Power Analysis for the Test on the Location of Quadratic Growth Curves

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Abstract
Quadratic growth curves of 2^{nd} degree polynomial are widely used in longitudinal studies. For a 2^{nd} degree polynomial, the vertex displays the location of the curve in the XY plane. Under some models, an indirect test on the location of the vertex can be based on the intercept and slope parameters; but in other models, a direct test on the vertex is required. In this paper, we derive a quadratic-form statistic for a test of the null hypothesis that there is no shift in the location of the vertex in a linear mixed model. The statistic has a large sample chi-square distribution. For 2^{nd} degree polynomials from two independent groups, another chi-square statistic is derived for a test that there is no difference of location between the two curves, and it is compared to an F statistic. Power functions are presented for both the indirect F test and the direct chi-square test. We calculate the theoretical power and propose a simulation study to investigate the power of the tests. An analysis is also presented using the TELL efficacy longitudinal study, in which sound identification scores for children are modeled as quadratic growth curves for two groups, TELL and control curriculum.

Key Words: Power Function, Random Effect, Mixed Model, Quadratic Growth Curve, Vertex

1. Introduction

Many longitudinal studies are designed to investigate a characteristic of an individual, where the characteristic is measured repeatedly over the occasion for each study subject. Often the individuals are considerably correlated across observations [Fitzmaurice, Laird, and Ware, 2004]. A multivariate model with general unrestricted covariance structure may be applied to analyze these correlated data, but the growth curve model is usually considered. The analysis of growth curves provides an explanation of within-individual variation by the aging process or natural development. In some longitudinal studies, the relation between the time measurement t and response y cannot be adequately described by a linear trend model. Adding a square term of the fixed effect time t to the model gives a quadratic growth curve model, which can often describe the true unknown model better. The coefficient parameters of the fixed effect are necessary to determine the growth curve. The vertex of a quadratic curve provides the location of such a curve, which is interesting. For two independent groups, such as control and treatment, difference of vertices of two quadratic growth curves are useful to compare the locations. Both the x-coordinate and y-coordinate of the vertex are given by a non-linear combination of the model fixed regression coefficients, not simply only one of them. However, common statistical computer packages usually display statistical inferences for the fixed regression coefficient, but not for any of their functions.

For a one-sample study, the test of the null hypothesis of no shift in the location can be performed indirectly with an F test on the model parameters. The location of the vertex is a function of the model parameters, and a statistic for a direct test on the location of the

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[¶]This research was supported by the U.S. Department of Education, Institute of Education Sciences Grant R324A110048.

vertex is also presented. Power calculations are performed to investigate the performance of the indirect F test and the direct test. For a two-sample study, the null hypothesis of no difference in the location of the vertices can sometimes be conducted with the indirect F test, but sometimes only the direct test is available. Power calculations for comparing the F test and direct test will also be performed for the two-sample study.

Two models, linear mixed effects models and growth curve models will be reviewed in Chapter 2. Power functions and analysis will be derived and performed in Chapter 3 and Chapter 4 for a one-sample quadratic growth curve and two independent samples respectively. In Chapter 5, an application of analysis, TELL Efficacy Study, will be presented. The conclusion and discussion for future research are presented in Chapter 6.

2. Linear Mixed Models and Growth Curves

A mixed model is a statistical model containing mixed effects, where the mixed effects consist of both fixed effects and random effects. They are appropriate in settings where repeated measurements are provided on the same individual, or where measurements are made on clusters of related individuals. Mixed models are based on explicit identification of individual and population characteristics; many mixed models in the literature can be described either as growth models or as repeated-measures models. Growth-curve analyses emphasize the explanation of within-subject variation by the natural developmental or aging process [Ware, 1985]. These analyses often compare growth characteristics for different populations, emphasizing the contribution of experimental conditions to between-subject variability [Laird and Ware, 1982].

A linear mixed model for longitudinal data can be expressed in matrix notation,

$$\boldsymbol{y}_i = \boldsymbol{X}_i\boldsymbol{\beta} + \boldsymbol{Z}_i\boldsymbol{\alpha}_i + \boldsymbol{\epsilon}_i, \tag{1}$$

where $\boldsymbol{y}_i$ is a known vector of observations for subject i, $\boldsymbol{Y}' = [\boldsymbol{y}'_1, \cdots, \boldsymbol{y}'_N]$, $\boldsymbol{X}_i$ and $\boldsymbol{Z}_i$ are known model matrices of regressors for subject i relating the observations $\boldsymbol{y}_i$ to $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}_i$, $\boldsymbol{X}' = [\boldsymbol{X}'_1, \cdots, \boldsymbol{X}'_N]$, $\boldsymbol{\beta}$ is an unknown vector of fixed effects parameters, $\boldsymbol{\alpha}_i$ is an unknown vector of random effects with mean $E(\boldsymbol{\alpha}_i) = \mathbf{0}$ and covariance $\mathrm{Cov}(\boldsymbol{\alpha}_i) = \boldsymbol{G}$; the covariance matrix $\boldsymbol{G}$ is usually identical for all the subjects, $\boldsymbol{\epsilon}_i$ is an unknown vector of random error terms with mean $E(\boldsymbol{\epsilon}_i) = \mathbf{0}$ and covariance $\mathrm{Cov}(\boldsymbol{\epsilon}_i) = \boldsymbol{R}_i$; the set of unknown parameters in $\boldsymbol{R}_i$ do not depend on the subject i, only the dimension of $\boldsymbol{R}_i$ depends on the subject i; $\boldsymbol{\alpha}_i$ and $\boldsymbol{\epsilon}_i$ are independent, $\mathrm{Cov}(\boldsymbol{\alpha}_i, \boldsymbol{\epsilon}_i) = \mathbf{0}$.

Assuming $\boldsymbol{\alpha}_i$ is normally distributed with mean $\mathbf{0}$ and covariance $\boldsymbol{G}$. The marginal density function of the random vector $\boldsymbol{y}_i$ is given by [Verbeke and Molenberghs, 2009],

$$f(\boldsymbol{y}_i) = \int f(\boldsymbol{y}_i|\boldsymbol{\alpha}_i) f(\boldsymbol{\alpha}_i) d\boldsymbol{\alpha}_i$$

which is multivariate normal distributed with the dimension of time measurements n, i.e., the marginal model of $\boldsymbol{y}_i$ is, $\boldsymbol{y}_i \sim \boldsymbol{N}_n(\boldsymbol{X}_i\boldsymbol{\beta}, \boldsymbol{Z}_i\boldsymbol{G}\boldsymbol{Z}'_i + \boldsymbol{R}_i)$. When all the covariance parameters are known, the maximum likelihood (ML) function of $\boldsymbol{\theta} = (\boldsymbol{\beta}, \boldsymbol{\alpha}_i)'$ is [Verbeke and Molenberghs, 2009],

$$L_{ML}(\boldsymbol{\theta}) = \prod_{i=1}^{N} \left\{ (-2\pi)^{-n/2} \left| \boldsymbol{Z}_i\boldsymbol{G}\boldsymbol{Z}'_i + \boldsymbol{R}_i \right|^{-1/2} \times exp\left(\sum_{i=1}^{n} (\boldsymbol{y}_i - \boldsymbol{X}_i\hat{\boldsymbol{\beta}}_{ML})'(\boldsymbol{Z}_i\boldsymbol{G}\boldsymbol{Z}'_i + \boldsymbol{R}_i)^{-1}(\boldsymbol{y}_i - \boldsymbol{X}_i\hat{\boldsymbol{\beta}}_{ML}) \right) \right\},$$

where N is the sample size. The ML estimator for fixed regression coefficients and their variance are [Laird and Ware, 1982],

$$\hat{\beta}_{ML} = \left(\sum_{i=1}^{N} \boldsymbol{X}'_i(\boldsymbol{Z}_i\boldsymbol{G}\boldsymbol{Z}'_i + \boldsymbol{R}_i)^{-1}\boldsymbol{X}_i \right)^{-1} \left(\sum_{i=1}^{N} \boldsymbol{X}'_i(\boldsymbol{Z}_i\boldsymbol{G}\boldsymbol{Z}'_i + \boldsymbol{R}_i)^{-1}\boldsymbol{y}_i \right) \tag{2}$$

$$\Sigma_{\hat{\beta}_{ML}} = \left(\sum_{i=1}^{N} \boldsymbol{X}_i' (\boldsymbol{Z}_i \boldsymbol{G} \boldsymbol{Z}_i' + \boldsymbol{R}_i)^{-1} \boldsymbol{X}_i \right)^{-1}. \tag{3}$$

