Estimations of the Roman Life Expectancy Using Ulpian´s Table

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Abstract
In this paper a life table for the Roman population is constructed using Ulpian’s table. This table can be regarded as a tool to compute the value of an annuity taking into account the age of the beneficiary. The Gompertz distribution and some of its extensions are applied for the life table construction. It is shown that the Roman life table can be represented by a five-parameter formula, which consists of three terms. Since the life expectancy at birth depends on the unknown infant mortality, different assumptions are made. Simulations show that a range of the life expectancy between 20 and 30 years is quite possible. Finally, it is discussed whether Ulpian´s table represents annuities or life expectancies. It cannot be excluded that the values in Ulpian’s table represent annuities premiums based on an interest rate of about 1.5%.

Key Words: Life Table, Mortality Law, Gompertz-Makeham, Roman Demography

I. Introduction

This paper continues a research on Ulpian´s table of Pflaumer (2014). Ulpian’s scheme can be summarized as a table which illustrates the relation between age and the present value of a life-time annuity of 1. If the interest rate is zero, which is in general assumed, the present value corresponds to the life expectancy at age x (see Table 1).

Table 1: Ulpian’s table

Age x	[0,20)	[20,25)	[25,30)	[30,35)	[35,40)	[40,50)	[50,55)	[55,60)	60+
Life expectancy at age x	30	28	25	22	20	59 - x	9	7	5

Pflaumer (2014) found that Ulpian´s table between the ages 20 and 70 can be approximated quite well by a Gompertz distribution $l(x) = \exp\left(\frac{A}{k} - \frac{A}{k} e^{k \cdot x}\right)$ with A=0.002410 and k=0.058923, assuming that the figures are median life expectancies.

2. Lazarus Distribution

The Gompertz and Makeham laws are partial, because they do not apply to life tables with high mortality at young ages. We now turn to a life table distribution which was first proposed by Lazarus (1867). Wilhelm Lazarus (1825-1890) was an actuary in Hamburg and Trieste (see, e.g., Pitacco 2009, p. 410 or Loewy 1930). It is a general law of mortality that applies to all ages. The force of mortality function is defined by: $\mu(x) = B \cdot e^{-g \cdot x} + C + A \cdot e^{k \cdot x}$ with B>0, g>0, C>0, k>A>0.

Lazarus' law has three terms and five parameters, and covers the whole of the age range. The first term represents infant mortality, which declines steeply after birth; the second term represents age-independent mortality, and the third term is the Gompertz law with increasing mortality. This model has also been proposed and applied to primates by Siler (1979). An extension can be found in Thiele (1871), where C is replaced by an age-dependent function C(x). Special cases of the Lazarus model are the Gompertz formula (C=0, B=0), the Makeham formula (B=0), and the Gauss Mortality Formula (C=0, cf., Loewy 1906 or Pflaumer 2013). This particular form of the force of mortality function is called a bathtub curve in reliability engineering, because it is comprised of three parts: decreasing, constant, and increasing failure rates (Hjorth, 1980). The name is derived from the cross-sectional shape of a bathtub.

Since $l(x) = e^{-\int_0^x \mu(u)du}$, we get: $l(x) = \exp\left(\frac{A}{k} - \frac{A}{k}\cdot e^{kx} - \frac{B}{g} + \frac{B}{g}\cdot e^{-gx} - C\cdot x\right)$.

If the force of mortality decreases sharply after birth, then $\frac{B}{g}\cdot e^{-gx} \approx 0$ for ages $x > x_A$.

The remaining part of the survivor function can be represented by:

$l_A(x) = \exp\left(\frac{A}{k} - \frac{B}{g} - \frac{A}{k}\cdot e^{kx} - C\cdot x\right) = \exp\left(-\frac{B}{g}\right)\cdot \exp\left(\frac{A}{k} - \frac{A}{k}\cdot e^{kx} - C\cdot x\right)$ for $x>x_A$,

or simply by: $l_A(x) = \exp\left(-\frac{B}{g}\right)\cdot l_M(x)$, where $l_M(x)$ is the Makeham survivor function.

In this special case, we can estimate the modal value of the Lazarus distribution by:

$m = \frac{\ln\left(\frac{\sqrt{k\cdot(k - 4\cdot C))} - 2\cdot C + k}{2\cdot A}\right)}{k}$ for $k>4\cdot C$.

The maximum age can be estimated from the Lazarus model as the age of the last and single survivor of a population of size N. Solving $l(\omega) = \frac{1}{N}$ yields an implicit equation:

$A\cdot e^{k\cdot\omega} + C\cdot k\cdot\omega - k\cdot \ln N + \frac{A\cdot g - B\cdot g}{g} = 0$. If $C\cdot k\cdot\omega \approx 0$, we can find an approximation

formula for the maximum life span: $\omega = \frac{\ln\left(\frac{g\cdot k\cdot \ln N + A\cdot g - B\cdot k}{A\cdot g}\right)}{k}$.

