Asymptotic Properties of Bootstrap Parameter Estimator for the AR(2) Model

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Abstract
This paper is the extension of our research about asymptotic distribution of the bootstrap parameter estimator for the AR(1) model. We investigate the asymptotic distribution of the bootstrap parameter estimator of a second order autoregressive AR(2) model by applying the delta method. The asymptotic distribution is the crucial property in inference of statistics. We conclude that the bootstrap parameter estimators of the AR(2) model asymptotically converge in distribution to the normal distribution.

Key Words: convergence, delta method, limiting distribution, multivariate normal distribution

1. Introduction

Consider the following stationary second order autoregressive AR(2) process:

$$X_t = \theta_1 X_{t-1} + \theta_2 X_{t-2} + \epsilon_t,$$

where ϵ_t is a zero mean white noise process with constant variance σ^2. Let the vector $\widehat{\boldsymbol{\theta}} = (\widehat{\theta_1}, \widehat{\theta_2})^T$ is the estimator of the parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$, and $\widehat{\boldsymbol{\theta}}^*$ be the bootstrap version of $\widehat{\boldsymbol{\theta}}$. Studying of estimation of the unknown parameter involves: (i) what estimator should be used? (ii) having chosen a particular estimator, is this consistent? (iii) how accurate is the chosen estimator? (iv) what is the asymptotic behaviour of such estimator? (v) what is the method used in proving the asymptotic properties?

Bootstrap is a general methodology for answering the second and third questions, while the delta method is one of tools used to answer the last two questions. Consistency theory is needed to ensure that the estimator is consistent to the actual parameter as desired, and thereof the asymptotic behaviour of such estimator will be studied. The consistency theories of parameter of autoregressive model have studied in Bibi and Aknounche (2010), Brockwell and Davis (1991), and Brouste and Kleptsyna (2014), and for bootstrap version of the same topic, see *e.g.* Bickel and Freedman (1981), Efron and Tibshirani (1993), Freedman (1985), Hardle, *et. al* (2003), Politis (2003), and Singh (1981). They deal with the bootstrap approximation in various senses (*e.g.,* consistency of estimator, simulation results, limiting distribution, applying of Edgeworth expansions, etc.), and they reported that the bootstrap works usually very well. The accuracy of the bootstrapping method for autoregressive model studied in Bose (1988) and Sahinler and Topuz (2007). They showed that the parameter estimates of the autoregressive model can be bootstrapped with accuracy that outperforms the normal approximation. The asymptotic result for the AR(1) model has been exhibited in Suprihatin *et. al* (2015). We concluded that the bootstrap parameter estimator for the AR(1) model converges in distribution to the normal distribution. A good perform of the bootstrap estimator is applied to study the asymptotic distribution of $\widehat{\boldsymbol{\theta}}^*$ using the delta method. We describe the asymptotic distribution of the autocovariance function and investigate the bootstrap limiting distribution of $\widehat{\boldsymbol{\theta}}^*$. Section 2 reviews

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the asymptotic distribution of estimator of mean and autocovariance function for the autoregression model. Section 3 describes the bootstrap and delta method. Section 4 deals with the main results, *i.e.* the asymptotic distribution of $\widehat{\boldsymbol{\theta}}^*$ by applying the delta method. Section 5 briefly describes the conclusions of the paper.

2. Estimator of Mean and Autocovariance for the Autoregressive Model

Suppose we have the observed values $X_1, X_2, \ldots, X_n$ from the stationary AR(2) process. Mean and autocovariance are two important statistics in investigating the consistency properties of the estimator $\widehat{\boldsymbol{\theta}} = \left(\widehat{\theta}_1, \widehat{\theta}_2\right)^T$ for the parameter $\boldsymbol{\theta}$ of the AR(2) model. A natural estimators for parameters mean, covariance and correlation function are $\widehat{\mu}_n = \overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$, $\widehat{\gamma}_n(h) = \frac{1}{n}\sum_{t=1}^{n-h}(X_{t+h} - \overline{X}_n)(X_t - \overline{X}_n)$, and $\widehat{\rho}_n(h) = \widehat{\gamma}_n/\widehat{\gamma}_n(0)$ respectively. These all three estimators are consistent (see, *e.g.*, Brockwell and Davis (1991) and Van der Vaart (2012). The following theorem describes the property of the estimator $\overline{X}_n$, is stated in Brockwell and Davis (1991).

