Bernoulli's *Ars Conjectandi* and Its Pedagogical Implications

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1. State of Probability Prior to Bernoulli

Although the Italian scholar Gerolamo Cardano's (1501-1576) 1525 book ***Liber de ludo aleae*** (Book on Games of Chance), which was published posthumously in 1663, can be considered a harbinger of mathematical theory of probability (Dunham 1990), the date historians accept as the beginning of modern probability theory is 1654, when two of the most prolific and resourceful mathematicians of the time, Blaise Pascal (1623-1662) and Pierre de Fermat (1601-1665), began a correspondence addressing the ***problem of points***: suppose a game of chance is to be played several times by two players each of whom have equal chances of winning and each of whom contribute equally to the pot. Each time a player wins the game, he gets one point. The first player to win a certain number of points collects the entire prize. Now suppose that the game is interrupted by external circumstances before either player has achieved the required number of points. How does one then divide the pot fairly?

The Fermat-Pascal correspondence aroused the interest of several scholars in the topic. Among them was the Dutch mathematician Christiaan Huygens (1629-1695), whose reflections on the issue culminated in a 1657 treatise titled ***De ratiociniis in aleae ludo*** (Calculations in Games of Chance) (Dunham 1990). This tome is considered to be the first comprehensive and authoritative text on probabilities involved in games of chance.

It is important to note that at this juncture in history, there was no mention of the word "probability" and no effort was being made for abstract generalizations – this new theory was being developed solely in terms of specific games of chance.

2. Jacob Bernoulli and Probability

Jacob Bernoulli (1654-1705), a well-known member of the celebrated Swiss family from Basel that dominated mathematics for nearly two centuries, produced much of his probability-related results between 1684 and 1689, and recorded them in his diary ***Meditationes*** (Shafer 2006).

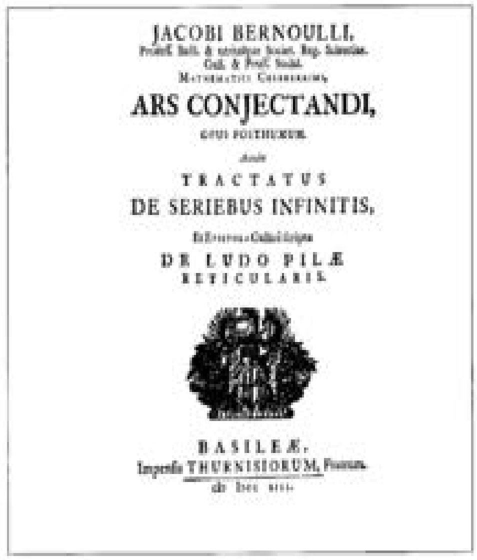
Jacob Bernoulli

It is clear from the ***Meditationes*** that Bernoulli was intrigued by combinatorial and probabilistic problems, and had thoroughly studied the works of his predecessors, namely those of Gerolamo Cardano, Blaise Pascal, Pierre de Fermat, and Christiaan Huygens (Collani 2006). In fact, this had led to his producing a number of treaties related to the topic such as ***Parallelismus ratiocinii logici et algebraici*** in 1685 and ***De Arte Combinatoria Oratio Inauguralis*** in 1692 while still working on his ***Ars Conjectandi*** (Maĭstrov 1974, Shafer 2006).

3. Ars Conjectandi

Ars Conjectandi (The Art of Conjecturing), Jacob Bernoulli's seminal work, was written between 1684 and 1689. The book was finally published in 1713, eight years after his death, by his nephew Nikolaus Bernoulli (1687-1759).

Presumably, his brother Johann (1667-1748) would have been better suited for this task, but according to most accounts, the two brothers did not have a very cordial relationship and spent their lives in professional rivalry and even in open animosity. That is probably why the responsibility of publishing the book was undertaken by Nikolaus. Moreover, Jacob's practice of applying the "art of conjecturing" to practical matters must have been well known and well appreciated by Nikolaus, who had already taken parts verbatim out of ***Ars Conjectandi*** for his own dissertation entitled ***De Usu Artis Conjectandi in Jure*** published in 1709 (Collani 2006). To his credit, Nikolaus later on also edited Jacob Bernoulli's complete works and supplemented it with results taken from ***Meditationes***.