Other important parameters can only be obtained by numerical integration.

3. A Roman life table over the whole span of life based on Ulpian's table[1]

For the complete discrete life table l_x^C we first suppose the same survivorship rates l_x^F as did Frier (1982) for young ages: $l_x^C = l_x^F$ for x=0,1..20. The basis for adult ages x>20 is the Gompertz median model with A=0.002410 and k=0.058923 (see Pflaumer 2014, section 4).

[1] An alternative approach is suggested by Thieme (2003, pp. 194-195), who fits a rational function to Ulpian´s Table and derives the survivor function from a well-known relationship between life expectancy at age x and the l(x)-function. His model yields a life expectancy at birth of 30.6 years, a survivorship rate l(20) = 0.555, and unrealistically low infant mortality, e.g., $\mu(0) = 0.038$.

In order to obtain the complete life table for the remaining ages, the age specific rates of Frier l_x^F and Gompertz l_x^G are concatenated by: $l_x^C = \frac{l_x^G}{l_{20}^G} \cdot l_{20}^F = l_x^G \cdot \frac{0.40385}{0.9120}$ x=20,25,….80.

Table 2: Survivorship rates of Frier's (1982, p. 245), Gompertz's (A=0.002410 and k=0.058923), and the complete (concatenated) life table

x	l_x^F (Frier)	l_x^G (Gompertz)	l_x^C (Complete)
0	1		1.0000
1	0.64178		0.6418
5	0.48968		0.4897
10	0.45828		0.4583
15	0.43618		0.4362
20	*0.40385*	*0.91210*	*0.4039*
25	0.37047	0.87150	0.3859
30	0.33604	0.81982	0.3630
35	0.30055	0.75522	0.3344
40	0.26401	0.67642	0.2995
45	0.22642	0.58339	0.2583
50	0.18777	0.47829	0.2118
55	0.14807	0.36632	0.1622
60	0.11096	0.25606	0.1134
65	0.07459	0.15833	0.0701
70	0.04377	0.08303	0.0368
75	0.02067	0.03490	0.0155
80	0.00671	0.01090	0.0048

In the next step, Lazarus models were fitted to the observed life table data sets of the l_x and x values of Table 2, both for Frier's and the complete life table, using non-linear least squares. The results are given in Tables 3 and 4.

Table 3: Estimation Results
a) Complete life table (Lazarus model based on Gompertz model)

Dependant variable l_x^C
Independant variable x=0, 1, 5, 10….,75,80

Parameter	Estimate	Std Error	Lower 95% C.I.	Upper 95% C.I.
A	7.72117E-04	1.9454E-04	3.51844E-04	1.19239E-03
B	0.688977	0.017303	0.651597	0.726357
k	0.076330	4.3765E-03	0.066875	0.085785
g	1.008889	0.032522	0.938630	1.079148
C	8.27207E-03	8.3727E-04	6.46325E-03	0.010081

Convergence criterion met after 9 iterations.

Residual SS (SSE)	1.886E-04
Residual MS (MSE)	1.451E-05
Standard Deviation	3.809E-03
Degrees of Freedom	13
AICc	-186.76
Pseudo R²	0.9998

Cases Included 18 Missing Cases 0

b) Frier’s life table (Lazarus model)

Dependant variable l_x^F
Independant variable x=0, 1, 5, 10….,75,80

Parameter	Estimate	Std Error	95% C.I.	95% C.I.
A	1.95738E-03	3.2308E-04	1.25941E-03	2.65536E-03
B	0.705528	0.011452	0.680788	0.730268
k	0.057015	2.7278E-03	0.051122	0.062908
g	1.074100	0.022925	1.024573	1.123627
C	9.15156E-03	8.0287E-04	7.41706E-03	0.010886

Convergence criterion met after 8 iterations.

Residual SS (SSE)	7.053E-05
Residual MS (MSE)	5.426E-06
Standard Deviation	2.329E-03
Degrees of Freedom	13
AICc	-204.46
Pseudo R²	0.9999