Theorem 1. *If $\{X_t\}$ is stationary process with mean μ and autocovariance function $\gamma(\cdot)$, then as $n \to \infty$,*

$$Var\left(\overline{X}_n\right) = E\left(\overline{X}_n - \mu\right)^2 \to 0 \quad \textit{if}\ \gamma(n) \to 0,$$

and

$$nE\left(\overline{X}_n - \mu\right)^2 \to \sum_{j=-\infty}^{\infty} \gamma(h) \quad \textit{if} \sum_{j=-\infty}^{\infty} |\gamma(h)| < \infty.$$

It is not a loss of generality to assume that $\mu_X = 0$. Under some conditions (see, *e.g.,* Van der Vaart (2012), the sample autocovariance function can be written as

$$\widehat{\gamma}_n(h) = \frac{1}{n}\sum_{t=1}^{n-h} X_{t+h}X_t + O_p(1/n).$$

It can be seen that the sequence $\sqrt{n}\,(\widehat{\gamma}_n(h) - \gamma_X(h))$ having asymptotic behaviour depends only on $n^{-1}\sum_{t=1}^{n-h} X_{t+h}X_t$. Note that a change of $n - h$ by n or vice versa, is asymptotically negligible, so that, for simplicity of notation, we can equivalently study the average

$$\widetilde{\gamma}_n(h) = \frac{1}{n}\sum_{t=1}^{n} X_{t+h}X_t.$$

Both $\widehat{\gamma}_n(h)$ and $\widetilde{\gamma}_n(h)$ are unbiased estimators of $E(X_{t+h}X_t) = \gamma_X(h)$, under the condition that $\mu_X = 0$. Their asymptotic distribution then can be derived by applying a central limit theorem to the averages $\overline{Y}_n$ of the variables $Y_t = X_{t+h}X_t$. As in Van der Vaart (2012), the autocovariance function of the series Y_t can be written as

$$V_{h,h} = \kappa_4(\varepsilon)\gamma_X(h)^2 + \sum_g \gamma_X(g)^2 + \sum_g \gamma_X(g+h)\gamma_X(g-h),$$

where $\kappa_4(\varepsilon) = E\left(\varepsilon_1^4\right) - 3\left(E\left(\varepsilon_1^2\right)\right)^2$, the fourth cumulant of ε_t. The following theorem is due to Van der Vaart (2012) that gives the asymptotic distribution of the sequence $\sqrt{n}\,(\widehat{\gamma}_n(h) - \gamma_X(h))$.

Theorem 2. *If $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$ holds for an i.i.d. sequence ε_t with mean zero and $E\left(\varepsilon_t^4\right) < \infty$ and numbers ψ_j with $\sum_j |\psi_j| < \infty$, then $\sqrt{n}\,(\widehat{\gamma}_n(h) - \gamma_X(h)) \to_d N(0, V_{h,h})$.*

