A simulation study for pairwise multiple comparisons under heteroscedasticity

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Abstract

Fiducial generalized pivotal quantity and the parametric bootstrap approach are a type of Monte Carlo method applied on observed data. These approaches can be applied in situations where samples or sample statistics are not easy to derive. In this paper, we consider multiple comparison produces for one-way ANOVA under unequal variances. We compare two methods based on fiducial generalized pivotal quantity and a parametric bootstrap approach. A Monte Carlo simulation study is conducted to evaluate type I error probabilities and powers of these methods under different scenarios.

Keywords: Fiducial generalized pivotal quantity, parametric bootstrap, pairwise multiple comparison.

1. Introduction

Consider the usual one-way fixed-effects model:

$$X_{ij} = \mu_i + \varepsilon_{ij};\ \ i = 1, \ldots, m; k = 1, \ldots, n_i$$

where the ε_{ij} are independent $N(0, \sigma_i^2)$ random variable. The means μ_i and variances σ_i^2 are assumed to be unknown. When variances are all equal, the one-way layout has been studied extensively in the literature. The existing methods for such cases include Scheffe's method, Bonferroni inequality-based method and Tukey-Kramer method, etc. When the variances are unequal, research of multiple comparisons is limited. Games and Howell (1976) developed a method for constructing simultaneous confidence interval based on the Behrens-Fisher statistic and an estimated degree of freedom. Kaiser and Bowden (1983) discussed simultaneous confidence intervals for all linear contrasts in a one-way layout with unequal variance. Witkovsky (2002) developed the methods to calculate the *p*-values required for deriving the conservative joint confidence interval estimates for the pairwise mean differences, referred to as the generalized Scheffé intervals. Chang et al. (2009) proposed simultaneous fiducial generalized confidence intervals for pairwise comparisons of means in the one-way fixed-effect model. Li (2009) investigated an exact method that extends Dunnett's method on the multiple comparisons with a control (MCC) to the case of unequal error variances when the ratios of population variances of knew treatments to that of the control group are known from previous experience. Xiong and Mu (2009) developed two kinds of simultaneous confidence intervals for one-way layout based on generalized pivotal quantities. Zhang (2015) presented the parametric bootstrap approach to a multiple comparison procedure. Sezer et al. (2015) compared confidence intervals based on classical and generalized approach for The Behrens-Fisher problem.

In this research, we compare two methods based on generalized approach by Xiong and Mu (2009) and the parametric bootstrap approach by Zhang (2015) for one-way fixed-effects model under heteroscedasticity and unequal sample size. A Monte Carlo simulation study is conducted to evaluate type I error probabilities and powers of these methods under different scenarios.

2. A review of the procedures

Let $X_{i1},\ldots,X_{in_i}$ be random variable from $N(\mu_i,\sigma_i^2), i=1,\ldots,k$. Suppose that all X_{ik} are independent, $k=1,\ldots,n_i$. The parameters of interest are $\mu_i-\mu_j$ for all $i<j$. Denote,

$$N=\sum_{i=1}^{m} n_i,\ \ \bar{X}_i=\frac{1}{n}\sum_{k=1}^{n_i} X_{ik} \text{ and } S_i^2=\frac{1}{n_i-1}\sum_{k=1}^{n_i}(X_{ik}-\bar{X}_i)^2\ ,=1,\ldots,m \qquad (1)$$

Recall that

$$\sqrt{n_i}\frac{\bar{X}_i-\mu_i}{\sigma_i}\sim N(0,1),\ \ \frac{(n_i-1)S_i^2}{\sigma_i^2}\sim\chi^2_{n_i-1}, i=1,\ldots,m \qquad (2)$$

are jointly independent.

