# A Robustness Study of the AEWMA Control Chart

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## Abstract

The in-control average run-length (ICARL) robustness is crucial for the application and the interpretation of any control chart. In this paper, in an extensive simulation study, the ICARL robustness of the well-known adaptive exponentially weighted moving average (AEWMA) chart of Capizzi and Masarotto (2003) is examined with respect to the underlying assumption of normality. The ICARL profiles of the AEWMA chart are calculated for a family of distributions of various shapes, including light-tailed, heavy-tailed, symmetric and skewed distributions. Our results show that the AEWMA chart is quite sensitive to the normality assumption and may not maintain the nominal ICARL well under non-normality. This raises questions about the application of the AEWMA chart in some practical situations. As an alternative, a nonparametric analog of the AEWMA (NPAEWMA) chart is proposed based on average ranks. The NPAEWMA chart shows good ICARL-robustness against non-normality. Performance comparisons are made between the NPAEWMA chart and an available nonparametric EWMA (NEWMA) chart of Li et al. (2010). It is seen that the proposed NPAEWMA has better shift detection properties in some situations. An illustration with some data is provided.

**Key words:** Nonparametric, AEWMA, ARL, robustness.

## 1. Introduction

Traditional control charts in the literature include the Shewhart charts, the EWMA charts and the CUSUM charts. These three charts are typically used to monitor the process mean. The Shewhart chart is easy to implement in practice, but is less efficient in detecting smaller shifts, as it uses information from the current sample. By contrast, the EWMA and CUSUM charts use the information in a sequence of samples up to the point of comparison by combining the previous and present observations. These charts are proven to be effective for small to moderate shifts with an assumed knowledge of the magnitude of the shifts. In this paper we focus on the EWMA type charts as they are often preferred by the users. In practice however, it may be difficult to use the EWMA chart since the magnitude and the direction of the shift may be unknown and unpredictable so it's not clear what tuning (weight) parameter to use. In such situations, Capizzi and Masarotto (2003) considered an adaptive EWMA (AEWMA) chart, where the weight parameter in a standard EWMA chart is adapted, over time, as each data point becomes available, so that the chart is able to detect a variety of shifts more effectively.

Because the AEWMA chart holds a lot of promise, over the last decade, a lot of research effort has been spent on the AEWMA chart and its various generalizations. Among these, Woodall and Mahmoud (2005) studied the inertia properties and evaluate the 'signal resistance' of several control charts. They stated that 'Likewise, the AEWMA procedure proposed by Capizzi and Masarotto (2003) has much better worst-case performance than the omnibus EWMA chart'. Reynolds and Stoumbos (2006) compared different charts and chart combinations for monitoring the process mean and/or variance. They re-defined the 'error' in the AEWMA by Capizzi and Masarotto (2003), and developed and evaluated AEWMA-type charts based on squared deviations from target for monitoring process mean and/or variance. Shu (2008) also extended the idea of the AEWMA chart for monitoring process locations to the case of monitoring process dispersion. Mahmoud and Zahran (2010) proposed a multivariate AEWMA (MAEWMA) control chart to detect shifts in process mean vector. They made performance comparison between MEWMA chart and the combined Shewhart-MEWMA chart in terms of the standard and worst-case average run length profiles and proved the effectiveness of their proposed MAEWMA. Simoes, Epprecht and Costa (2010) compared the performance of the AEWMA, the combined EWMA-Shewhart scheme and the combined AEWMA-Shewhart scheme optimized for the same pair of shifts and concluded 'First, there is no practical benefit in combining AEWMA chart with a Shewhart chart. Second, the performances of the AEWMA chart and of the combined EWMA-Shewhart scheme are practically identical'. Liu et al (2013) proposed a sequential rank-based nonparametric adaptive EWMA (NAE) control chart for detecting the persistent shifts in location parameter. Their NAE chart is claimed to be a self-starting procedure with no requirement of any prior knowledge of the underlying distributions, while Capizzi and Masarotto's AEWMA is developed under the assumption of normality. Liu, Tsung and Zhang (2014) proposed a nonparametric adaptive CUSUM control chart based on the sequential rank as well. Saleh, Mahmoud and Abdel-Salam (2013) pointed out that the AEWMA control chart was studied under the situation where the process parameters are known, but in practice the process parameters are usually unknown and are required to be estimated. They considered the performance of the AEWMA chart with estimated parameters and showed that the effect of different standard deviation estimators on the chart performance and recommended the use of the AEWMA over the EWMA especially when a small number of Phase I samples is available. Tang et al (2014) applied the AEWMA control chart method to detect low-rate denial of service (LDoS) for internet network and showed the priority of the AEWMA than EWMA by an experiment on a real dataset. Huang, Shu and Su (2014) improved the computational method for estimating the run length performance of the AEWMA chart.