3. Bootstrap and Delta Method

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with common distribution F, and let $T(X_1, X_2, \ldots, X_n; F)$ be the specified random variable or statistic of interest, possibly depending upon the unknown distribution F. Let F_n denote the empirical distribution function of $X_1, X_2, \ldots, X_n$, *i.e.,* the distribution putting probability 1/n at each of the points $X_1, X_2, \ldots, X_n$. A bootstrap sample is defined to be a random sample of size n drawn from F_n, say $X^* = X_1^*, X_2^*, \ldots, X_n^*$. The bootstrap sample at first bootstrapping is usually denoted by X^{*1}. In general, the bootstrap sample at Bth bootstrapping is denoted by X^{*B}. The bootstrap data set $X^{*b} = X_1^{*b}, X_2^{*b}, \ldots, X_n^{*b}$, $b = 1, 2, \ldots, B$ consists of members of the original data set $X_1, X_2, \ldots, X_n$, some appearing zero times, some appearing once, some appearing twice, etc. The bootstrap method is to approximate the distribution of $T(X_1, X_2, \ldots, X_n; F)$ under F by that of $T(X_1^*, X_2^*, \ldots, X_n^*; F_n)$ under F_n.

Let a functional T is defined as $T(X_1, X_2, \ldots, X_n; F) = \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$, where $\widehat{\boldsymbol{\theta}}$ is the estimator for the coefficient $\boldsymbol{\theta}$ of a stationary AR(2) model. The bootstrap version of T is $T(X_1^*, X_2^*, \ldots, X_n^*; F_n) = \sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}})$, where $\widehat{\boldsymbol{\theta}}^*$ is a bootstrap version of $\widehat{\boldsymbol{\theta}}$ computed from sample bootstrap $X_1^*, X_2^*, \ldots, X_n^*$. The residuals bootstrapping procedure was proposed in Freedman (1985) to obtain $X_1^*, X_2^*, \ldots, X_n^*$ for the time series data. In bootstrap view, the key of bootstrap terminology says that the population is to the sample as the sample is to the bootstrap samples. Therefore, when we want to investigate the asymptotic distribution of bootstrap estimator $\widehat{\boldsymbol{\theta}}^*$, we investigate the distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}})$ contrast to the distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$. Thus, the bootstrap method is a device for estimating $P_F\big(\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \le x\big)$ by $P_{F_n}\big(\sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}}) \le x\big)$. We propose the delta method in estimating for such distribution.

The delta method consists of using a Taylor expansion to approximate a random vector of the form $\phi(T_n)$ by the polynomial $\phi(\theta) + \phi'(\theta)(T_n - \theta) + \cdots$ in $T_n - \theta$. This method is useful to deduce the limit law of $\phi(T_n) - \phi(\theta)$ from that of $T_n - \theta$, which is guaranteed by the following theorem, as stated in Van der Vaart (2000).

Theorem 3. *Let $\phi : \mathbf{D}_\phi \subset \mathbf{R}^k \to \mathbf{R}^m$ be a map defined on a subset of $\mathbf{R}^k$ and differentiable at θ. Let T_n be random vector taking their values in the domain of ϕ. If $r_n(T_n - \theta) \to_d T$ for numbers $r_n \to \infty$, then $r_n\big(\phi(T_n) - \phi(\theta)\big) \to_d \phi'_\theta(T)$. Moreover, $\Big| r_n\big(\phi(T_n) - \phi(\theta)\big) - \phi'_\theta\big(r_n(T_n - \theta)\big)\Big| \to_p 0$.*

Assume that $\widehat{\theta}_n$ is a statistic, and that ϕ is a given differentiable map. The bootstrap estimator for the distribution of $\phi(\widehat{\theta}_n - \phi(\theta)$ is $\phi(\widehat{\theta}_n^* - \phi(\widehat{\theta}_n)$. If the bootstrap is consistent for estimating the distribution of $\sqrt{n}(\widehat{\theta}_n - \theta)$, then it is also consistent for estimating the distribution of $\sqrt{n}\big(\phi(\widehat{\theta}_n) - \phi(\theta)\big)$, as given in the following theorem. The theorem is due to Van de Vaart (2000).