2.1 Two methods based on fiducial generalized pivotal quantity (FGPQ)

Weerahandi (1993) defines a generalized pivotal quantity as a statistic that has a distribution free of unknown parameters and an observed value that does not depend on nuisance parameters. The possibility of exact confidence interval can be achieved by extending the definition of confidence interval. The generalized pivotal quantity is allowed to be a function of nuisance parameters. Combining Fisher's fiducial arguments and the generalized *p*-value approach, Hannig (2009) and Hannig et al. (2006) developed a fiducial recipe for generalized confidence intervals. Using the FGPQ, Xiong and Mu (2009) present two new kinds of simultaneous confidence interval of all-pairwise differences.

Let $E_i\sim N(0,1), C_i^2\sim\chi^2_{n_i-1}, i=1,\ldots,m$, be jointly independent and be independent of the observation $X_{ik}, i=1,\ldots,m, k=1,\ldots,n_i$. Using Equation (2), the FGPQs for μ_i and σ_i^2, $i=1,\ldots,m$ are obtained as

$$R_{\mu_i}=\bar{X}_i-\sqrt{\frac{n_i-1}{n_i}}\frac{S_iE_i}{C_i},\quad R_{\sigma_i^2}=\frac{(n_i-1)S_i^2}{C_i^2},\qquad i=1,\ldots,m$$

Some calculations yield that for all $i\neq j$,

$$R_{\mu_i}-R_{\mu_j}=\bar{X}_i-\sqrt{\frac{n_i-1}{n_i}}\frac{S_iE_i}{C_i}-\left(\bar{X}_j-\sqrt{\frac{n_j-1}{n_j}}\frac{S_jE_j}{C_j}\right)$$

$$E^*\left(R_{\mu_i}-R_{\mu_j}\right)=\bar{X}_i-\bar{X}_j,$$

$$\mathrm{Var}^*\left(R_{\mu_i}-R_{\mu_j}\right)=\frac{(n_i-1)S_i^2}{n_i(n_i-3)}+\frac{(n_j-1)S_j^2}{n_j(n_j-3)},$$

$$\zeta_{ij}=\mathrm{Var}\left(E^*\left(R_{\mu_i}-R_{\mu_j}\right)\right)=\frac{\sigma_i^2}{n_i}+\frac{\sigma_j^2}{n_j}$$

$$R_{\zeta_{ij}} = \frac{(n_i - 1)S_i^2}{n_i C_i^2} + \frac{(n_j - 1)S_j^2}{n_j C_j^2}$$

Where E^* and Var^* represent conditional expectation and conditional variance conditional on $T = (\bar{X}_1, \dots, \bar{X}_m, S_1^2, \dots, S_m^2)$, respectively, and $R_{\zeta_{ij}}$ is the FGPQ of ζ_{ij}. Note that the distribution of

$$max_{i<j} \left| \frac{\mu_i - \mu_j - E^*\left(R_{\mu_i} - R_{\mu_j}\right)}{\left(\mathrm{Var}^*\left(R_{\mu_i} - R_{\mu_j}\right)\right)^{1/2}} \right| \tag{3}$$

can be approximated by the conditional distributions of

$$Q_1 = max_{i<j} \left| \frac{R_{\mu_i} - R_{\mu_j} - E^*\left(R_{\mu_i} - R_{\mu_j}\right)}{\left(\mathrm{Var}^*\left(R_{\mu_i} - R_{\mu_j}\right)\right)^{1/2}} \right| \tag{4}$$

or

$$Q_2 = max_{i<j} \left| \frac{R_{\mu_i} - R_{\mu_j} - E^*\left(R_{\mu_i} - R_{\mu_j}\right)}{{R_{\zeta_{ij}}}^{1/2}} \right| \tag{5}$$

conditional on T. $Q_1$and Q_2 can be computed by the Monte Carlo method.