Denote $\boldsymbol{\zeta}$ as the vector of variance and covariance parameters found in $\boldsymbol{R}_i$ and $\boldsymbol{G}$. The restricted maximum likelihood (REML) function of $\boldsymbol{\zeta}$ is [Verbeke and Molenberghs, 2009],

$$L_{REML}(\zeta) = (2\pi)^{-(n-k)/2} \left| \sum_{i=1}^{N} \boldsymbol{X}_i' \boldsymbol{X}_i \right|^{1/2} \times \left| \sum_{i=1}^{N} \boldsymbol{X}_i' (\boldsymbol{Z}_i \boldsymbol{G} \boldsymbol{Z}_i' + \boldsymbol{R}_i)^{-1} \boldsymbol{X}_i \right|^{-1/2} \prod_{i=1}^{N} \left| \boldsymbol{Z}_i \boldsymbol{G} \boldsymbol{Z}_i' + \boldsymbol{R}_i \right|^{-1/2}$$
$$\times exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} (\boldsymbol{y}_i - \boldsymbol{X}_i \hat{\boldsymbol{\beta}}_{ML})' (\boldsymbol{Z}_i \boldsymbol{G} \boldsymbol{Z}_i' + \boldsymbol{R}_i)^{-1} (\boldsymbol{y}_i - \boldsymbol{X}_i \hat{\boldsymbol{\beta}}_{ML}) \right\}.$$

$\boldsymbol{\zeta}$ is a function of a set of error contrasts $\boldsymbol{U} = \boldsymbol{A}'\boldsymbol{Y}$ where $\boldsymbol{A}$ is any $(n \times (n-k))$ full-rank matrix with columns orthogonal to the columns of the $\boldsymbol{X}$ matrix. Then for each individual i, the REML estimator through an empirical bayesian algorithm for the random effect and its variance are [Laird and Ware, 1982],

$$\hat{\boldsymbol{\alpha}}_{i(REML)} = \boldsymbol{G} \boldsymbol{Z}_i' (\boldsymbol{Z}_i \boldsymbol{G} \boldsymbol{Z}_i' + \boldsymbol{R}_i)^{-1} (\boldsymbol{y}_i - \boldsymbol{X}_i \hat{\boldsymbol{\beta}}_{ML}) \tag{4}$$

$$\Sigma_{\hat{\boldsymbol{\alpha}}_{i(REML)}} = \boldsymbol{G} \boldsymbol{Z}_i' \left\{ (\boldsymbol{Z}_i \boldsymbol{G} \boldsymbol{Z}_i' + \boldsymbol{R}_i)^{-1} - (\boldsymbol{Z}_i \boldsymbol{G} \boldsymbol{Z}_i' + \boldsymbol{R}_i)^{-1} \boldsymbol{X}_i \hat{\Sigma}_{\hat{\beta}_{ML}} \boldsymbol{X}_i' (\boldsymbol{Z}_i \boldsymbol{G} \boldsymbol{Z}_i' + \boldsymbol{R}_i)^{-1} \right\} \boldsymbol{Z}_i \boldsymbol{G},$$

assuming that the necessary matrix inverses exist when it is implied. For the case of less than full rank, we could work out the relevant formulas using generalized inverses. When the covariance matrices are unknown, the literature on the estimation of variance components is extensive.

3. Power Analysis for One Quadratic Growth Curve

Two specific quadratic models for the growth curves from model (1) are explored, one is a mixed model with second-order polynomial and random intercept, named the random intercept model; the other is a mixed model with second-order polynomial and both random intercept and random slope, named the random slope model. They are defined as follows:

Second-order mixed model with random intercept (random intercept model),

$$y_{ij} = \beta_0 + \beta_1 t_{ij} + \beta_2 t_{ij}^2 + \alpha_{0i} + \varepsilon_{ij} \qquad i = 1, ..., N \quad j = 1, ..., n_i \tag{5}$$

where N is the number of individuals, n_i is the number of occasions for the i^{th} individual, β_0, β_1 and β_2 are fixed regression coefficients, assuming $\beta_2 \neq 0$, α_{0i} is random effect of the i^{th} individual, $\alpha_{0i} \sim N(0, \sigma_{\alpha_0}^2)$, ε_{ij} is the random error term of the i^{th} individual at the j^{th} occasion, $\varepsilon_{ij} \sim N(0, \sigma_e^2)$, α_{0i} and ε_{ij} are independent, i.e. $\text{Cov}(\alpha_{0i}, \varepsilon_{ij}) = 0$ for all i, y_{ij} is the response at j^{th} occasion of i^{th} individual, and t_{ij} is a time measurement.

To derive the covariance structure for the random intercept model (5), the variance for each response and the marginal covariance and correlation between any pair of responses, $y_{i,j}$ and $y_{i,j'}$, are,

$$\text{Var}(y_{ij}) = \text{Var}(\boldsymbol{X}_{ij}\boldsymbol{\beta} + \alpha_{0i} + \varepsilon_{ij}) = \sigma_{\alpha_0}^2 + \sigma_e^2.$$

$$\text{Cov}(y_{i,j}, y_{i,j'}) = \text{Cov}(\boldsymbol{X}_{ij}\boldsymbol{\beta} + \alpha_{0i} + \varepsilon_{ij},\ \boldsymbol{X}_{ij'}\boldsymbol{\beta} + \alpha_{1i} + \varepsilon_{ij'}) = \sigma_{\alpha_0}^2,$$

The intraclass correlation is,

$$\rho = Corr(y_{i,j}, y_{i,j'}) = \frac{\sigma_{\alpha_0}^2}{\sigma_{\alpha_0}^2 + \sigma_e^2}.$$

Therefore the marginal covariance matrix of the repeated measurements has the following compound symmetry pattern, $\Sigma_{\boldsymbol{y}_i} = \sigma_e^2 \boldsymbol{I} + \sigma_{\alpha_0}^2 \boldsymbol{J}$.

Second-order mixed model with random intercept and random slope (random slope model),

$$y_{ij} = \beta_0 + \beta_1 t_{ij} + \beta_2 t_{ij}^2 + \alpha_{0i} + \alpha_{1i} t_{ij} + \varepsilon_{ij} \qquad i = 1, \ldots, N \quad j = 1, \ldots, n_i \tag{6}$$

where α_{0i} and α_{1i} are random effects of individual i, $\alpha_{0i} \sim N(0, \sigma^2_{\alpha_0})$, $\alpha_{1i} \sim N(0, \sigma^2_{\alpha_1})$ and $\mathrm{Cov}(\alpha_{0i}, \alpha_{1i}) = \sigma_{\alpha_0\alpha_1}$; ε_{ij}, β_0, β_1, β_2, n_i, N, y_{ij} and t_{ij} are defined the same as in model (5), α_{0i}, and α_{1i} are independent of ε_{ij}, i.e. $\mathrm{Cov}(\alpha_{0i}, \varepsilon_{ij}) = 0$ and $\mathrm{Cov}(\alpha_{1i}, \varepsilon_{ij}) = 0$ for all i.

To derive the covariance structure for the random slope model (6), the variance of each response, and the marginal covariance and correlation between any pair of responses, $y_{i,j}$ and $y_{i,k}$, are

$$\mathrm{Var}(y_{ij}) = \mathrm{Var}(\boldsymbol{X}_{ij}\boldsymbol{\beta} + \boldsymbol{Z}_{ij}\boldsymbol{\alpha_i} + \varepsilon_{ij}) = g_{11} + 2t_{ij}g_{12} + t_{ij}^2 g_{22} + \sigma_e^2 \ ,$$

$$\mathrm{Cov}(y_{i,j}, y_{i,k}) = \mathrm{Cov}(\boldsymbol{X}_{ij}\boldsymbol{\beta} + \boldsymbol{Z}_{ij}\boldsymbol{\alpha_i} + \varepsilon_{ij},\ \boldsymbol{X}_{ik}\boldsymbol{\beta} + \boldsymbol{Z}_{ik}\boldsymbol{\alpha_i} + \varepsilon_{ik}) = g_{11} + (t_{ij} + t_{ik})g_{12} + t_{ij}t_{ik}g_{22},$$

where $\mathrm{Var}(\varepsilon_{ij}) = \sigma_e^2$, $g_{11} = \sigma^2_{\alpha_0}$, $g_{22} = \sigma^2_{\alpha_1}$ and $g_{12} = \sigma_{\alpha_0\alpha_1}$; g_{11} and g_{22} are the diagonal elements of $\boldsymbol{G}$, and g_{12} is the off diagonal element of $\boldsymbol{G}$. The intraclass correlation is,

$$\rho = Corr(Y_{i,j}, Y_{i,k}) = \frac{g_{11} + (t_{ij} + t_{ik})g_{12} + t_{ij}t_{ik}g_{22}}{\sqrt{g_{11} + 2t_{ij}g_{12} + t_{ij}^2 g_{22} + \sigma^2}\sqrt{g_{11} + 2t_{ik}g_{12} + t_{ik}^2 g_{22} + \sigma^2}}$$

which is close to the unstructured covariance pattern.