Theorem 4 (Delta Method For Bootstrap). *Let $\phi : \mathbf{R}^k \to \mathbf{R}^m$ be a measurable map defined and continuously differentiable in a neighborhood of θ. Let $\widehat{\theta}_n$ be random vector taking their values in the domain of ϕ that converge almost surely to θ. If $\sqrt{n}\big(\widehat{\theta}_n - \theta\big) \to_d T$, and $\sqrt{n}\big(\widehat{\theta}_n^* - \widehat{\theta}_n\big) \to_d T$ conditionally almost surely, then both $\sqrt{n}\big(\phi(\widehat{\theta}_n) - \phi(\theta)\big) \to_d \phi'_\theta(T)$ and $\sqrt{n}\big(\phi(\widehat{\theta}_n^*) - \phi(\widehat{\theta}_n)\big) \to_d \phi'_\theta(T)$ conditionally almost surely.*

4. Results

We now address our main results. The Yule-Walker equation system for the AR(2) model is,

$$\begin{pmatrix} \sum_{t=1}^n X_t^2 & \sum_{t=2}^n X_t X_{t-1} \\ \sum_{t=2}^n X_t X_{t-1} & \sum_{t=1}^n X_t^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \sum_{t=2}^n X_t X_{t-1} \\ \sum_{t=3}^n X_t X_{t-2} \end{pmatrix},$$

or

$$\begin{aligned} \theta_1\gamma_0 + \theta_2\gamma_1 &= \gamma_1 \\ \theta_1\gamma_1 + \theta_2\gamma_0 &= \gamma_2. \end{aligned}$$

Dividing both sides by $\gamma_0 > 0$ we obtain

$$\begin{aligned} \theta_1 + \theta_2\rho_1 &= \rho_1 \\ \theta_1\rho_1 + \theta_2 &= \rho_2. \end{aligned}$$

By the moment method, we obtain the estimator for $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$ as follows:

$$\begin{aligned} \widehat{\boldsymbol{\theta}} = \begin{pmatrix} \widehat{\theta}_1 \\ \widehat{\theta}_2 \end{pmatrix} &= \begin{pmatrix} 1 & \widehat{\rho}_1 \\ \widehat{\rho}_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{\rho}_1 \\ \widehat{\rho}_2 \end{pmatrix} \\ &= \frac{1}{1-\widehat{\rho}_1^2} \begin{pmatrix} 1 & -\widehat{\rho}_1 \\ -\widehat{\rho}_1 & 1 \end{pmatrix} \begin{pmatrix} \widehat{\rho}_1 \\ \widehat{\rho}_2 \end{pmatrix} \\ &= \frac{1}{1-\widehat{\rho}_1^2} \begin{pmatrix} \widehat{\rho}_1 - \widehat{\rho}_1\widehat{\rho}_2 \\ -\widehat{\rho}_1^2 + \widehat{\rho}_2 \end{pmatrix}. \end{aligned}$$

The estimator $\widehat{\theta}_1$ can be described as follows:

$$\begin{aligned} \widehat{\theta}_1 &= \frac{\widehat{\rho}_1 - \widehat{\rho}_1\widehat{\rho}_2}{1-\widehat{\rho}_1^2} \\ &= \frac{\frac{\sum_{t=2}^n X_t X_{t-1}}{\sum_{t=1}^n X_t^2} - \frac{\sum_{t=2}^n X_t X_{t-1}}{\sum_{t=1}^n X_t^2} \frac{\sum_{t=3}^n X_t X_{t-2}}{\sum_{t=1}^n X_t^2}}{1 - \left(\frac{\sum_{t=2}^n X_t X_{t-1}}{\sum_{t=1}^n X_t^2}\right)^2} \\ &= \frac{\sum_{t=2}^n X_t X_{t-1}\left(\sum_{t=1}^n X_t^2 - \sum_{t=3}^n X_t X_{t-2}\right)}{\left(\sum_{t=1}^n X_t^2\right)^2 - \left(\sum_{t=2}^n X_t X_{t-1}\right)^2}. \end{aligned}$$