2.2 The parametric bootstrap (PB) method for multiple comparison

The parametric bootstrap (PB) approach is a type of Monte Carlo method applied on observed data (Efron and Tibshirani 1993). The parametric bootstrap involves sampling from the estimated models. That is, samples or sample statistics are generated from parametric models with the parameters replaced by their estimates. Krishnamoorthy et al. (2007) developed the PB approach for one-way ANOVA under unequal variances. The PB approach has been carried out to solve a number of problems when exact solutions are not available satisfactorily for example in Ma and Tian (2009), Krishnamoorthy and Lu (2010), Xu et al. (2013).

Zhang (2015) extended the PB approach for a multiple comparison procedure. He took common mean to be zero and developed the PB method for multiple comparisons as follows. Let $\bar{X}_{Bi} \sim N\left(0, \frac{S_i^2}{n_i}\right)$ and $S_{Bi}^2 \sim \frac{\chi^2_{n_i - 1}}{(n_i - 1)}$, $i = 1, \dots, k$. Hence,

$$T_{ij} = \frac{\left|\bar{X}_{Bi} - \bar{X}_{Bj}\right|}{\sqrt{\frac{S_{Bi}^2}{n_i} + \frac{S_{Bj}^2}{n_j}}} \tag{6}$$

has the same distribution as

$$T_{ij} = \frac{\left|\bar{X}_{Bi} - \bar{X}_{Bj}\right|}{\sqrt{\frac{S_i^2}{n_i(n_i-1)}\chi^2_{n_i-1} + \frac{S_j^2}{n_j(n_j-1)}\chi^2_{n_j-1}}} \text{ for } i < j, i, j = 1, \dots, r \qquad (7)$$

T_{ij} can be computed by the Monte Carlo method.

3. Simulation

In this section, we use simulation to compare the two methods based on fiducial generalized pivotal quantity and a parametric bootstrap approach under the assumption of heteroscedastic variances and unequal sizes. For a given sample size and parameter configurations, we generated 2000 observed vectors $(\bar{x}_1, \dots, \bar{x}_3; s_1^2, \dots, s_3^2)$ and used 2000 runs to estimate the p-value. The estimates of type I error rates and powers of tests for pairwise multiple comparisons under heteroscedasticity are presented in Table 1 and Table 2.

4. Conclusions

In this paper, we consider multiple comparison produces for one-way ANOVA under unequal variances. We compare two methods based on fiducial generalized pivotal quantity and a parametric bootstrap approach. Simulation studies show that the type I error of method based on PB approach are closer to the nominal level. The power of the method based on equation (4) test is best among the three tests.

5. References

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Table 1. Type I error rates of the proposed tests for multiple comparison procedure with $\alpha = 0.05$ and $(\mu_1, \mu_2, \mu_3) = (0,0,0)$.

$(\sigma_1^2, \sigma_2^2, \sigma_3^2)$	PB	Q1	Q2	PB	Q1	Q2
	$(n_1, n_2, n_3) = (5,5,5)$			$(n_1, n_2, n_3) = (10,10,10)$		
(1,1,1)	0.044	0.103	0.130	0.061	0.086	0.091
(2,2,2)	0.044	0.103	0.130	0.061	0.086	0.091
(1,2,3)	0.050	0.102	0.126	0.055	0.078	0.094
(3,2,1)	0.038	0.102	0.127	0.060	0.081	0.101
(1,3,5)	0.056	0.107	0.132	0.060	0.083	0.095
(5,3,1)	0.039	0.110	0.134	0.059	0.083	0.094
	$(n_1, n_2, n_3) = (10,20,30)$			$(n_1, n_2, n_3) = (25,50,75)$		
(1,1,1)	0.056	0.069	0.075	0.047	0.053	0.058
(2,2,2)	0.056	0.068	0.075	0.047	0.053	0.058
(1,2,3)	0.048	0.061	0.063	0.053	0.057	0.058
(3,2,1)	0.059	0.079	0.089	0.043	0.050	0.056
(1,3,5)	0.043	0.051	0.058	0.056	0.056	0.059
(5,3,1)	0.057	0.082	0.092	0.044	0.053	0.057
	$(n_1, n_2, n_3) = (30,30,30)$			$(n_1, n_2, n_3) = (50,50,50)$		
(1,1,1)	0.048	0.054	0.055	0.048	0.054	0.055
(2,2,2)	0.048	0.054	0.053	0.045	0.054	0.055
(1,2,3)	0.048	0.054	0.057	0.049	0.048	0.053
(3,2,1)	0.052	0.058	0.064	0.046	0.051	0.053
(1,3,5)	0.046	0.053	0.056	0.049	0.053	0.054
(5,3,1)	0.053	0.056	0.063	0.045	0.052	0.053
	$(n_1, n_2, n_3) = (50,100,150)$			$(n_1, n_2, n_3) = (100,100,100)$		
(1,1,1)	0.041	0.044	0.046	0.042	0.045	0.044
(2,2,2)	0.041	0.044	0.046	0.042	0.045	0.044
(1,2,3)	0.043	0.043	0.052	0.046	0.051	0.051
(3,2,1)	0.042	0.047	0.044	0.042	0.043	0.044
(1,3,5)	0.040	0.043	0.051	0.056	0.057	0.058
(5,3,1)	0.048	0.051	0.051	0.047	0.049	0.049