For the random intercept model (5) and the random slope model (6), denote $\boldsymbol{b}' = (b_0, b_1, b_2)$ as the maximum likelihood estimator (MLE) , defined in equation (2), of fixed regression coefficients $\boldsymbol{\beta}' = (\beta_0, \beta_1, \beta_2)$. As proved in Section 2, under some situations such as all the covariance parameters of random effects are known, the distribution of $\boldsymbol{b}$ is exactly normal. More generally, such as the covariance parameters of random effects are unknown, $\boldsymbol{b}$ is approximately normally distributed in large samples with mean $\boldsymbol{\beta}$ and covariance $\Sigma_{\boldsymbol{b}}$, defined in equation (3), $\Sigma_{\boldsymbol{b}} = (\sum\limits_i \boldsymbol{X}_i' \Sigma_{\boldsymbol{y}_i}^{-1} \boldsymbol{X}_i)^{-1}$. The corresponding estimated covariance of $\Sigma_{\boldsymbol{b}}$ is, $\hat{\Sigma}_{\boldsymbol{b}} = (\sum\limits_i \boldsymbol{X}_i' \hat{\Sigma}_{\boldsymbol{y}_i}^{-1} \boldsymbol{X}_i)^{-1}$. Denote $\Omega_{\boldsymbol{b}} = \frac{1}{N}(\sum\limits_i \boldsymbol{X}_i' \Sigma_{\boldsymbol{y}_i}^{-1} \boldsymbol{X}_i)^{-1}$, then $\sqrt{N}(\boldsymbol{b} - \boldsymbol{\beta}) \xrightarrow{L} N_3(\boldsymbol{0}, \Omega_{\boldsymbol{b}})$.

Power analysis plays an important role to reject the null hypothesis if it specifies a vertex point that is actually not the true vertex point for quadratic growth curve. Consider the hypotheses,

$$H_0 : \boldsymbol{V} = \boldsymbol{V}_0 \quad \text{v.s.} \quad H_a : \boldsymbol{V} = \boldsymbol{V}_a \tag{7}$$

where $\boldsymbol{V}_a$ is the true vertex and $\boldsymbol{V}_0$ is the hypothesized vertex point under the null hypothesis. The power function of a statistical test is the probability that the test statistic falls in the rejection region R [Kenward and Roger, 1997]. The chi-square approximation can be used to obtain a direct method to test the hypothesis (7). The power function of the test will be presented in Section 3.3.

An indirect method to test the hypotheses (7) would use an F statistic with respect to $\boldsymbol{\beta}$'s, since the x and y-coordinates of the vertex are nonlinear functions of $\boldsymbol{\beta}$'s. Transform the hypotheses (7) to the hypotheses with regard to $\boldsymbol{\beta}$'s; the new hypotheses are stated as follows,

$$H_0 : \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\beta_{0,1}\beta_{0,2}^{-1} \\ \beta_0 - \frac{1}{4}\beta_{0,1}^2\beta_{0,2}^{-1} \end{pmatrix} \quad v.s. \quad \begin{pmatrix} V_x \\ V_y \end{pmatrix} \neq \begin{pmatrix} -\frac{1}{2}\beta_{0,1}\beta_{0,2}^{-1} \\ \beta_0 - \frac{1}{4}\beta_{0,1}^2\beta_{0,2}^{-1} \end{pmatrix}$$

where V_x and V_y are the coordinates of $\boldsymbol{V}$. Alternatively, the null hypothesis may be simply stated as,

$$H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0, \tag{8}$$

where $\beta_0' = (\beta_{0,0}, \beta_{0,1}, \beta_{0,2})$ and $V_{0x} = -\frac{1}{2}\beta_{0,1}\beta_{0,2}^{-1}$ and $V_{0y} = \beta_{0,0} - \frac{1}{4}\beta_{0,1}^2\beta_{0,2}^{-1}$. Power functions of the indirect F test will be presented in Section 3.1 and Section 3.2 for the random intercept model (5) and the random slope model (6).

The two null hypotheses (7) and (8) are not necessarily equivalent. For the $x-$coordinate of the vertex, $V_x = -\frac{1}{2}\beta_1\beta_2^{-1}$; if β_2 is shifted by amount Δ, V_x can remain unchanged by changing β_1 with certain amount Δ , i.e. the change of β_2 can be offset by the change of β_1. Similarly, for the $y-$coordinate of the vertex, $V_y = \beta_0 - \frac{1}{4}\beta_1^2\beta_2^{-1}$; if the ratio $\beta_1^2\beta_2^{-1}$ is shifted amount Δ, V_y can remain the same by shifting the same amount Δ for β_0, i.e. the change of ratio $\beta_1^2\beta_2^{-1}$ can be offset by the change of β_0. In conclusion, "do not reject $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$" implies "do not reject $H_0 : \boldsymbol{V} = \boldsymbol{V}_0$", while "reject $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$" does not necessarily imply "reject $H_0 : \boldsymbol{V} = \boldsymbol{V}_0$".

3.1 Power Function of F Test for Random Intercept Model

To derive the power function for testing the hypothesis (8) with respect to $\boldsymbol{\beta}$ for the random intercept model (5), a randomized block design with random block can be presented since it is applicable to model the longitudinal data. Repeated measurements on a single sample from a population can be represented by a randomized block model,

$$y_{ij} = \mu_{..} + \alpha_{0i} + \tau_j + \varepsilon_{ij} \tag{9}$$

where, y_{ij} is the response at jth occasion for ith subject with $E(y_{ij}) = \mu_{..} + \tau_j$, $\mu_{..}$ is a constant for grand mean of all the observations, α_{0i} is the random effect, and α_{0_i} are independent $N(0, \sigma_{\alpha_0}^2)$, τ_j is the fixed effect, and τ_j's are constants subject to the restriction $\Sigma\tau_j = 0$, ε_{ij} are independent $N(0, \sigma_e^2)$, and independent of the α_{0i}, $i = 1, 2, ..., N$; $j = 1, 2, ...n_i$. N is sample size, and n_i is number of occasions assuming to be same for all the subjects as n.

Testing hypothesis (8) for random intercept model (5) is equivalent to testing a potential quadratic trend for the randomized block model (9). The null hypothesis of no potential trend for model (9) can be stated as $H_0 : \boldsymbol{\tau} = \mathbf{0}$. Under the assumption of the compound symmetry covariance structure, $\Sigma_{\boldsymbol{y}_i} = \sigma_e^2 \cdot \boldsymbol{I}_{n\times n} + \sigma_{\alpha_0}^2 \cdot \boldsymbol{J}_{n\times n}$, the test statistic for $H_0 : \boldsymbol{\tau} = 0$ is an F statistic based on sum of squares error and sum of squares treatment (occasion), where $SS(Occasion) = N \cdot \sum_j (\bar{y}_{.j} - \bar{y}_{..})^2$ and $SS(Error) = \sum_i \sum_j (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2$. The F statistic is exact and uniformly most powerful (UMP) [Casella and Berger, 2002]. Sum of squares occasion can be partitioned into sum of squares for polynomial trend using Gram-Schmidt orthonormalization or the Cholesky factorization of $\boldsymbol{X}_i'\boldsymbol{X}_i$, where $\boldsymbol{X}_i$ is the model matrix for subject i.

The null hypothesis $H_0 : \boldsymbol{\beta}_{q\times 1} = \boldsymbol{\beta_0}$, testing a potential $(q-1)^{\text{th}}$ order polynomial trend, is a component of the null hypothesis $H_0 : \boldsymbol{\tau} = 0$, testing all polynomial trends; then the sum of squares for $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta_0}$ can be obtained from $H_0 : \boldsymbol{L\tau} = 0$ by reparametrization, where $\boldsymbol{L}$ contains coefficients for orthogonal polynomial contrasts. Denote l_m as the mth row for $\boldsymbol{L}$, the sum of squares for each contrast is,

$$SS(\text{Contrast}_k) = \frac{N \cdot (\sum_j l_{mj}\bar{y}_{.j})^2}{\sum_j l_{mj}^2} = \left(\boldsymbol{L}\hat{\boldsymbol{\beta}}\right)' \left(\boldsymbol{L}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{L}'\right)^{-1} \left(\boldsymbol{L}\hat{\boldsymbol{\beta}}\right) .$$

$SS(\text{Contrast}_k)/\sigma^2 \sim \chi^2(n, \lambda)$, where $\lambda = \left(\boldsymbol{L}\hat{\boldsymbol{\beta}}\right)' \left(\boldsymbol{L}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{L}'\right)^{-1} \left(\boldsymbol{L}\hat{\boldsymbol{\beta}}\right)/(2\sigma^2)$. Then the test is based on $F = \frac{MS(\text{Contrast}_k)}{MS(\text{Error})}$; it is an exact test [Khuri et al., 2011].