According to Theorem 2, the random vector $\left(\frac{1}{n}\sum_{t=1}^n X_t^2, \frac{1}{n}\sum_{t=2}^n X_t X_{t-1}, \frac{1}{n}\sum_{t=3}^n X_t X_{t-2}\right)^T$ has limiting distribution

$$\begin{aligned} &\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{t=1}^n X_t^2 \\ \frac{1}{n}\sum_{t=2}^n X_t X_{t-1} \\ \frac{1}{n}\sum_{t=3}^n X_t X_{t-2} \end{pmatrix} - \begin{pmatrix} \gamma_X(0) \\ \gamma_X(1) \\ \gamma_X(2) \end{pmatrix} \right) \\ &\qquad \to_d N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} V_{0,0} & V_{0,1} & V_{0,2} \\ V_{1,0} & V_{1,1} & V_{1,2} \\ V_{2,0} & V_{2,1} & V_{2,2} \end{pmatrix} \right). \qquad (1) \end{aligned}$$

Moreover, $\widehat{\theta}_1$ can be expressed as $\phi_1\left(\sum_{t=1}^n X_t^2, \sum_{t=2}^n X_t X_{t-1}, \sum_{t=3}^n X_t X_{t-2}\right)$ for a measurable function $\phi_1 : \mathbf{R}^3 \to \mathbf{R}$ defined as $\phi_1(u,v,w) = \frac{v(u-w)}{u^2-v^2}$. The map ϕ_1 is differentiable with the derivative matrix

$$\begin{aligned} \phi_1' &= \begin{pmatrix} \frac{\partial}{\partial u}\phi_1(u,v,w) & \frac{\partial}{\partial v}\phi_1(u,v,w) & \frac{\partial}{\partial w}\phi_1(u,v,w) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{v(u^2+v^2-2uw)}{(u^2-v^2)^2} & \frac{(u-w)(u^2+v^2}{(u^2-v^2)^2} & \frac{-v}{u^2-v^2} \end{pmatrix}, \end{aligned}$$

and
$\phi'_{1(\gamma_X(0),\gamma_X(1),\gamma_X(2))} =$

$$\begin{pmatrix} \frac{-\gamma_X(1)\left(\gamma_X(0)^2+\gamma_X(1)^2-2\gamma_X(0)\gamma_X(2)\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{\left(\gamma_X(0)-\gamma_X(2)\right)\left(\gamma_X(0)^2+\gamma_X(1)^2\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{-\gamma_X(1)}{\gamma_X(0)^2-\gamma_X(1)^2} \end{pmatrix}.$$

Similarly, the estimator $\widehat{\theta}_2$ can be derived as follows:

$$\begin{aligned} \widehat{\theta}_2 &= \frac{\widehat{\rho}_1^2+\widehat{\rho}_2}{1-\widehat{\rho}_1^2} \\ &= \frac{-\left(\frac{\sum_{t=2}^n X_tX_{t-1}}{\sum_{t=1}^n X_t^2}\right)^2+\frac{\sum_{t=3}^n X_tX_{t-2}}{\sum_{t=1}^n X_t^2}}{1-\left(\frac{\sum_{t=2}^n X_tX_{t-1}}{\sum_{t=1}^n X_t^2}\right)^2} \\ &= \frac{-\left(\sum_{t=2}^n X_tX_{t-1}\right)^2+\sum_{t=1}^n X_t^2\sum_{t=3}^n X_tX_{t-2}}{\left(\sum_{t=1}^n X_t^2\right)^2-\left(\sum_{t=2}^n X_tX_{t-1}\right)^2}. \end{aligned}$$