However, having performed a thorough literature review of some 56 papers related to the AEWMA chart, we noticed that none of these authors considered the ICARL robustness of the AEWMA chart to the assumption of normality. Without the ICARL robustness the value of a control chart is highly diminished because, for example, too many or too few false alarms can ruin the efficacy of the chart and the out-of-control shift detection property becomes somewhat meaningless. It is true that non-normal distributions are often encountered in practice, so a robustness study of any chart to the normality (or whatever distributional) assumption is essential. Liu, Zi, Zhang and Wang(2013) suspected the robustness of the AEWMA, stating 'However, these control charts often assume that data come from some parametric distribution, most commonly the normal distribution. When the underlying process is unknown or not normal, these charts may not be appropriate', but they neither performed any experiments nor provided any evidences to prove the statement.

Borror, Montgomery and Runger (1999) studied the robustness of the EWMA control chart to non-normality. Human, Kritzinger and Chakraborti (2011) also examined the robustness of the EWMA chart and suggested using caution against its overuse, particularly in situations where the shape of the underlying process distribution is not sufficiently known. But to the best of our knowledge, no researcher has examined the ICARL robustness of the AEWMA control chart to non-normality. In our paper, we examine the robustness of the AEWMA chart for six different shapes of distributions within a family of distributions. This includes the 1) normal, 2) heavy tailed symmetric, 3) light tailed symmetric, 4) slightly right-skewed, 5) slightly right-skewed and heavy tailed, 6) highly left-skewed. Numerical results reveal that the AEWMA chart is highly sensitive to non-normality so one has to be very careful while applying it in practice. As a follow up, we will propose a nonparametric AEWMA (NPAEWMA) chart based on Wilcoxon rank sum statistics. We further access the robustness of the NPAEWMA and make performance comparisons with the W.EWMA chart proposed by Li, Tang and Ng (2010). First we start with a brief introduction to the AEWMA chart.

## 2. The AEWMA Control Chart

The basic idea of the adaptive EWMA (AEWMA) control chart, proposed by Capizzi and Masarotto (2003) (hereafter CM), is to "adapt" the weights given to the past observations in a standard EWMA chart. More specifically, in a standard EWMA control chart the weight parameter λ of the current observations changes along with every new observation coming in. In an AEWMA chart this constant weight λ is replaced by a suitable function of the current 'error', which is the difference between the observed variable and the previous monitored value (the EWMA value at the previous time point). If the error value is small, the weight assigned to the current observation is small, and thus the chart can detect a small shift quickly, that is the chart behaves close to a standard EWMA chart. Otherwise, the current observation will be given a larger weight, and the chart will perform more like a Shewhart-type chart.