The generalized F statistic for testing $H_0 : \boldsymbol{\beta}_{q\times 1} = \boldsymbol{\beta_0}$ is,

$$F = \frac{(\boldsymbol{b} - \boldsymbol{\beta}_0)'((\sum_i \boldsymbol{X}_i'\Sigma_{\boldsymbol{y}_i}^{-1}\boldsymbol{X}_i)^{-1})^{-1}(\boldsymbol{b} - \boldsymbol{\beta}_0)}{q}, \tag{10}$$

where the numerator degrees of freedom is $\mathrm{ndf}_1 = q$ and the denominator degrees of freedom is $\mathrm{ddf}_1 = N \cdot (n-1) - (q-1)$; it is an approximate test. The non-centrality parameter is $\lambda_1 = (\boldsymbol{\beta} - \boldsymbol{\beta}_0)'((\sum_i \boldsymbol{X}_i' \Sigma_{\boldsymbol{y}_i}^{-1} \boldsymbol{X}_i)^{-1})^{-1}(\boldsymbol{\beta} - \boldsymbol{\beta}_0)$. Under H_0, $\lambda_1 = 0$; on the other hand, given H_a is true, $\lambda_1 > 0$. Therefore the power function is

$$\text{Power} \approx \text{Prob}\{F(\mathrm{ndf}_1,\ \mathrm{ddf}_2,\ \lambda_1) > F_{1-\alpha,\ \mathrm{ndf}_1,\ \mathrm{ddf}_1}\}$$

where λ_1 is the value of the non-centrality parameter and $F_{1-\alpha}$ is the critical value of the central F at the designated α level.

3.2 Power Function of F Test for Random Slope Model

For random slope model (6), the variance of each response and covariance between any two responses of same subject are given in equation (3). The test of $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta_0}$ using a F-type statistic (10) is an approximate test in that the denominator degrees of freedom ddf_{1a} is not exact. The power function for the approximate F test is

$$\text{Power} \approx \text{Prob}\{F(\mathrm{ndf}_1,\ \mathrm{ddf}_{1a},\ \lambda_1) > F_{1-\alpha,\ \mathrm{ndf}_1,\ \mathrm{ddf}_{1a}}\}$$

where $F_{1-\alpha}$ is the critical value of the central F distribution with the approximate denominator degrees of freedom. Two main methods for computing denominator degrees of freedom are Satterthwaite and Kenward-Roger Method [Kenward and Roger, 1997].

3.3 Power Function for Chi-Square Test

The non-central chi-square distribution can be applied to compute power for the hypotheses (7), since $\boldsymbol{V}$ has an asymptotic multivariate normal distribution. $\hat{\boldsymbol{V}}'\Sigma_{\hat{\boldsymbol{V}}}^{-1}\hat{\boldsymbol{V}}$ distributes as a non-central chi-square with 2 degrees of freedom with the non-centrality parameter $\lambda_2 = (\boldsymbol{V} - \boldsymbol{V}_0)'\Sigma_{\hat{\boldsymbol{V}}}^{-1}(\boldsymbol{V} - \boldsymbol{V}_0)$. Namely, $\hat{\boldsymbol{V}}'\Sigma_{\hat{\boldsymbol{V}}}^{-1}\hat{\boldsymbol{V}} \overset{a}{\sim} \chi^2_{2,\lambda_2}$. Under the null hypothesis, the non-centrality parameter $\lambda_2 = 0$. The decision rule is reject the null hypothesis if

$$\begin{pmatrix} \hat{V}_x - V_{0x} \\ \hat{V}_y - V_{0y} \end{pmatrix}' \hat{\Sigma}_{\hat{\boldsymbol{V}}}^{-1} \begin{pmatrix} \hat{V}_x - V_{0x} \\ \hat{V}_y - V_{0y} \end{pmatrix} > \chi^2_{1-\alpha,2}\ , \tag{11}$$

otherwise do not reject the null hypothesis, where $\chi^2_{1-\alpha,2}$ is the critical value given test size level α and the estimated covariance $\hat{\Sigma}_{\hat{\boldsymbol{V}}}$, the consistent statistic for $\Sigma_{\hat{\boldsymbol{V}}}$. The power function for the test is

$$\text{Power} \approx \text{Prob}\left\{\chi^2(2,\ \lambda_2) > \chi^2_{1-\alpha,2}\right\}.$$

4. Power Analysis for the Difference of Location of Two Quadratic Growth Curves

In this section, we investigate the difference of vertices for growth curves from two independent samples, such as the control and treatment groups. Similar to the one sample case, two growth curve models are explored; one is the second-order random intercept model, and the other is the second-order random slope model. They are defined as follows,

Second-order mixed model with random intercept (random intercept model),

$$y_{ij} = \beta_0^{(\text{mid})} + \beta_0^{(\text{eff})} I_i + \beta_1^{(\text{mid})} t_{ij} + \beta_1^{(\text{eff})} I_i t_{ij} + \beta_2^{(\text{mid})} t_{ij}^2 + \beta_2^{(\text{eff})} I_i t_{ij}^2 + \alpha_{0i} + \varepsilon_{ij} \tag{12}$$

where

$$I_i = \begin{cases} -1 & \text{if } y_{ij} \text{ comes from control group C,} \\ +1 & \text{if } y_{ij} \text{ comes from treatment group T.} \end{cases}$$

is a dummy variable to indicate the group, $i = 1,\ldots,N,\ j = 1,\ldots,n_i,\ N = N_1 + N_2$ is the total number of individuals, N_1 and N_2 are sample sizes for treatment group and control group, n_i is the number of time measurements for subject i, β's are fixed regression coefficients, α_{0i} is a random effect, $\alpha_{0i} \sim N(0,\sigma^2_{\alpha_0})$, $0 < \sigma^2_{\alpha_0} < \infty$, assuming the variance for individual across groups are same, i.e. homogeneous variances, ε_{ij} is the random error term for the i^{th} individual at the j^{th} occasion, $\varepsilon_{ij} \sim N(0,\sigma^2_e)$, $0 < \sigma^2_e < \infty$, α_{0i} and ε_{ij} are independent, $\text{Cov}(\alpha_{0i},\varepsilon_{ij}) = 0$ for all i, y_{ij} denotes response at j^{th} occasion for the i^{th} individual, and t_{ij} is the time measurement. From model (12), the distinct models for the control and the treatment groups are

$$y_{ij} = \beta_0^{(\text{C})} + \beta_1^{(\text{C})} t_{ij} + \beta_2^{(\text{C})} t_{ij}^2 + \alpha_{0i} + \varepsilon_{ij} \qquad \text{for group C},$$

$$y_{ij} = \beta_0^{(\text{T})} + \beta_1^{(\text{T})} t_{ij} + \beta_2^{(\text{T})} t_{ij}^2 + \alpha_{0i} + \varepsilon_{ij} \qquad \text{for group T},$$

where

$$\beta_k^{(\text{C})} = \beta_k^{(\text{mid})} - \beta_k^{(\text{eff})} \qquad \text{for } k = 0,1,2,$$
$$\beta_k^{(\text{T})} = \beta_k^{(\text{mid})} + \beta_k^{(\text{eff})} \qquad \text{for } k = 0,1,2.$$

Second-order mixed model with random intercept and random slope (random slope model),

$$y_{ij} = \beta_0^{(\text{mid})} + \beta_0^{(\text{eff})} I_i + \beta_1^{(\text{mid})} t_{ij} + \beta_1^{(\text{eff})} I_i t_{ij} + \beta_2^{(\text{mid})} t_{ij}^2 + \beta_2^{(\text{eff})} I_i t_{ij}^2 + \alpha_{0i} + \alpha_{1i} t_{ij} + \varepsilon_{ij}, \quad (13)$$

where

$$I_i = \begin{cases} -1 & \text{if } y_{ij} \text{ comes from control group C}, \\ +1 & \text{if } y_{ij} \text{ comes from treatment group T}, \end{cases}$$

is a dummy variable to indicate the group. α_{0i} and α_{1i} are random effects, $\alpha_{0i} \sim N(0,\sigma^2_{\alpha_0})$, $\alpha_{1i} \sim N(0,\sigma^2_{\alpha_1})$, $0 < \sigma^2_{\alpha_0} < \infty$, assuming the variances for individual are homogeneous. ε_{ij}, β_0's, n, N, y_{ij} and t_{ij} are defined the same as in model (12), α_{0i}, α_{1i} are independent of ε_{ij}, $\text{Cov}(\alpha_{0i},\varepsilon_{ij}) = 0$, and $\text{Cov}(\alpha_{1i},\varepsilon_{ij}) = 0$. From model (13), the distinct models for control and treatment group are,

$$y_{ij} = \beta_0^{(\text{C})} + \beta_1^{(\text{C})} t_{ij} + \beta_2^{(\text{C})} t_{ij}^2 + \alpha_{0i} + \alpha_{1i} t_{ij} + \varepsilon_{ij} \qquad \text{for group C},$$

$$y_{ij} = \beta_0^{(\text{T})} + \beta_1^{(\text{T})} t_{ij} + \beta_2^{(\text{T})} t_{ij}^2 + \alpha_{0i} + \alpha_{1i} t_{ij} + \varepsilon_{ij} \qquad \text{for group T},$$

where

$$\beta_k^{(\text{C})} = \beta_k^{(\text{mid})} - \beta_k^{(\text{eff})} \qquad \text{for } k = 0,1,2,$$
$$\beta_k^{(\text{T})} = \beta_k^{(\text{mid})} + \beta_k^{(\text{eff})} \qquad \text{for } k = 0,1,2.$$