Note that $\widehat{\theta}_2$ can be expressed as $\phi_2\left(\sum_{t=1}^n X_t^2, \sum_{t=2}^n X_tX_{t-1}, \sum_{t=3}^n X_tX_{t-2}\right)$ for a measurable function $\phi_2 : \mathbf{R}^3 \to \mathbf{R}$ defined as $\phi_2(u,v,w) = \frac{-v^2+uw}{u^2-v^2}$. The derivative matrix for ϕ_2 is

$$\begin{aligned} \phi'_2 &= \begin{pmatrix} \frac{\partial}{\partial u}\phi_2(u,v,w) & \frac{\partial}{\partial v}\phi_2(u,v,w) & \frac{\partial}{\partial w}\phi_2(u,v,w) \end{pmatrix} \\ &= \begin{pmatrix} \frac{2uv^2-u^2w-v^2w}{(u^2-v^2)^2} & \frac{2uv(w-u)}{(u^2-v^2)^2} & \frac{u}{u^2-v^2} \end{pmatrix}, \end{aligned}$$

and
$\phi'_{2(\gamma_X(0),\gamma_X(1),\gamma_X(2))} =$

$$\begin{pmatrix} \frac{2\gamma_X(0)\gamma_X(1)^2-\gamma_X(0)^2\gamma_X(2)-\gamma_X(1)^2\gamma_X(2)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{2\gamma_X(0)\gamma_X(1)\left(\gamma_X(2)-\gamma_X(0)\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{\gamma_X(0)}{\gamma_X(0)^2-\gamma_X(1)^2} \end{pmatrix}.$$

The next step, we investigate the asymptotic distribution of the random variables $\widehat{\theta}_1^*$ and $\widehat{\theta}_2^*$, the bootstrapped version of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ respectively. For simplicity of notation, let $A_1 = \frac{-\gamma_X(1)\left(\gamma_X(0)^2+\gamma_X(1)^2-2\gamma_X(0)\gamma_X(2)\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2}$, $A_2 = \frac{\left(\gamma_X(0)-\gamma_X(2)\right)\left(\gamma_X(0)^2+\gamma_X(1)^2\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2}$, and $A_3 = \frac{-\gamma_X(1)}{\gamma_X(0)^2-\gamma_X(1)^2}$. By applying Theorem 3, we obtain

$$\sqrt{n}\Bigg(\phi_1\Big(\tfrac{1}{n}\textstyle\sum_{t=1}^n X_t^2, \tfrac{1}{n}\sum_{t=2}^n X_{t-1}X_t, \tfrac{1}{n}\sum_{t=3}^n X_{t-2}X_t\Big) - \phi_1\big(\gamma_X(0), \gamma_X(1), \gamma_X(2)\big)\Bigg)$$

$$\begin{aligned}
&= \phi'_{1(\gamma_X(0),\gamma_X(1),\gamma_X(2))}\begin{pmatrix} \sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^n X_t^2 - \gamma_X(0)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^n X_{t-1}X_t - \gamma_X(1)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^n X_{t-2}X_t - \gamma_X(2)\Big) \end{pmatrix} + o_p(1) \\
&= \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}\begin{pmatrix} \sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^n X_t^2 - \gamma_X(0)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^n X_{t-1}X_t - \gamma_X(1)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^n X_{t-2}X_t - \gamma_X(2)\Big) \end{pmatrix} + o_p(1) \\
&= A_1\sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^n X_t^2 - \gamma_X(0)\Big) + A_2\sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^n X_{t-1}X_t - \gamma_X(1)\Big) + \\
&\quad A_3\sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^n X_{t-2}X_t - \gamma_X(2)\Big) + o_p(1).
\end{aligned}$$

In view of Theorem 3, if $(Z_1, Z_2, Z_3)^T$ possesses the multivariate normal distribution as in (1), then

$$\begin{aligned}
&A_1\sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^n X_t^2 - \gamma_X(0)\Big) + A_2\sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^n X_{t-1}X_t - \gamma_X(1)\Big) + \\
&A_3\sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^n X_{t-2}X_t - \gamma_X(2)\Big) \to_d A_1Z_1 + A_2Z_2 + A_3Z_3 \sim N(0, \tau_1^2),
\end{aligned}$$

where

$$\begin{aligned}
\tau_1^2 &= Var(A_1Z_1 + A_2Z_2 + A_3Z_3) \\
&= A_1^2Var(Z_1) + A_2^2Var(Z_2) + A_3^2Var(Z_3) + \\
&\quad 2A_1A_2Cov(Z_1, Z_2) + 2A_1A_3Cov(Z_1, Z_3) + 2A_2A_3Cov(Z_2, Z_3) \\
&= A_1^2V_{0,0} + A_2^2V_{1,1} + A_3^2V_{2,2} + 2A_1A_2V_{0,1} + 2A_1A_3V_{0,2} + 2A_2A_3V_{1,2}.
\end{aligned}$$