Cases Included 18   Missing Cases 0

Table 4: Estimated ($\hat{l}_x$) and original values (l_x)

x	l_x^F (Frier)	$\hat{l}_x^F$ (Frier)	l_x^G (Complete: Lazarus-Gompertz*)	$\hat{l}_x^G$ (Complete: Lazarus-Gompertz)
0	1	1.0000	1	1
1	0.64178	0.6417	0.6418	0.6421
5	0.48968	0.4912	0.4897	0.4845
10	0.45828	0.4608	0.4583	0.4597
15	*0.43618*	0.4315	0.4362	0.4366
20	0.40385	0.4013	0.4039	0.4128
25	0.37047	0.3701	0.3859	0.3876
30	0.33604	0.3372	0.3630	0.3603
35	0.30055	0.3026	0.3344	0.3300
40	0.26401	0.2660	0.2995	0.2958
45	0.22642	0.2274	0.2583	0.2569
50	0.18777	0.1875	0.2118	0.2131
55	0.14807	0.1472	0.1622	0.1651
60	0.11096	0.1084	0.1134	0.1159
65	0.07459	0.0732	0.0701	0.0703
70	0.04377	0.0441	0.0368	0.0345
75	0.02067	0.0228	0.0155	0.0124
80	0.00671	0.0097	0.0048	0.0028

Both life tables can be fitted well by a Lazarus distribution. With the Lazarus distribution it is possible to represent the life tables by a five-parameter model. Life table functions and characteristic parameters can easily be calculated (see Table 5). The difference between the actual and the estimated survivor rates is small. The main difference between Frier’s life table and our life table can clearly be seen in Table 5 and in Figure 1. The assumptions mean that the curves are identical up to the age of 20. Between 20 and 55 Frier’s life table shows a slightly higher mortality. Thus, the modal age for adults and the normal age is less (see Table 5). But for older ages Frier’s mortality rates are significantly lower than those of our model. This is difficult to see in the graphs of the survivor functions, but is easy to see in the graphs of the force of mortality functions and in parameters such as maximum age or old age ratio. Consequently, Frier's life table overestimates life expectancies above the age of around 50 in Ulpian's table (see Figure 2). Those old age estimates are better if one uses a model based on a Gompertz distribution. Although both models lead to the same life expectancy at birth (about 21 years), our model predicts higher life expectancies at ages between 1 and 20. A child surviving the first few years would have a median life expectancy of more than 40 years (see Figure 2).

Table 5: Characteristic life table parameters[1)]

	Complete Life Table (Lazarus-Gompertz)	Frier's Life Table
Mean life expectancy	21.8	21.1
Median life expectancy	3.4	4.0
Modal age for adult ages	56.7	51.3
Mean of the stationary population	25.7	25.5
Maximum age ($N=10^5$)	90.5 (92.2)[2)]	99.5
Maximum age ($N=10^6$)	93.2 (94.6)	103.1
Maximum age ($N=10^7$)	95.4 (96.6)	106.1
Keyfitz entropy[3)]	1.060	1.109
Youth ratio (ages 0 to 15)	0.342	0.356
Old age ratio (ages 65+)	0.019	0.027

[1)] An overview of the calculation and the interpretation of the parameters can be found in Pflaumer (2011).

[2)] The figures in parentheses are obtained by the approximation formula.

[3)] The Keyfitz entropy H (Keyfitz, 1977) is one of the best-known mortality measures in demography. H can be interpreted as follows: if the death rates at all ages increase by 1%, the mean expectation of life diminishes by 1.06% or 1.109%.

4. Sensitivity analyses

According to our model results, the mean life expectancy at birth is between 21 and 22 years (see Table 5). This depends on the assumption of the mortality between 0 and 20. Since Ulpian's table did not reveal realistic assumptions in this age class, we used the mortality pattern of Model West, level 2. As Frier (1982, p. 246) pointed out, this is the weakest point in the life table, since very little is known about the exact pattern of juvenile mortality. In order to analyze the uncertainty about this point, we make different assumptions about juvenile mortality.

A proportional change of the force of mortality function in the age class 0 to 20 is assumed:

$$\mu^*(x) = f \cdot \mu(x) = f \cdot \left(B \cdot e^{-g \cdot x} + C + A \cdot e^{k \cdot x}\right) \qquad 0 \le x \le 20; f > 0 .$$