JACOBI BERNOULLI,

ARS CONJECTANDI,

TRACTATUS

DE SERIEBUS INFINITIS,

DE LUDO PILÆ

RETICULARIS.

BASILEÆ,

Impensis THURNISIORUM, Fratrum.

Front Cover of Ars Conjectandi

Why this title? There are two possible reasons.

As for the first possibility, let us keep in mind that, Bernoulli wished to demonstrate through his book that a theory that would help one compute the likelihood of chance occurrences would also serve as a rigorous process of reasoning in cases involving inadequate information and would be a far-reaching and consistent method of analysis in rather complex and intricate situations (Sylla 1998). Meanwhile, in 1662, a book titled ***La Logique ou l'Art de Penser*** was published anonymously in Paris and immediately became a popular and prominent text on logic. The Latin title of this book was ***Ars cogitandi***. It consisted of four parts, last of which dealt with decision-making under uncertainty, much the same way as in a game of chance. Bernoulli certainly possessed a copy or at least knew the contents of ***Ars cogitandi***. Thus, he might have chosen the title to imply that his work was, in part, inspired by ***Ars cogitandi*** and was, in fact, an amendment of it (Collani 2006).

The second possibility is a bit more philosophical. In the 1600s, there was considerable interest in developing ***mathesis universalis*** (from *mathesis* meaning science in Greek and *universalis* meaning universal in Latin), a hypothetical universal science modeled on mathematics, introduced by René Descartes (1596-1650) and Gottfried Wilhelm Leibniz (1646-1716). Descartes' definition of *mathesis universalis* was given in Rule IV of the ***Rules for the Direction of the Mind***, written about 1628: there is need of a method for finding out the truth. The chief notion of *mathesis universalis* was ***ars inveniendi*** (art of invention): ascertaining truth through the use of mathematics (Marciszewski 1984). Thus, the title ***Ars Conjectandi*** might have been chosen to show Bernoulli's interest in the concept of *ars inveniendi* (Collani 2006).

In Bernoulli's own words, the term *ars conjectandi* is defined in Chapter II of Part IV of the book as:

> The art of measuring, as precisely as possible, probabilities of things, with the goal that we would be able always to choose or follow in our judgments and actions that course, which will have been determined to be better, more satisfactory, safer or more advantageous.

Here let us emphasize a very significant development, namely, the use of the term "probability" for the first time in a mathematical setting. Moreover, the goal of the book is stated as "measuring … probabilities of things…" and not necessarily only outcomes of games of chance. Thus, now we have a more modern, more abstract and more general approach to the theory and a complete liberation from games of chance.

As for its contents, let us see how they were identified by Francis Maseres in his 1795 English translation:

> The doctrine of Permutations and Combinations, Being an Essential and Fundamental Part of the Doctrine of Chances; As is it delivered by Mr. James Bernoulli, in his excellent Treatise on the Doctrine of Chances, intitled [sic], Ars Conjectandi… (Alexanderson 2013)

Ars Conjectandi was divided into four parts (Schneider 2006).

The first part was an in-depth expository on Huygens' ***De ratiociniis in aleae ludo***. In this section, Bernoulli provided solutions to the five problems Huygens posed at the end of his work. The fifth of these problems is what is now known as the ***Gambler's Ruin*** problem. In John Arbuthnot's (1667-1735) 1692 translation of ***De ratiociniis in aleae ludo*** with the English title ***Of the Laws of Chance*** the problem was given as follows:

> A and B taking 12 counters, each play with three dice after this manner, that if 11 comes up A shall give one counter to B; but if 14 comes up B shall give one to A, and that he shall gain who first has all the counters. A's hazard to B is 244140625 to 282429536481 (Arbuthnot 1692).

However, possibly the most notable concept expounded in this section was that of the ***expected value*** (Dunham 1994).

Huygens had somewhat described the concept as the weighted average of all possible outcomes of an event:

$$\frac{p_0a_0 + p_1a_1 + \cdots + p_na_n}{p_0 + p_1 + \cdots + p_n}$$

where for $j = 0, \ldots n$, p_j are the probabilities of attaining the values a_j. Assuming that p_j are the probabilities of disjoint outcomes, Bernoulli derived the formula

$$E = p_0a_0 + p_1a_1 + \cdots + p_na_n$$

for the expected value of a finite distribution.

