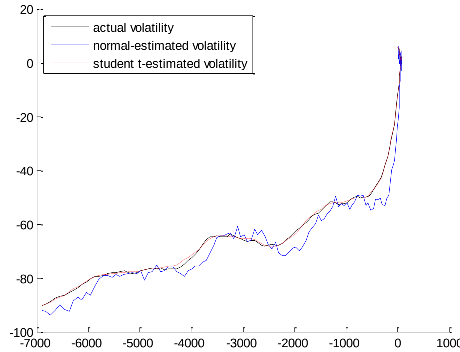


Fig. 7: Brownian Path for OANDO

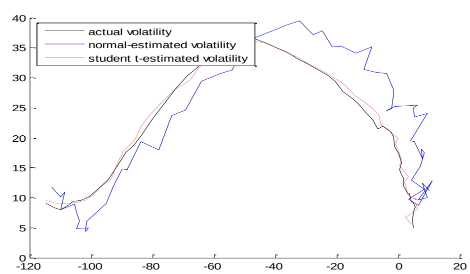


Fig. 8: Brownian Path for JBERGER

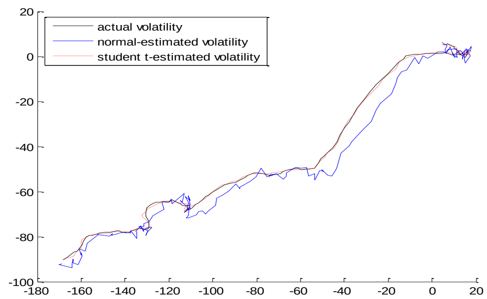


Fig. 9: Brownian Path for GLAXOSMITH

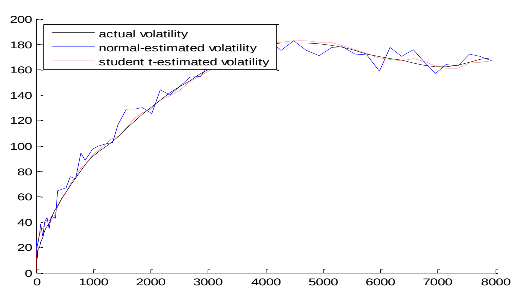


Fig. 10: Brownian Path for CAP

Figure 6-10 shows the volatility estimation. The plots show that the GBM student-t based particle filter estimate (red line) and the actual volatility (black line) lie close together in comparison to the GBM normal estimate (blue line).

5. Conclusion

This work presented an extension of the random noise process, dB_t in the GBM model from normal to student-t distribution. The goal is to compare and contrast the two models in five different stock market periods in terms of their predictability of such exceptional movements in the NSE market. We have revealed that the GBM Student's-t distribution performed expressively better at estimating both the volatility and the parameters of the model than the GBM normal. A particle filter technique based on student-t distribution is developed to estimate the random effects and parameters for the extended model. The functions provided by MATLAB enabled us to develop the techniques based on the student-t GBM model and a strategy for fitting the model. This change to the proposed model allows for a more robust fit, giving us a new tool to explore the tail fit. The student-t GBM model was compared and evaluated with the normal GBM model. The experimental outcome of the simulation and real data analyses confirms the viability of the proposed method. The evaluation statistics are calculated to compare the fit of distributions. Student-t Geometric Brownian motion is highly accurate than the normal GBM as proved by the MAPE value which is lower than 10%.The results, based on daily stock prices reveal that the student-t GBM is comparable to the normal GBM model but empirically more successful.

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