Assume that $y_1, y_2, \dots, y_n$ are independent and identically distributed normal random variables. The AEWMA control chart is based on the statistics below:

$$x_t = \big(1 - w(e_t)\big)x_{t-1} + w(e_t)y_t = x_{t-1} + w(e_t)(y_t - x_{t-1}) = x_{t-1} + \phi(e_t)$$

where $x_0$=0, $e_t = y_t - x_{t-1}$ is the "error", $w(e_t) = \phi(e_t)/e_t$, where $\phi(e_t)$ is the 'score' function, which varies with the $e_t$. CM suggested three choices of the score function, among which the Huber score function is favored for its efficiency and simplicity. An AEWMA control chart with a Huber score function is the one that is widely studied and made comparisons with other charts by CM and also many researchers.

$$\phi(e) = \begin{cases} e + (1-\lambda)k & if\ e < -k \\ \lambda e & if\ |e| \le k \\ e - (1-\lambda) & if\ e > k \end{cases}$$

The parameters $\lambda$ and $k$ are the two of the three chart design parameters. The third chart design parameter is the control limit h, that is chosen to guarantee a specified in-control average run length. A monitored value $x_i$ exceeding above h or falling below –h will provide an out-of-control signal.

The key advantage of the AEWMA chart stem from the fact that with the flexible weighting, it can effectively detect a variety of magnitudes of shifts. However, as we noted earlier, the AEWMA chart is developed under the normality assumption, meaning that under non-normal process distributions the performance of the chart is not guaranteed. To study the in-control run length (RL) distribution of the AEWMA chart under non-normality, six different shapes of underlying distributions are selected within the g-and-k distribution family (Hoaglin, 1986, and Haynes et al., 1997). They are 1) normal, 2) heavy tailed symmetric, 3) light tailed symmetric, 4) slightly right-skewed, 5) slightly right-skewed and heavy tailed, 6) highly left-skewed. We start with a description of the g-and-k family of distributions:

## 3. The g-and-k Distribution Family

The family of g-and-k distributions is defined by quantile function:

$$Q_x(u|A,B,g,k) = A + Bz_u(1 + c\frac{1-e^{-gz_u}}{1+e^{-gz_u}})(1+z_u^2)^k$$

where A and B >0 are the location and scale parameter, respectively, g measures the skewness of the distribution, and k>-0.5 measures the kurtosis (in the general sense of peakedness/heavy or light-tailedness), $z_u$ is the u$^{th}$ quantile of standard normal distribution and c is a normalizing constant to help produce a proper distribution. Approximately, c $\leq$ 0.83 guarantees a completely proper distribution. Here we use c=0.8 as researchers normally do.

Below is the chart of the parameters and the graph of shapes of the six selected distributions.

**Table 1**: parameters for the chosen six distributions

| Distribution | gk1 | gk2 | gk3 | gk4 | gk5 | gk6 |
|---|---|---|---|---|---|---|
| Shape | normal | symmetric heavy tailed | symmetric light tailed | slightly skewed | slightly skewed heavy tailed | highly skewed |
| A | 0 | 0 | 0 | 0 | 0 | 0 |
| B | 1 | 1 | 1 | 1 | 1 | 1 |
| c | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 |
| g | 0 | 0 | 0 | 0.5 | 0.5 | -2 |
| k | 0 | 0.5 | -0.1 | 0 | 0.5 | 0 |