Hence, by Theorem 3 we deduce that

$$\sqrt{n}\Bigg(\phi_1\Big(\frac{1}{n}\sum_{t=1}^n X_t^2, \frac{1}{n}\sum_{t=2}^n X_{t-1}X_t, \frac{1}{n}\sum_{t=3}^n X_{t-2}X_t\Big) - \phi_1\big(\gamma_X(0), \gamma_X(1), \gamma_X(2)\big)\Bigg)$$

$\to_d N(0, \tau_1^2)$.

An analogous representation holds for the bootstrapped version. By applying Theorem 4 we obtain

$$\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{t=1}^n X_t^{*2} \\ \frac{1}{n}\sum_{t=2}^n X_{t-1}^*X_t^* \\ \frac{1}{n}\sum_{t=3}^n X_{t-2}^*X_t^* \end{pmatrix} - \begin{pmatrix} \widehat{\gamma}_X(0) \\ \widehat{\gamma}_X(1) \\ \widehat{\gamma}_X(2) \end{pmatrix}\right)$$

$$\to_d N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} V_{0,0}^* & V_{0,1}^* & V_{0,2}^* \\ V_{1,0}^* & V_{1,1}^* & V_{1,2}^* \\ V_{2,0}^* & V_{2,1}^* & V_{2,2}^* \end{pmatrix}\right)$$

and

$$\begin{aligned}\sqrt{n}\big(\widehat{\theta}_1^* - \widehat{\theta}_1\big) &= \sqrt{n}\bigg(\phi_1\Big(\frac{1}{n}\sum_{t=1}^{n} X_t^{*2}, \frac{1}{n}\sum_{t=2}^{n} X_{t-1}^* X_t^*, \frac{1}{n}\sum_{t=3}^{n} X_{t-2}^* X_t^*\Big) - \\ &\quad \phi_1\Big(\frac{1}{n}\sum_{t=1}^{n} X_t^2, \frac{1}{n}\sum_{t=2}^{n} X_{t-1}X_t, \frac{1}{n}\sum_{t=3}^{n} X_{t-2}X_t\Big)\bigg) \\ &\quad \to_d N(0, \tau_1^{2*}),\end{aligned}$$

where τ_1^{2*} is the bootstrapped version of τ_1^2.

Analog with the previous discussion, the asymptotic distribution for $\widehat{\theta}_2^*$ is obtained as follows. For sake of the simplicity of notation, write $B_1 = \frac{2\gamma_X(0)\gamma_X(1)^2 - \gamma_X(0)^2\gamma_X(2) - \gamma_X(1)^2\gamma_X(2)}{\big(\gamma_X(0)^2 - \gamma_X(1)^2\big)^2}$, $B_2 = \frac{2\gamma_X(0)\gamma_X(1)\big(\gamma_X(2) - \gamma_X(0)\big)}{\big(\gamma_X(0)^2 - \gamma_X(1)^2\big)^2}$, and $B_3 = \frac{\gamma_X(0)}{\gamma_X(0)^2 - \gamma_X(1)^2}$. By Theorem 3, we obtain