Power analysis plays an important role to reject the null hypothesis of identical vertex for two groups given that the vertices of two groups are actually different. The power function is interesting to be developed for testing the difference of two vertices. Consider the null hypothesis,

$$H_0 : \boldsymbol{V}^{(\text{C})} = \boldsymbol{V}^{(\text{T})} \text{ v.s. } H_a : \boldsymbol{V}^{(\text{C})} \neq \boldsymbol{V}^{(\text{T})} \quad (14)$$

where $\boldsymbol{V}^{(\text{C})}$ and $\boldsymbol{V}^{(\text{T})}$ are distinct vertices of control and treatment groups. Since the vertices are nonlinear functions of $\boldsymbol{\beta}$, the null hypothesis can also be expressed as

$$H_0 : \begin{pmatrix} \frac{-\beta_1^{(\text{C})}}{2\beta_2^{(\text{C})}} \\ \beta_0^{(\text{C})} - \frac{[\beta_1^{(\text{C})}]^2}{4\beta_2^{(\text{C})}} \end{pmatrix} = \begin{pmatrix} \frac{-\beta_1^{(\text{T})}}{2\beta_2^{(\text{T})}} \\ \beta_0^{(\text{T})} - \frac{[\beta_1^{(\text{T})}]^2}{4\beta_2^{(\text{T})}} \end{pmatrix}.$$

Under some conditions,

$$H_0 : \boldsymbol{\beta}^{(\mathrm{C})} = \boldsymbol{\beta}^{(\mathrm{T})} \text{ v.s. } H_a : \boldsymbol{\beta}^{(\mathrm{C})} \neq \boldsymbol{\beta}^{(\mathrm{T})} \tag{15}$$

is an equivalent hypothesis to (14). Therefore the difference of two vertices may be tested either indirectly by an F test with respect to regression coefficients $\boldsymbol{\beta}$'s or directly by a chi-square test with regard to vertices $\boldsymbol{V}$'s.

Comparing the hypotheses (14) and (15), provided that the quadratic terms of two populations are equal, $\beta_2^{(\mathrm{C})} = \beta_2^{(\mathrm{T})} = \beta_2$, the null hypothesis (15) becomes

$$H_0 : \begin{pmatrix} \frac{-\beta_1^{(\mathrm{C})}}{2\beta_2} \\ \beta_0^{(\mathrm{C})} - \frac{[\beta_1^{(\mathrm{C})}]^2}{4\beta_2} \end{pmatrix} = \begin{pmatrix} \frac{-\beta_1^{(\mathrm{T})}}{2\beta_2} \\ \beta_0^{(\mathrm{T})} - \frac{[\beta_1^{(\mathrm{T})}]^2}{4\beta_2} \end{pmatrix}.$$

For the $x-$coordinate of the vertex, if $\beta_1^{(\mathrm{C})} = \beta_1^{(\mathrm{T})}$, then $V_x^{(\mathrm{C})} = V_x^{(\mathrm{T})}$ and vice versa. Similarly, for the $y-$coordinate of the vertex, if the $\beta_1^{(\mathrm{C})} = \beta_1^{(\mathrm{T})}$ and $\beta_0^{(\mathrm{C})} = \beta_0^{(\mathrm{T})}$ then $V_y^{(\mathrm{C})} = V_y^{(\mathrm{T})}$. Therefore the two null hypotheses $H_0 : \boldsymbol{\beta}^{(\mathrm{C})} = \boldsymbol{\beta}^{(\mathrm{T})}$ and $H_0 : \boldsymbol{V}^{(\mathrm{C})} = \boldsymbol{V}^{(\mathrm{T})}$ are necessarily equivalent. More specifically, comparing a chi-square statistic χ_p^2 with p degrees of freedom, and a F statistic $F_{p,\,q}$ with numerator degrees of freedom p and denominator degrees of freedom q, when q tends to infinity, $\chi_p^2 \to p \cdot F_{p,\,q}$ [Casella and Berger, 2002]. On the other hand, if the quadratic terms of two samples are different, $\beta_2^{(\mathrm{C})} \neq \beta_2^{(\mathrm{T})}$, the two null hypothesis $H_0 : \boldsymbol{\beta}^{(\mathrm{C})} = \boldsymbol{\beta}^{(\mathrm{T})}$ and $H_0 : \boldsymbol{V}^{(\mathrm{C})} = \boldsymbol{V}^{(\mathrm{T})}$ are not necessarily equivalent. Since for the $x-$coordinate of vertex, the ratio $\frac{\beta_1^{(\mathrm{C})}}{\beta_2^{(\mathrm{C})}} = \frac{\beta_1^{(\mathrm{T})}}{\beta_2^{(\mathrm{T})}}$ leads to $V_x^{(\mathrm{C})} = V_x^{(\mathrm{T})}$, i.e. even $\beta_1^{(\mathrm{C})} \neq \beta_1^{(\mathrm{T})}$ and $\beta_2^{(\mathrm{C})} \neq \beta_2^{(\mathrm{T})}$ may result in $V_x^{(\mathrm{C})} = V_x^{(\mathrm{T})}$. Similarly, for the $y-$coordinate of the vertex, the difference of the ratios $\frac{[\beta_1^{(\mathrm{C})}]^2}{\beta_2}$ and $\frac{[\beta_1^{(\mathrm{T})}]^2}{\beta_2}$ can be offset by the difference of $\beta_0^{(\mathrm{C})}$ and $\beta_0^{(\mathrm{T})}$. Namely, even $\beta_0^{(\mathrm{C})} \neq \beta_0^{(\mathrm{T})}$, $\beta_1^{(\mathrm{C})} \neq \beta_1^{(\mathrm{T})}$ and $\beta_2^{(\mathrm{C})} \neq \beta_2^{(\mathrm{T})}$ may not preclude $V_y^{(\mathrm{C})} = V_y^{(\mathrm{T})}$.

4.1 Power Function of F Test for Growth Curves with Common Quadratic Term

Repeated measurements on two independent samples, control and treatment, can be presented by a split plot design model,

$$y_{ijk} = \mu_{\ldots} + \alpha_{0i(k)} + \tau_j + \gamma_k + (\tau\gamma)_{jk} + \varepsilon_{ijk}, \tag{16}$$

where y_{ijk} is the response at j^{th} occasion for i^{th} subject from group k, $\mu_{\ldots}$ is a constant for grand mean of all the observations, $\alpha_{0i(k)}$ is the random effect for subject i nested within group k, and $\alpha_{0i(k)} \sim N(0, \sigma_{\alpha_0}^2)$, τ_j is the fixed time effect and τ_j's are constants subject to the restriction $\sum \tau_j = 0$, γ_k is the fixed group effect and γ_k's are constants subject to the restriction $\sum \gamma_k = 0$, $\varepsilon_{ijk} \sim N(0, \sigma_e^2)$, and independent of the $\alpha_{0i(k)}$, $i = 1, 2, \ldots, N$; $N = N_1 + N_2$; $j = 1, 2, \ldots n_i$; and $k = 1, 2$. N is the total sample size, N_1 and N_2 are sample sizes for control and treatment groups and n_i is the number of occasions assuming to be same for all the subjects as n.

The corresponding 2^{nd} order random intercept model with compound symmetry covariance structure with respect to model (16) is model (12), Given the common quadratic term for control and treatment groups, $\beta_2^{(\mathrm{C})} = \beta_2^{(\mathrm{T})} = \beta_2$, the equivalent null hypothesis to test $H_0 : \beta^{(\mathrm{C})} = \beta^{(\mathrm{T})}$ with regard to the F test is $H_0 : \boldsymbol{C}_1\boldsymbol{\beta} = \boldsymbol{0}$, where

$$\boldsymbol{C}_1 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_0^{(\mathrm{C})} \\ \beta_1^{(\mathrm{C})} \\ \beta_0^{(\mathrm{T})} \\ \beta_1^{(\mathrm{T})} \\ \beta_2 \end{pmatrix}$$

The F test statistic is,

$$F = \frac{(\boldsymbol{C}_1\hat{\boldsymbol{\beta}})'[\boldsymbol{C}_1(\sum_i \boldsymbol{X}_i'\hat{\Sigma}_{\boldsymbol{y}_i}^{-1}\boldsymbol{X}_i)^{-1}\boldsymbol{C}_1']^{-1}(\boldsymbol{C}_1\hat{\boldsymbol{\beta}})}{\text{rank}(\boldsymbol{C}_1)}, \tag{17}$$

with the non-centrality parameter $\lambda_3 = (\boldsymbol{C}_1\boldsymbol{\beta})'[\boldsymbol{C}_1(\sum_i \boldsymbol{X}_i'\Sigma_{\boldsymbol{y}_i}^{-1}\boldsymbol{X}_i)^{-1}\boldsymbol{C}_1']^{-1}(\boldsymbol{C}_1\boldsymbol{\beta})$, where $\Sigma_{\boldsymbol{y}_i} = \sigma_e^2 \cdot \boldsymbol{I}_{n\times n} + \sigma_{\alpha_0}^2 \cdot \boldsymbol{J}_{n\times n}$ and $\boldsymbol{X}_i$ is the model matrix for control group and treatment group,

$$\boldsymbol{X}_i^{(\mathrm{C})} = \begin{pmatrix} 1 & t_{i1} & 0 & 0 & t_{i1}^2 \\ 1 & t_{i2} & 0 & 0 & t_{i2}^2 \\ 1 & t_{i3} & 0 & 0 & t_{i3}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_{in} & 0 & 0 & t_{in}^2 \end{pmatrix}, \qquad \boldsymbol{X}_i^{(\mathrm{T})} = \begin{pmatrix} 0 & 0 & 1 & t_{i1} & t_{i1}^2 \\ 0 & 0 & 1 & t_{i2} & t_{i2}^2 \\ 0 & 0 & 1 & t_{i3} & t_{i3}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & t_{in} & t_{in}^2 \end{pmatrix}.$$