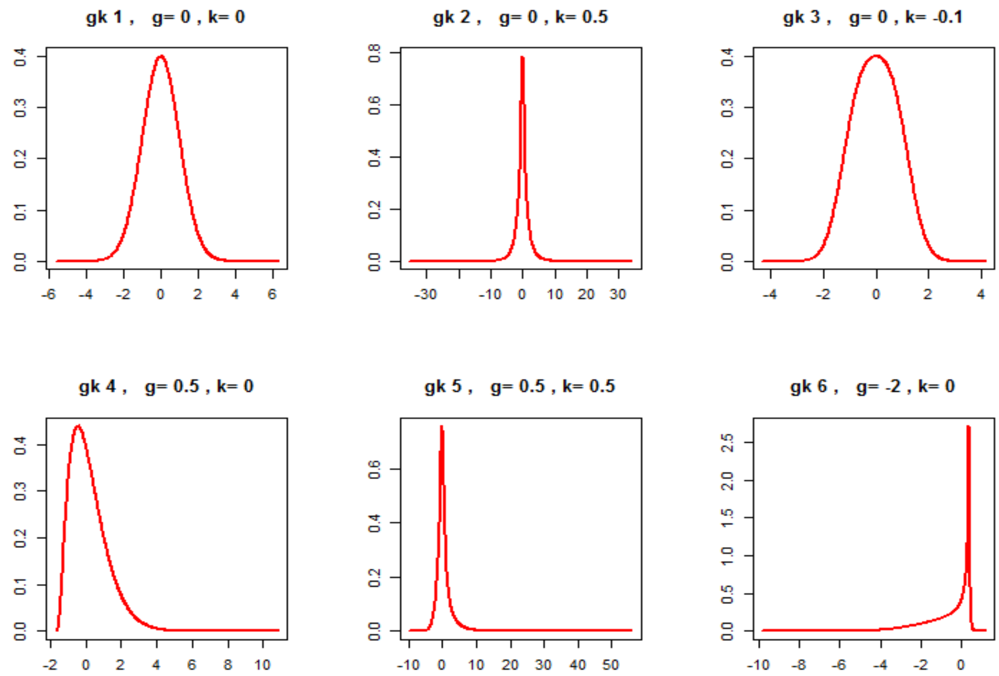


**Figure 1:** shapes of the chosen six distributions

## 4. Robustness Study of the AEWMA Control Chart

As for the design parameters (λ, k and h) in the AEWMA, we consider two optimal combinations given by CM that produce an in-control ARL of 500 for normal distribution. Table 2 displays the design parameters used in our robustness study.

**Table 2:** Optimal design parameters of two AEWMAs that produce in-control ARL of 500

| Shifts pairs | | Optimal design parameters | | |
|---|---|---|---|---|
| $\mu_1$ | $\mu_2$ | λ | k | h |
| 1.0 | 5.0 | 0.1354 | 3.2587 | 0.7931 |

Usually, zero-state or/and steady-state ARL measures are used to evaluate the performance of a control chart. In our study, we investigate the robustness of the AEWMA in terms of zero-state in-control ARL using Monte Carlo simulation.

**Table 3:** The in-control ARL comparison of the six different shapes of underlying distributions. The design parameters used are those producing an in-control nominal ARL of 500 when normality assumption is met.

| Shifts pairs | | Underlying distributions | | | | | |
|---|---|---|---|---|---|---|---|
| $\mu_1$ | $\mu_2$ | normal | symmetric heavy tailed | symmetric light tailed | slightly skewed | slightly skewed heavy tailed | highly skewed |
| 1.0 | 5.0 | 506.13 | 10.00 | 5356.58 | 71.43 | 11.05 | 23.39 |

Table 3 shows the performance of the two selected AEWMA charts for six different shapes of distributions: 1) normal, 2) symmetric with heavy tailed, 3) symmetric with light tailed, 4) slightly skewed, 5) slightly skewed with heavy tailed, and 6) highly skewed. All monitored random variables are generated from the g-and-k distribution. All six distributions have the location parameter 0, scale parameter 1 and skewness/kurtosis as listed in table 1. Each numerical result is calculated by 10,000 simulation times. According to the numerical results in table 3, the effect of the shapes of the underlying distributions on the in-control ARL is large. The exact in-control ARL with normal underlying distribution is very close to the nominal in-control ARL. However, when the underlying in-control distribution is not normal, the exact in-control ARL is far from the nominal in-control ARL=500. For heavy tailed distributions and highly skewed distribution, the exact in-control ARLs are significantly shorter than the nominal in-control ARL, say less than 30, which provides us with a significantly higher false alarm rate. For the light tailed distribution, the exact in-control ARL is remarkably longer than the nominal in-control ARL, say more than 2000, which results in a much lower false alarm rate. The rapid decrease in the in-control ARL indicates a much higher false alarm rate in reality. The results make us believe that the AEWMA control chart only performs well for the normal distribution and could be highly sensitive to non-normality, which leads to the conclusion of its non-robustness in general.