The first part concluded with the introduction of a distribution now known as the ***Bernoulli distribution***. Using this concept, Bernoulli investigated the probability of achieving a certain number of successes from a number of binary events given that the probability of success in each event was the same (Dunham 1994). Of course, these are what we refer to as the ***Bernoulli trials*** today. Bernoulli showed by mathematical induction that, under these conditions, if a was the number of favorable outcomes in each event, b the number of total outcomes in each event, x the desired number of successful outcomes, and n the number of events, the probability P of at least x successes would be

$$P = \sum_{j=0}^{n-x} \binom{n}{x+j} \left(\frac{a}{b}\right)^{x+j} \left(\frac{b-a}{b}\right)^{n-x-j}$$

i.e. derived the formula for the cumulative distribution function of the binomial distribution.

In the second part of the book, Bernoulli elaborated on combinatorics. He addressed problems that dealt with the classification of combinatorial counting problems by considering different ways of putting balls into urns. He noted that different setups arose depending on whether the balls were labeled or unlabeled, whether the urns were labeled or unlabeled, and whether each urn could contain any number of balls, at most one ball or at least one ball, leading to $2 \times 2 \times 3 = 12$ cases. This systematic classification is called the ***twelvefold way*** in modern combinatorics.

In this section of the book, themes somewhat unconnected to probability were also covered. For example, here, Bernoulli also gave the first non-inductive proof of the binomial expansion for integer exponent using combinatorial arguments. But, the prime example of a non-probabilistic topic covered in this section was the derivation of numbers, now called the ***Bernoulli numbers***, which turned out to possess numerous applications in number theory (Schneider 2006). Indeed, Bernoulli numbers are considered to be one of Jacob Bernoulli's more notable achievements.

Let for $m, n \geq 0$

$$f(m,n) = \sum_{k=1}^{n} k^m$$

As is well known, $f(m,n)$ is an $(m+1)$th degree polynomial in n. The coefficients of these polynomials are related to the Bernoulli numbers B_j by ***Bernoulli's formula***:

$$f(m,n) = \frac{1}{m+1} \sum_{j=0}^{m} \binom{m+1}{j} B_j n^{m+1-j}$$

Here we should note that Bernoulli's formula is sometimes called ***Faulhaber's formula*** after the German mathematician Johann Faulhaber (1580 – 1635) who also found remarkable ways to calculate sums of powers (Knuth 1993).

So, from

$$f(1,n) = \frac{1}{2}(B_0 n^2 + 2B_1 n) = \frac{1}{2}(n^2 + n)$$

we deduce that

$$B_0 = 1$$

and

$$B_1 = \frac{1}{2}$$

Similarly, from

$$f(2, n) = \frac{1}{3}(B_0 n^3 + 3B_1 n^2 + 3B_2 n) = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right)$$

we deduce that

$$B_2 = \frac{1}{6}$$

and so on.

Bernoulli numbers can be explicitly computed as

$$B_j = \sum_{k=0}^{j}\sum_{i=0}^{k}(-1)^i \binom{k}{i}\frac{i^j}{k+1}$$

or by using the generating function identity

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}$$

In the third part of the book, Bernoulli succumbed to what seems to be in vogue at the times and applied probabilistic techniques developed in the first section to some common games of chance. But even then, he seems to be interested more in the generalized and more abstractly conceptualized versions of these games. For example, instead of dealing with a game where five cards are dealt from a 52 card deck containing 12 face cards, he devises a "generalized game" with a deck containing c cards f of which are face cards, and players are dealt d cards (Hald 2003).

In the fourth section we see applications of probability to matters concerning *civilibus* (personal), *moralibus* (judicial), and *oeconomicis* (financial) circumstances. Thus, pondering all these applications, it is easy to see why ***Ars Conjectandi*** was acknowledged to be as such a pivotal occurrence; not only did it affect mathematical study of combinatorics and probability, but even non-mathematical topics were covered and were impacted.

Most importantly, we should note that in this section, Bernoulli dissociated from the empirical definition of probability and instead produced a result tantamount to the ***law of large numbers***, which he stated as follows:

> *The results of an observation would approach theoretical probability as more trials are held.*

This was the first time a comment regarding the convergence of empirical probability to theoretical probability appeared in print.