The numerator degrees of freedom is $\text{ndf}_2 = \text{rank}(\boldsymbol{C}_1)$, and the between-within denominator degrees of freedom $\text{ddf}_2 = N\cdot(n-1) - \text{rank}(\boldsymbol{C}_1)$, and the power function is,

$$\text{Power} \approx \text{Prob}\left\{F(\text{ndf}_2,\ \text{ddf}_2,\ \lambda_3) > F_{1-\alpha,\ \text{ndf}_2,\ \text{ddf}_2}\right\},$$

where $F_{1-\alpha}$ is the critical value for the central F distribution with Type I error rate α.

For the 2nd order random slope model (13), the test of $H_0 : \boldsymbol{\beta}^{(\mathrm{C})} = \boldsymbol{\beta}^{(\mathrm{T})}$ using an F-type statistic (17) is approximate since the denominator degrees of freedom ddf_{2a} are not known. The commonly used methods to compute the denominator degrees of freedom are Satterthwaite and Kenward-Roger. The power function for the approximate F test is

$$\text{Power} \approx \text{Prob}\left\{F(\text{ndf}_2,\ \text{ddf}_{2a},\ \lambda_3) > F_{1-\alpha,\ \text{ndf}_2,\ \text{ddf}_{2a}}\right\},$$

where $F_{1-\alpha}$ is the critical value of the central F distribution with the approximate denominator degrees of freedom.

4.2 Power Function of F Test for Growth Curves with Heterogeneity of the Quadratic Term

Assume the quadratic terms of two growth curves are not identical, $\beta_2^{(\mathrm{C})} \neq \beta_2^{(\mathrm{T})}$, for the 2nd order random intercept model (12), the equivalent null hypothesis to test $H_0 : \boldsymbol{\beta}^{(\mathrm{C})} = \boldsymbol{\beta}^{(\mathrm{T})}$ is $H_0 : \boldsymbol{C}_2\boldsymbol{\beta} = \boldsymbol{0}$ where

$$\boldsymbol{C}_2 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_0^{(\mathrm{C})} \\ \beta_1^{(\mathrm{C})} \\ \beta_2^{(\mathrm{C})} \\ \beta_0^{(\mathrm{T})} \\ \beta_1^{(\mathrm{T})} \\ \beta_2^{(\mathrm{T})} \end{pmatrix}$$

The F test statistic and the corresponding non-centrality parameter is,

$$F = \frac{(\boldsymbol{C}_2\hat{\boldsymbol{\beta}})'[\boldsymbol{C}_2(\sum_i \boldsymbol{X}_i'\hat{\Sigma}_{\boldsymbol{y}_i}^{-1}\boldsymbol{X}_i)^{-1}\boldsymbol{C}']^{-1}(\boldsymbol{C}_2\hat{\boldsymbol{\beta}})}{\text{rank}(\boldsymbol{C}_2)}, \tag{18}$$

$$\lambda_4 = (\boldsymbol{C}_2\boldsymbol{\beta})'[\boldsymbol{C}_2(\sum_i \boldsymbol{X}_i'\Sigma_{\boldsymbol{y}_i}^{-1}\boldsymbol{X}_i)^{-1}\boldsymbol{C}_2']^{-1}(\boldsymbol{C}_2\boldsymbol{\beta}),$$

where $\Sigma_{\boldsymbol{y}_i} = \sigma_e^2 \cdot \boldsymbol{I}_{n\times n} + \sigma_{\alpha_0}^2 \cdot \boldsymbol{J}_{n\times n}$ and $\boldsymbol{X}_i$ is the model matrix for control group or treatment group,

$$\boldsymbol{X}_i^{(\mathrm{C})} = \begin{pmatrix} 1 & t_{i1} & t_{i1}^2 & 0 & 0 & 0 \\ 1 & t_{i2} & t_{i2}^2 & 0 & 0 & 0 \\ 1 & t_{i3} & t_{i3}^2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_{in} & t_{in}^2 & 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{X}_i^{(\mathrm{T})} = \begin{pmatrix} 0 & 0 & 0 & 1 & t_{i1} & t_{i1}^2 \\ 0 & 0 & 0 & 1 & t_{i2} & t_{i2}^2 \\ 0 & 0 & 0 & 1 & t_{i3} & t_{i3}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & t_{in} & t_{in}^2 \end{pmatrix}.$$

The numerator degrees of freedom is $\mathrm{ndf}_3 = \mathrm{rank}(\boldsymbol{C}_2)$, and the denominator degrees of freedom $\mathrm{ddf}_3 = N\cdot(n-1) - \mathrm{rank}(\boldsymbol{C}_2)$ for between-within method [Schluchter and Elashoff, 1990], and the power function is,

$$\text{Power} \approx \text{Prob}\left\{F(\mathrm{ndf}_3,\ \mathrm{ddf}_3,\ \lambda_4) > F_{1-\alpha,\ \mathrm{ndf}_3,\ \mathrm{ddf}_3}\right\},$$

where $F_{1-\alpha}$ is the critical value for the central F distribution with Type I error rate α.

For the 2$^{\text{nd}}$ order random slope model (13), the F distribution for test statistic (18) becomes approximate in that the denominator degrees of freedom is not exact. The approximate power function is,

$$\text{Power} \approx \text{Prob}\left\{F(\mathrm{ndf}_3,\ \mathrm{ddf}_{3a},\ \lambda_4) > F_{1-\alpha},\ \mathrm{ndf}_3,\ \mathrm{ddf}_{3a}\right\},$$

where ddf_{3a} is the approximate denominator degrees of freedom that can be calculated by the Satterthwaite or Kenward-Roger method.

4.3 Power Function for Chi-Square Test

The non-central chi-square distribution is used to compute power for the null hypothesis for a direct test $H_0 : \boldsymbol{V}^{(\mathrm{C})} = \boldsymbol{V}^{(\mathrm{T})}$. Since $\hat{\boldsymbol{V}}^{(\mathrm{diff})} \overset{a}{\sim} N_2\left(\boldsymbol{V}^{(\mathrm{diff})}, \Sigma_{\hat{\boldsymbol{V}}^{(\mathrm{diff})}}\right)$, $\hat{\boldsymbol{V}}^{(\mathrm{diff})\prime}\Sigma^{-1}_{\hat{\boldsymbol{V}}^{(\mathrm{diff})}}\hat{\boldsymbol{V}}^{(\mathrm{diff})}$ distributes approximately as a non-central chi-square with 2 degrees of freedom with the non-centrality parameter $\lambda_5 = \boldsymbol{V}^{(\mathrm{diff})\prime}\Sigma^{-1}_{\hat{\boldsymbol{V}}^{(\mathrm{diff})}}\boldsymbol{V}^{(\mathrm{diff})}$. That is, $\hat{\boldsymbol{V}}^{(\mathrm{diff})\prime}\Sigma^{-1}_{\hat{\boldsymbol{V}}^{(\mathrm{diff})}}\hat{\boldsymbol{V}}^{(\mathrm{diff})} \overset{a}{\sim} \chi^2_{2,\lambda_5}$. Under null hypothesis, the non-centrality parameter $\lambda_5 = 0$. The approximate power function is,

$$\text{Power} \approx \text{Prob}\left\{\chi^2(2,\lambda_5) > \chi^2_{1-\alpha,2}\right\},$$

where $\chi^2_{1-\alpha,2}$ is the critical value given test size level α. Using $\hat{\Sigma}_{\hat{\boldsymbol{V}}^{(\mathrm{diff})}}$, the consistent statistic for $\Sigma_{\hat{\boldsymbol{V}}^{(\mathrm{diff})}}$, the decision rule is, reject the null hypothesis if

$$\begin{pmatrix} \boldsymbol{V}_x^{(\mathrm{T})} - \boldsymbol{V}_x^{(\mathrm{C})} \\ \boldsymbol{V}_y^{(\mathrm{T})} - \boldsymbol{V}_y^{(\mathrm{C})} \end{pmatrix}' \hat{\Sigma}^{-1}_{\hat{\boldsymbol{V}}^{(\mathrm{diff})}} \begin{pmatrix} \boldsymbol{V}_x^{(\mathrm{T})} - \boldsymbol{V}_x^{(\mathrm{C})} \\ \boldsymbol{V}_y^{(\mathrm{T})} - \boldsymbol{V}_y^{(\mathrm{C})} \end{pmatrix} > \chi^2_{1-\alpha,2},$$

otherwise do not reject the null hypothesis.