There are some solutions to consider:

- One can modify the chart and search for optimal design parameters in the AEWMA control chart which guarantee a nominal in-control ARL for a specified distribution. However, this solution requires a full knowledge of the underlying distribution.
- One can also consider pre-processing the data, such as transforming it to normal and then applying the AEWMA control chart to monitor the process.
- Applying nonparametric method to the AEWMA control chart, thus the chart is based on a distribution-free statistics, which does not require the normality assumption. For example, Li, Tang and Ng (2010) considered the idea of using a nonparametric Wilcoxon rank sum statistic in a standard EWMA chart and proposed the W.EWMA chart.

In the following sections of this paper, we pursue the third solution and propose a new AEWMA control chart based on a nonparametric statistic.

## 5. Nonparametric AEWMA (NPAEWMA) Control Chart

Wilcoxon (1945) proposed the Wilcoxon rank sum test (WRS) based on the sum of the ranks of one of the samples say the Ys in the combined independent samples of Xs and Ys. Suppose $X = (x_1,\ x_2, \dots x_m)$ and $Y = (y_1,\ y_2, \dots y_n)$ are two independent random samples from two independent continuous variables. Combine the two samples together and put them in ascending order. Let V stand for the sum of the ranks of the Y sample in the combined ordered sample and let $a_{(i)} = 1$ if the i[th] smallest observation is from sample Y, and $a_{(i)} = 0$ otherwise, then

$$V = \sum_{i=1}^{n+m} (i \cdot a_{(i)})$$

If the distribution of the Y is stochastically larger than that of X, then V will tend to be large; otherwise V will be small. Under the null hypothesis that the X's and the Y's are identically distributed, the expectation and the variance of V are (see, e.g., Gibbons and Chakraborti, 2010) :

$$E(V) = \frac{n(m+n+1)}{2}$$

$$Var(V) = \frac{mn(m+n+1)}{12},$$

respectively. To adopt this idea for a distribution-free control chart, first, a reference sample from an in-control process needs to be obtained. Then, at each time point during the future monitoring of the process, subgroups are obtained and compared to the reference sample. A nonparametric AEWMA control chart can be constructed as follow:

- Collect a reference sample of size m, $X = (x_1, x_2, \dots, x_m)$, from an in-control process.
- Collect subgroups, each of size n, from the monitoring process. And denote the subgroup collected at time t as $Y_t = (y_1, y_2, \dots, y_n)$ .
- Compute the Wilcoxon rank sum statistic for $Y_t$ against X, denote it as $V_t$. Let $V'_t = \frac{V_t - E(V)}{\sqrt{var(V)}}$ be the standardized $V_t$:
- Construct the NPAEWMA control chart as below, where $T_t$ is the monitoring statistic at time t and $T_0 = 0$. The design parameters for the NPAEWMA chart are the same as those for the AEWMA of CM .
$$T_t = \left(1 - w(e_t)\right)T_{t-1} + w(e_t)V'_t$$
- An out-of-control signal is will be given when $T_t$ falls on or outside the control limits $\pm h$.

## 6. The Robustness of the NPAEWMA Control Chart

The asymptotic distribution of the $V'_t$ is close to normal when the reference sample size m and the subgroup sample size n is large. From a practical point of view, we prefer the value of subgroup size n to be small. Meanwhile, we would like to have a modest to large size of Phase I reference sample to gather enough information about the assumed in-control process. A common selection of m and n values by researchers are m=100, 300, and 500, and n=5, respectively. Following this tradition, we examine the robustness of the proposed NPAEWMA control chart and performance of the chart when there is a shift in the location of the monitored process. Table 5 shows in-control ARLs and out-of control ARLs of the NPAEWMA control chart. The data are simulated from standard normal distribution. The nominal in-control ARL is 500. The design parameters of the chart are from CM's paper. The number of the reference sample m=100, 300 and 500, and n=5. Each ARL is calculated from 20,000 simulated RL.