There were precursors of the law of large numbers. For example, Cardano had stated without proof that the accuracies of empirical statistics tend to improve as the number of trials increase (Mlodinow 2008). But the law of large numbers (albeit for a special case, namely for a binary random variable) was first proved in ***Ars Conjectandi***. Indeed, Bernoulli was so proud of this result, he named it the ***Golden Theorem*** and remarked that this was

> … a problem in which I have engaged myself for twenty years (Dunham 1994).

The theorem ended up being one of the more significant results given in the book. In fact, in 1913, Andrey Andreyevich Markov (1856-1922), the Russian mathematician best known for his work on stochastic processes, remarked “More than two hundred years passed since Bernoulli’s death, but he lives and he will live in his theorem.”

It should be mentioned that Bernoulli did not create the Golden Theorem in a complete vacuum. Not only were there predecessors as mentioned above, but between 1703 and 1705, Leibniz corresponded with Jacob Bernoulli and provided meticulous recommendations on the proof of the theorem (Sylla 1998).

The title Golden Theorem did not enjoy the same longevity as the theorem itself. The theorem was soon renamed as ***Bernoulli's Theorem*** and is now called ***Bernoulli’s Theorem*** or the ***weak law of large numbers***, as it is less rigorous and less general than the modern version.

After these four primary expository sections, almost as an afterthought, Bernoulli appended to ***Ars Conjectandi*** a tract on calculus, which concerned infinite series. It was a reprint of the five tracts he had published between 1686 and 1704 (Schneider 2006).

4. Impact of Ars Conjectandi on the Development of Probability

Ars Conjectandi is considered to be an innovative and revolutionary work in combinatorics and mathematical probability (Hald 2005), a work that inspired mathematicians throughout 18th and 19th centuries, and one that has been referred to as

> … [a] milestone of probability theory… (Dunham 1990)

and

> … Jacob Bernoulli's masterpiece… (Dunham 1990).

One mathematician that was profoundly influenced by ***Ars Conjectandi*** was Abraham de Moivre (1667-1754). For example, de Moivre adopted the term ***probability*** from Jacob Bernoulli - indeed, this term had not been used at all in any previous publication. De Moivre then wrote extensively on the subject in ***De mensura sortis: Seu de Probabilitate Eventuum in Ludis a Casu Fortuito Pendentibus*** of 1711 and its extension ***The***

Doctrine of Chances of 1718, which became another very popular book in probability theory. Of course, as is well known, de Moivre's most notable achievement was the discovery of the first instance of the Central Limit Theorem, by which he was able to approximate the binomial distribution with the normal distribution (Schneider 2006).

In 1812, Pierre-Simon Laplace (1749-1827) published his ***Théorie analytique des probabilités*** in which he consolidated and laid down many fundamental results in probability and statistics such as the moment generating function, method of least squares, and hypothesis testing, thus completing the final phase in the development of classical probability.

Arguably, the most significant impact of ***Ars Conjectandi*** on the ensuing generations of mathematicians was Bernoulli's Golden Theorem and the efforts to refine it. This was taken up by many notable mathematicians like Abraham de Moivre, Pierre-Simon Laplace, Siméon Denis Poisson (1781-1840), Pafnuty Chebyshev (1821-1894), Andrey Markov (1856-1922), Émile Borel (1871-1956), Aleksandr Khinchin (1894-1959), Francesco Paolo Cantelli (1875-1966), and Andrey Kolmogorov (1905-1987).

In 1837, Siméon Denis Poisson stated the theorem under the name ***la loi des grands nombres*** (the law of large numbers) on page 7 of his ***Probabilité des jugements en matière criminelle et en matière civile, précédées des règles générales du calcul des probabilitiés*** where he also attempted a proof. Thereafter, the theorem was known more commonly referred to as the law of large numbers.

These further studies have given rise to two prominent forms of the law of large numbers. One is called the ***weak*** law and the other the ***strong*** law, in reference to two different modes of convergence of the cumulative sample means to the expected value. Of course, as might be expected, the strong form implies the weak (Seneta 2013). Markov showed that the law can apply to a random variable that does not have a finite variance, and Khinchin showed in 1929 that if the series consists of independent identically distributed random variables, it suffices that the expected value exists for the weak law of large numbers to be true (Seneta 2013).

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