4.4 Power Results for Growth Curves with Common Quadratic Term

In this section, we investigate the indirect F test for $H_0 : \boldsymbol{\beta}^{(\mathrm{C})} = \boldsymbol{\beta}^{(\mathrm{T})}$ and the direct chi-square test for $H_0 : \boldsymbol{V}^{C} = \boldsymbol{V}^{(\mathrm{T})}$, assuming $\beta_2^{(\mathrm{C})} = \beta_2^{(\mathrm{T})}$. For the random intercept model (12) and parameter sets I, II, and III as shown in Table 1, twelve combinations of datasets are considered with different regression coefficients, variances of random effect, sample sizes, but the same time points. The six time points are $t_{ij} = 0, 1, 2, 3, 4, 5$; and sample sizes are selected to be 20 and 50. Two variance parameters chosen for the random effect are 10 and 80 with apparent difference between them. The vertices for parameter sets I, II, and III are

Table 1: Parameters for Power Analysis

		β_0	β_1	β_2	Vertex	Within
Parameter I	Control	6.05	3.0	-0.2	(7.5, 17.3)	No
	Treatment	4.5	3.3	-0.2	(8.25, 18.1125)	No
Parameter II	Control	5	1.7	-0.1	(8.5, 12.225)	No
	Treatment	4	1.9	-0.1	(9.5, 13.025)	No
Parameter III	Control	10	1.15	-0.06	(9.5833, 15.51)	No
	Treatment	9.5	1.45	-0.06	(12.0833, 18.26)	No
Parameter IV	Control	2	8	-1	(4, 18)	Yes
	Treatment	2	8.1	-1	(4.05, 18.4025)	Yes

outside the scope of occasions; while the vertices for parameter set IV is within the scope of occasions.

The simulated power and confidence intervals as well as the theoretical power are displayed in Table 2 and 3. In the tables, parameter sets (a) have the variances $\sigma^2_{\alpha_0} = 10$, $\sigma^2_e = 5$, and parameter sets (b) have $\sigma^2_{\alpha_0} = 80$, $\sigma^2_e = 5$. For parameter sets I, II, and III, with the smaller random effect variance $\sigma^2_{\alpha_0} = 10$, the F test has higher power than the chi-square test for every combination. When the variance of the random effect is larger, $\sigma^2_{\alpha_0} = 80$, it is more obvious that the F test has higher power than the chi-square test for every combination; and the power for both the F and the chi-square tests increases. Then, the increase of the variance $\sigma^2_{\alpha_0}$ would result in a decrease of power for both F and chi-square test. Parameter set IV is for a random intercept model with x-value of the vertex within the scope of the model. In this condition, the results show that there is a small difference between the theoretical power of the chi-square test and the F test even for small sample size. However, for parameter sets I, II, and III, with vertices outside the scope of the occasions, all the asymptotic F power are greater than the power of the chi-square test. As the vertices move further away from parameter set I to parameter set III, the power for both the F test and the chi-square test become lower. Hence the further the vertices are away from the scope of the occasions, the F and chi-square power becomes smaller; and it affects the chi-square power more. The theoretical power of the F test is always between the lower and upper bounds of the simulated power, for the vertex both within and outside the scope of occasions. As the sample size increases, the power will increase as a consequence. However, the theoretical power of chi-square test is between the lower and upper bounds of the simulated power only when the vertex is within the scope of the model. Even worse, when the vertex is further outside the occasions, the simulated power of the chi-square test decreases dramatically; and the difference between the simulated power and the theoretical power of chi-square test is very large. Therefore, when the vertex is far away from the scope of occasions, the use of chi-square test should be given more attention. For all the conditions, increasing sample size will lead to an increase in power. Table 2 and 3 provide little useful information to compare the denominator degrees of freedom for F test, since the simulated model is random intercept model which has an exact denominator degrees of freedom; the three different degrees of freedom methods, between-within, Satterthwaite and Kenward-Roger, provide similar power.

The random slope models (13) are generated using the fixed regression parameters listed in Table 1 with variances $\sigma^2_e = 5$, $\sigma^2_{\alpha_0} = 10$, and $\sigma^2_{\alpha_1} = 5$. The results are displayed in Table 4. Compared to Table 2 and 3, in all the conditions, the theoretical power and simulated power decrease simutaneously. Hence, adding a random slope term in the model results in a decrease of power for both the F and chi-square tests. Other findings are similar as the result of random intercept model.

Table 2: Power for Random Intercept Model with Common Quadratic Term

Parameters	Sample Size	Method	Simulated Power	Lower Bound	Upper Bound	Theoretical Power
I (a)	$N = 20$	BWF	0.374	0.344	0.404	0.386
		KRF	0.366	0.336	0.396	0.379
		SATF	0.369	0.339	0.399	0.376
		Chisq	0.208	0.183	0.233	0.329
	$N = 50$	BWF	0.771	0.745	0.797	0.785
		KRF	0.763	0.737	0.789	0.781
		SATF	0.766	0.739	0.792	0.767
		Chisq	0.708	0.680	0.736	0.700
I (b)	$N = 20$	BWF	0.339	0.310	0.368	0.339
		KRF	0.327	0.298	0.356	0.333
		SATF	0.326	0.297	0.355	0.330
		Chisq	0.165	0.142	0.188	0.280
	$N = 50$	BWF	0.725	0.697	0.753	0.718
		KRF	0.718	0.690	0.746	0.715
		SATF	0.718	0.690	0.746	0.714
		Chisq	0.635	0.605	0.665	0.616
II (a)	$N = 20$	BWF	0.212	0.187	0.237	0.188
		KRF	0.204	0.179	0.229	0.185
		SATF	0.207	0.182	0.232	0.184
		Chisq	0.061	0.046	0.076	0.149
	$N = 50$	BWF	0.431	0.400	0.462	0.420
		KRF	0.428	0.397	0.459	0.417
		SATF	0.429	0.398	0.460	0.406
		Chisq	0.163	0.140	0.186	0.317
II (b)	$N = 20$	BWF	0.169	0.146	0.192	0.170
		KRF	0.160	0.137	0.183	0.167
		SATF	0.160	0.137	0.183	0.166
		Chisq	0.042	0.030	0.054	0.131
	$N = 50$	BWF	0.374	0.344	0.404	0.374
		KRF	0.370	0.109	0.151	0.372
		SATF	0.369	0.339	0.399	0.370
		Chisq	0.130	0.109	0.151	0.270

5. Application: Tell Language Efficacy for Preschoolers with Developmental Speech and Language Impairment

We apply the direct F test and the indirect chi-square test for vertices on a study of growth of language and early literacy skills in preschoolers who have developmental speech and language impairment.

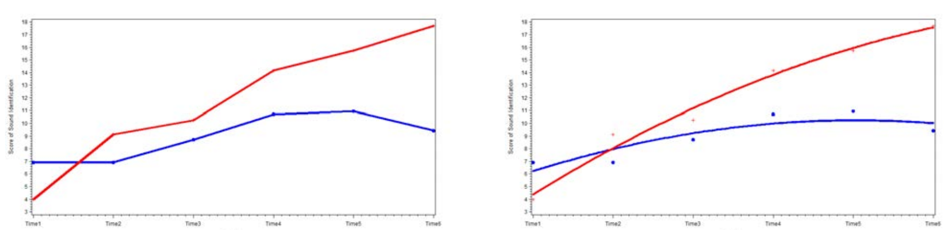
U.S. Department of Education data for the Individuals with Disabilities Education Act (IDEA) reported that 13% of four-year olds and five-year olds are receiving special education services in preschool and that 82% of these children show developmental speech and language impairment (DSLI) as a primary diagnosis [Wilcox et al., 2011]. One of these studies is examining the efficacy of “Teaching Early Literacy and Language”(TELL) curriculum in promoting the early literacy and oral language growth trajectories of preschoolers with DSLI. The variables in the TELL curriculum include a series of instructions, scripted teaching activities, materials for implementation of oral language and early literacy activities, and professional development for teachers. They targeted one specific skill (e.g., vocabulary, identification of beginning sounds in a word) or small set of skills (e.g., inferential language, print concepts, letter sounds and identification) over a relatively short period of time (e.g., weeks). The TELL curriculum has shown positive results in oral language and early literacy activities in an earlier small randomized controlled trial. Researchers compare those trajectories of children who were enrolled in the TELL curriculum with those who were randomly assigned to control classes [Wilcox et al., 2011].