$$\begin{aligned}&\sqrt{n}\Bigg(\phi_2\Big(\tfrac{1}{n}\textstyle\sum_{t=1}^{n} X_t^2, \tfrac{1}{n}\sum_{t=2}^{n} X_{t-1}X_t, \tfrac{1}{n}\sum_{t=3}^{n} X_{t-2}X_t\Big) - \phi_2\big(\gamma_X(0), \gamma_X(1), \gamma_X(2)\big)\Bigg) \\ &= \phi'_{2(\gamma_X(0),\gamma_X(1),\gamma_X(2))}\begin{pmatrix}\sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^{n} X_t^2 - \gamma_X(0)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^{n} X_{t-1}X_t - \gamma_X(1)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^{n} X_{t-2}X_t - \gamma_X(2)\Big)\end{pmatrix} + o_p(1) \\ &= \begin{pmatrix} B_1 & B_2 & B_3 \end{pmatrix}\begin{pmatrix}\sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^{n} X_t^2 - \gamma_X(0)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^{n} X_{t-1}X_t - \gamma_X(1)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^{n} X_{t-2}X_t - \gamma_X(2)\Big)\end{pmatrix} + o_p(1) \\ &= B_1\sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^{n} X_t^2 - \gamma_X(0)\Big) + B_2\sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^{n} X_{t-1}X_t - \gamma_X(1)\Big) + \\ &\quad B_3\sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^{n} X_{t-2}X_t - \gamma_X(2)\Big) + o_p(1).\end{aligned}$$

According to Theorem 3, we assert that

$$\begin{aligned}&B_1\sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^{n} X_t^2 - \gamma_X(0)\Big) + B_2\sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^{n} X_{t-1}X_t - \gamma_X(1)\Big) + \\ &B_3\sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^{n} X_{t-2}X_t - \gamma_X(2)\Big) \to_d B_1Z_1 + B_2Z_2 + B_3Z_3 \sim N(0, \tau_2^2),\end{aligned}$$

where

$$\begin{aligned}\tau_2^2 &= Var(B_1Z_1 + B_2Z_2 + B_3Z_3) \\ &= B_1^2V_{0,0} + B_2^2V_{1,1} + B_3^2V_{2,2} + 2B_1B_2V_{0,1} + 2B_1B_3V_{0,2} + 2B_2B_3V_{1,2}.\end{aligned}$$

Thus, applying Theorem 3 we conclude that
$\sqrt{n}\Bigg(\phi_2\Big(\frac{1}{n}\sum_{t=1}^{n} X_t^2, \frac{1}{n}\sum_{t=2}^{n} X_{t-1}X_t, \frac{1}{n}\sum_{t=3}^{n} X_{t-2}X_t\Big) - \phi_2\big(\gamma_X(0), \gamma_X(1), \gamma_X(2)\big)\Bigg)$
$\to_d N(0, \tau_2^2)$.
Both Bose (1988) and Freedman (1985) proved that the residuals bootstrapping work well

when it is applied to the autoregressive model. Hence, by applying Theorem 4 and employing the *plug-in* principle, we obtain the limiting distribution of bootstrapped version as follows

$$\begin{aligned}\sqrt{n}\big(\widehat{\theta}_2^* - \widehat{\theta}_2\big) &= \sqrt{n}\bigg(\phi_2\Big(\frac{1}{n}\sum_{t=1}^{n} X_t^{*2}, \frac{1}{n}\sum_{t=2}^{n} X_{t-1}^* X_t^*, \frac{1}{n}\sum_{t=3}^{n} X_{t-2}^* X_t^*\Big) - \\ &\quad \phi_2\Big(\frac{1}{n}\sum_{t=1}^{n} X_t^2, \frac{1}{n}\sum_{t=2}^{n} X_{t-1}X_t, \frac{1}{n}\sum_{t=3}^{n} X_{t-2}X_t\Big)\bigg) \\ &\quad \rightarrow_d N(0, \tau_2^{2*}),\end{aligned}$$

where τ_2^{2*} is the bootstrapped version of τ_2^2.

5. Conclusions

We conclude that the bootstrap parameter estimators of the AR(2) process are asymptotic and converge in distribution to the normal distribution.

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