Table 5: The in-control ARL and the out-of-control ARL of the NPAEWMA control chart. The shift in the location are in the unit of $/\sqrt{n}$ .

| SimN=20,000, rational subgroup n=5 | | | |
|---|---|---|---|
| shifts | number of reference samples | | |
| δ | m=100 | m=300 | m=500 |
| 0 | 428.3 | 469.9 | 502.8 |
| 0.05 | 416.2 | 450.8 | 482.9 |
| 0.1 | 387.1 | 413.1 | 426.3 |
| 0.25 | 282.7 | 219 | 197.1 |
| 0.5 | 99.1 | 52.7 | 46.3 |
| 1 | 13.3 | 10.9 | 10.7 |
| 1.5 | 5.9 | 5.5 | 5.4 |
| 2 | 3.8 | 3.5 | 3.6 |
| 3 | 2.2 | 2.1 | 2.1 |

According to the numerical results in Table 5, the in-control ARLs increase from 428.3 to 502.8 as the reference sample size increases from 100 to 500. When m=500, the exact in-control ARL equals the nominal in-control ARL. When there is a shift from the in-control distribution, the chart can detect it by producing an out-of-control alarm.

## 7. Performance Comparison between the NPAEWMA and WEWMA

Li et al. (2010) proposed an EWMA control chart based on the WRS statistics called the WEWMA chart. They also made performance comparisons among several nonparametric and parametric control charts. Among the nonparametric control charts, their WEWMA performed better than the other charts. Here comparisons are made between the NPAEWMA and the WEWMA control charts. Table 6 shows the out-of-control ARL for both charts at the same shift levels. In order to make the comparisons, we select the same in-control distributions: normal, t5 and gamma(3, 1) as in Li et al. (2010). From the results of Table 6 below, both of nonparametric charts are seen to be in-control robust to non-normality.. When the monitored process is out-of-control and the distribution is symmetric, the NPAEWMA is seen to perform better than the WEWMA chart. Especially when there is a small shift, for example, a shift of 0.25, the NPAEWMA chart is able to detect it quicker than the WEWMA. For larger shifts, greater than 2, the NPAEWMA and WEWMA charts perform similarly. For asymmetric distribution, the performance of the NPAEWMA chart is not as good as the WEWMA chart for smaller shifts, but the out-of-control ARLs are almost the same when the shift is greater than 3.

Table 6: The ARL for WEWMA and NPAEWMA control chart for normal, t5 and gamma(3,1) distribution. The data for WEWMA are from Li et al (2010). The reference sample size m=500 and subgroup size=5 for both charts. The shifts are in unit of $1/\sqrt{5}$. The ARLs of the NPAEWMA are calculated from 10,000 simulations.

| | normal | | t5 | | gamma(3,1) | |
|---|---|---|---|---|---|---|
| shifts | WEWMA | NPAEWMA | WEWMA | NPAEWMA | WEWMA | NPAEWMA |
| 0 | 502.94 | 509.08 | 501.61 | 518.21 | 501.13 | 505.37 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.25 | 321.52 | 197.04 | 288.19 | 238.56 | 340.5 | 534.84 |
| 0.5 | 103.15 | 45.76 | 75.53 | 61.82 | 103 | 141.06 |
| 1 | 14.29 | 10.55 | 11.35 | 13.43 | 11.79 | 26.49 |
| 1.5 | 7.52 | 5.44 | 6.44 | 6.64 | 6.5 | 11.33 |
| 2 | 5.3 | 3.57 | 4.71 | 4.11 | 4.75 | 6.85 |
| 3 | 3.64 | 2.06 | 3.39 | 2.54 | 3.41 | 3.72 |

Further work on this chart, including a detailed performance examination, is in progress.

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