Table 2. The powers of the proposed tests for multiple comparison procedure with $\alpha = 0.05$.

$(\sigma_1^2, \sigma_2^2, \sigma_3^2)$	(μ_1, μ_2, μ_3)	PB	Q1	Q2	PB	Q1	Q2
		$n = (5,5,5)$			$n = (10,10,10)$		
(1,1,1)	(0.5,0,0)	0.094	0.181	0.213	0.174	0.225	0.248
	(1,0,0)	0.247	0.388	0.435	0.546	0.622	0.661
	(1.5,0,0)	0.496	0.664	0.706	0.896	0.917	0.928
	(2,0,0)	0.745	0.869	0.903	0.994	0.996	0.997
(0.2,0.4,0.6)	(0.5,0,0)	0.161	0.280	0.325	0.312	0.392	0.414
	(1,0,0)	0.495	0.644	0.686	0.877	0.909	0.919
	(1.5,0,0)	0.824	0.928	0.942	0.999	1.00	1.00
	(2,0,0)	0.971	0.995	0.998	1.00	1.00	1.00
		$n = (10,20,30)$			$n = (20,20,20)$		
(1,1,1)	(0.5,0,0)	0.236	0.271	0.294	0.380	0.421	0.431
	(1,0,0)	0.678	0.730	0.746	0.940	0.952	0.956
	(1.5,0,0)	0.963	0.973	0.979	1.00	1.00	1.00
	(2,0,0)	0.999	0.999	0.999	1.00	1.00	1.00
(0.2,0.4,0.6)	(0.5,0,0)	0.508	0.669	0.560	0.630	0.544	0.560
	(1,0,0)	0.980	0.986	0.988	1.00	0.986	0.988
	(1.5,0,0)	1.00	1.00	1.00	1.00	1.00	1.00
	(2,0,0)	1.00	1.00	1.00	1.00	1.00	1.00
		$n = (25,50,75)$			$n = (50,50,50)$		
(1,1,1)	(0.5,0,0)	0.505	0.526	0.533	0.696	0.704	0.713
	(1,0,0)	0.984	0.988	0.988	1.00	1.00	1.00
	(1.5,0,0)	1.00	1.00	1.00	1.00	1.00	1.00
	(2,0,0)	1.00	1.00	1.00	1.00	1.00	1.00
(0.2,0.4,0.6)	(0.5,0,0)	0.905	0.916	0.913	0.957	0.957	0.958
	(1,0,0)	1.00	1.00	1.00	1.00	1.00	1.00
	(1.5,0,0)	1.00	1.00	1.00	1.00	1.00	1.00
	(2,0,0)	1.00	1.00	1.00	1.00	1.00	1.00