We focused on one specific item from TELL curriculum, Curriculum Based Measurement (CBM) Letter Sound Identification (SoundID) in year 2011. Fifty-seven children

Table 3: Power for Random Intercept Model with Common Quadratic Term

Parameters	Sample Size	Method	Simulated Power	Lower Bound	Upper Bound	Theoretical Power
III (a)	$N = 20$	BWF	0.345	0.315	0.275	0.337
		KRF	0.333	0.304	0.362	0.331
		SATF	0.335	0.306	0.364	0.328
		Chisq	0.025	0.015	0.035	0.120
	$N = 50$	BWF	0.720	0.692	0.748	0.716
		KRF	0.714	0.686	0.742	0.712
		SATF	0.716	0.688	0.744	0.698
		Chisq	0.110	0.091	0.129	0.239
III (b)	$N = 20$	BWF	0.359	0.329	0.389	0.332
		KRF	0.346	0.316	0.376	0.326
		SATF	0.345	0.315	0.375	0.323
		Chisq	0.043	0.030	0.056	0.116
	$N = 50$	BWF	0.716	0.688	0.744	0.708
		KRF	0.711	0.683	0.739	0.705
		SATF	0.711	0.683	0.739	0.703
		Chisq	0.089	0.071	0.107	0.228
IV (a)	$N = 20$	BWF	0.080	0.063	0.097	0.081
		KRF	0.079	0.058	0.090	0.081
		SATF	0.077	0.060	0.094	0.081
		Chisq	0.079	0.062	0.096	0.082
	$N = 50$	BWF	0.141	0.119	0.163	0.134
		KRF	0.135	0.114	0.156	0.133
		SATF	0.137	0.116	0.158	0.133
		Chisq	0.142	0.120	0.164	0.134
IV (b)	$N = 20$	BWF	0.082	0.065	0.099	0.077
		KRF	0.074	0.058	0.090	0.077
		SATF	0.074	0.058	0.090	0.077
		Chisq	0.083	0.066	0.100	0.078
	$N = 50$	BWF	0.129	0.108	0.150	0.123
		KRF	0.129	0.108	0.150	0.122
		SATF	0.128	0.107	0.149	0.122
		Chisq	0.131	0.110	0.152	0.123

with DSLI nested under teacher are randomly assigned to offer the TELL curriculum or accept those with business as usual (BAU). The efficacy variable, SoundID test score, was obtained by six follow-up time measurements (1, 2.25, 3.5, 5.25, 6.5, 7.75 months). The profile plot and smoothed profile plot for children with DSLI receiving TELL curriculum and BAU are shown in Figure 1, which indicate the quadratic curve for the trend.



(a) Profile Plot for TELL Efficacy Example (b) Smoothed Plot for TELL Efficacy Example

Figure 1: Profile and Smoothed Plots for TELL Efficacy Example

The joint random slope model for the TELL and control group is

$$
\begin{aligned}
y_{ijkl} = & \beta_0^{(\mathrm{mid})} + \beta_0^{(\mathrm{eff})} \cdot I_l + \beta_1^{(\mathrm{mid})} \cdot t_{ijkl} + \beta_0^{(\mathrm{eff})} \cdot I_l \cdot t_{ijkl} + \beta_2^{(\mathrm{mid})} \cdot t_{ijkl}^2 + \beta_2^{(\mathrm{eff})} \cdot I_l \cdot t_{ijkl}^2 \\
& + \beta_{c1} \cdot x_{1ijl} + \gamma_{0j(l)} + \alpha_{0i(jl)} + \alpha_{1i(jl)} t_{ijkl} + \varepsilon_{ijkl},
\end{aligned}
$$

Table 4: Power for Random Slope Model with Common Quadratic Term

Parameters	Sample Size	Method	Simulated Power	Lower Bound	Upper Bound	Theoretical Power
I (a)	$N = 20$	BWF	0.227	0.200	0.253	0.222
		KRF	0.186	0.161	0.210	0.210
		SATF	0.190	0.166	0.214	0.210
		Chisq	0.091	0.073	0.109	0.208
	$N = 50$	BWF	0.482	0.451	0.513	0.499
		KRF	0.470	0.439	0.501	0.489
		SATF	0.476	0.445	0.507	0.489
		Chisq	0.375	0.345	0.405	0.463
II (a)	$N = 20$	BWF	0.166	0.143	0.189	0.117
		KRF	0.140	0.118	0.162	0.112
		SATF	0.148	0.126	0.170	0.112
		Chisq	0.028	0.018	0.038	0.107
	$N = 50$	BWF	0.247	0.220	0.274	0.232
		KRF	0.247	0.220	0.247	0.227
		SATF	0.239	0.213	0.265	0.227
		Chisq	0.075	0.059	0.091	0.204
III (a)	$N = 20$	BWF	0.074	0.058	0.090	0.076
		KRF	0.061	0.046	0.076	0.074
		SATF	0.064	0.049	0.079	0.074
		Chisq	0.004	0.001	0.008	0.065
	$N = 50$	BWF	0.124	0.104	0.144	0.119
		KRF	0.111	0.091	0.131	0.117
		SATF	0.116	0.096	0.136	0.117
		Chisq	0.012	0.005	0.019	0.090
IV (a)	$N = 20$	BWF	0.066	0.051	0.081	0.051
		KRF	0.057	0.043	0.072	0.051
		SATF	0.060	0.045	0.075	0.051
		Chisq	0.061	0.046	0.076	0.051
	$N = 50$	BWF	0.058	0.043	0.073	0.053
		KRF	0.054	0.040	0.068	0.053
		SATF	0.054	0.040	0.068	0.053
		Chisq	0.058	0.043	0.073	0.054

where,

$$I_l = \begin{cases} 1 & \text{if } y_{ijkl} \text{ comes from the control group,} \\ 0 & \text{if } y_{ijkl} \text{ comes from the TELL group.} \end{cases}$$

y_{ijkl} is the sound identification score at the k^{th} time point for child i under teacher j and curriculum l; x_{1ijl} is a covariate of mother's education for child i under teacher j and curriculum l; $\gamma_{0j(l)}$ is the random effect of j^{th} teacher nested under curriculum, $\gamma_{0j(l)} \sim N(0,\ \sigma^2_{\gamma_0})$; $\alpha_{0i(jl)}$ and $\alpha_{1i(jl)}$ are the random intercept and random slope effects of i^{th} children nested under j^{th} teacher and l^{th} curriculum, $\alpha_{0i(jl)} \sim N(0,\ \sigma^2_{\alpha_0})$ and $\alpha_{1i(jl)} \sim N(0,\ \sigma^2_{\alpha_1})$; ε_{ijkl} is the random error term, $\varepsilon_{ijkl} \sim N(0,\ \sigma^2_e)$. The fitted regression model is

$$\hat{y}_{ijkl} = 1.766 + 1.109 \cdot I_l + 2.914 \cdot t_{ijkl} - 0.996 \cdot I_l \cdot t_{ijkl} - 0.113 \cdot t^2_{ijk} - 0.011 \cdot I_l \cdot t^2_{ijk} + 1.573 \cdot x_{1ijl},$$

with the estimates of variance components, $\sigma^2_{\alpha_0} = 38.209$, $\sigma^2_{\alpha_1} = 0.666$, and $\sigma^2_e = 7.136$. The estimated vertices are $(12.745,\ 20.430)$ and $(7.474,\ 10.145)$ for TELL and control groups respectively.

To compare the vertices from the TELL and the control groups for sound identification score, hypothesis testing is performed for a direct chi-square test $H_0 : \boldsymbol{V}^{(\text{C})} = \boldsymbol{V}^{(\text{T})}$, and a indirect F test $H_0 : \boldsymbol{\beta}^{(\text{C})} = \boldsymbol{\beta}^{(\text{T})}$. The test statistic of the chi-square test is $\chi^2_2 = 6.482$ with 2 degrees of freedom; and the p-value of the test is 0.039. At the significance level $\alpha = 0.05$, we reject the null hypothesis that the vertices from control and TELL group are identical, since the p-value is less than α. The test statistic of the F test is shown in Table 5 with different denominator degrees of freedom methods. All three p-values are less than the significance level $\alpha = 0.05$, therefore we reject the null hypothesis that the fixed regression

Table 5: F Test for the Difference of Vertices for Control and TELL Children

DDFM	Test Statistic	NDF	DDF	P-value
Between-Within	5.38	3	261	0.0013
Kenward-Roger	5.31	3	112	0.0019
Satterthwaite	5.38	3	112	0.0017

coefficients of TELL and control group are equivalent. The chi-square and F tests conclude the identical result.

6. Conclusion

Power functions were obtained for the test of difference of vertices in this project. Different power functions for chi-square and F test are applicable for quadratic growth curves. When the vertices are within the scope of occasions, both the F test and the chi-square test are valid to test the equality of the vertices of two groups. When the vertex is outside the scope of the model, the use of chi-square test should be given more attention. For the random intercept model, the larger the variance of random intercept, $\sigma^2_{\alpha_0}$, the lower the power for both F and chi-square tests. Increasing the sample size will always help to increase the power of both tests. For the random slope model, adding a random slope variance, $\sigma^2_{\alpha_0}$, the power of both tests will decrease as a consequence. When the fixed quadratic term, β_2, is close to zero, the vertex of the quadratic growth curve will be further away outside the occasions which will lead to reduce of power for both the F and the chi-square tests.

An interesting topic for further research can be dealing with vertices of quadratic growth curves under heterogeneity in the random effects population.

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