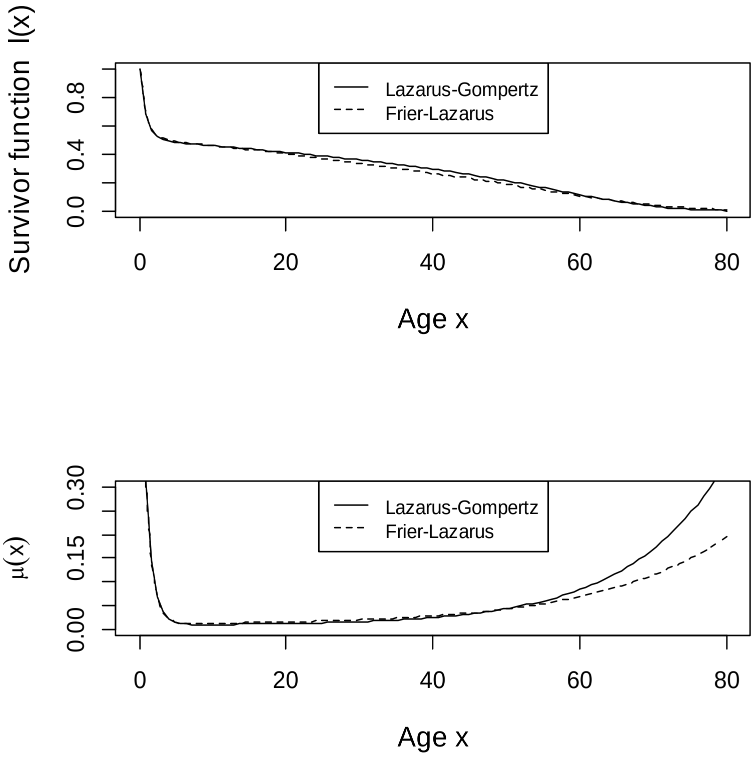

Figure 1: Survivor and force of mortality functions

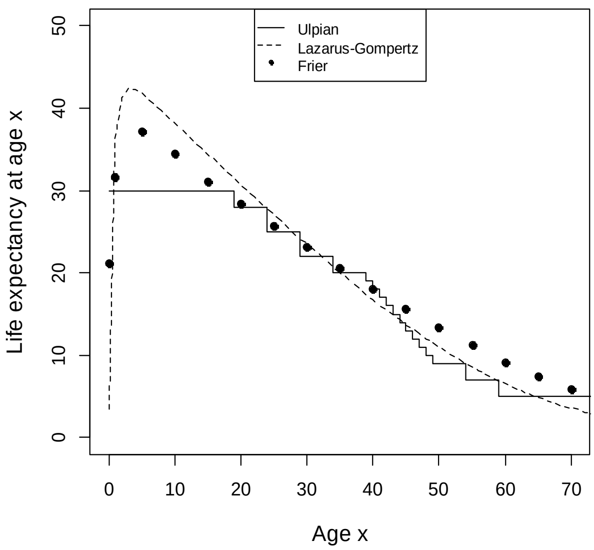

Figure 2: Median life expectancies

Mortality increases if the factor f is greater than one, and decreases if the factor is less than one. The assumption of a proportional change in the force of mortality functions implies the following relationship between the original and the new survivor function:

$$l^{*}_{0-20}(x)=\left(l(x)\right)^{f} \qquad 0\le x\le 20; f>0 .$$

At the age of 20 the original survivor function for ages > 20 is concatenated with the new survivorship function $l^{*}_{0-20}(x)$. Thus, the new survivor function over the whole span of life is

$$l^{*}(x)=\begin{cases}\left(l(x)\right)^{f} & 0\le x\le 20\\ l(x)\cdot\dfrac{\left(l(20)\right)^{f}}{l(20)} & x>20\end{cases} .$$

The life expectancy after the mortality change is:

$$\overset{0}{e}(x)^{*}=\int_{0}^{20} l^{*}(x)dx+\int_{20}^{\omega} l^{*}(x)dx .$$

In Figure 3 the effects of a proportional change in the force of mortality function are shown. The left part of Figure 3 relates the change factor f to the survivorship rate l(20), whereas the right part shows the relation between the survivorship rate l(20) and the mean life expectancy at birth. If, for example, f=0.8, then there is a decrease in the force of mortality of 20% in the age class from 0 to 20. The force of mortality at ages x>20 remains unchanged. A negligible jump discontinuity arises at that age (see Figure 4). As a result, all survivorship rates increase, especially $l(20)=0.4128$ to $\left(l(20)\right)^{0.8}=0.4927$. The survivor function is furthermore a continuous function, but is no longer differentiable at the point x=20 (see Figure 5). A survivorship rate $l(20)=0.4927$ implies a growth of the life expectancy at birth to $\overset{0}{e}(0)^{*}=25.7$, which can be seen in the right part of Figure 3. The simulated life expectancies have been calculated by numerical integration. If it is assumed to be realistic that l(20) ranges between 0.4 and 0.5 (which means that only 40% to 50% of all newborns would reach the age of 20), then the life expectancy at birth would be between 21 and 26 years (see Figure 3).

A comparison of selected historical life tables shows that the survivorship rate l(20) ranges roughly between 0.49 and 0.58. The assumed rate of l(20) of Ulpian’s life table, based on Model West, level 2, seems very low. If the l(20)-rates of the historical life tables were assumed for our complete life table the mean life expectancy would range between 25.6 and 30.1 years (see simulated life expectancies in Table 6). They are less than the life expectancies of the historical tables because of the very high force of mortality of our life table at older ages.

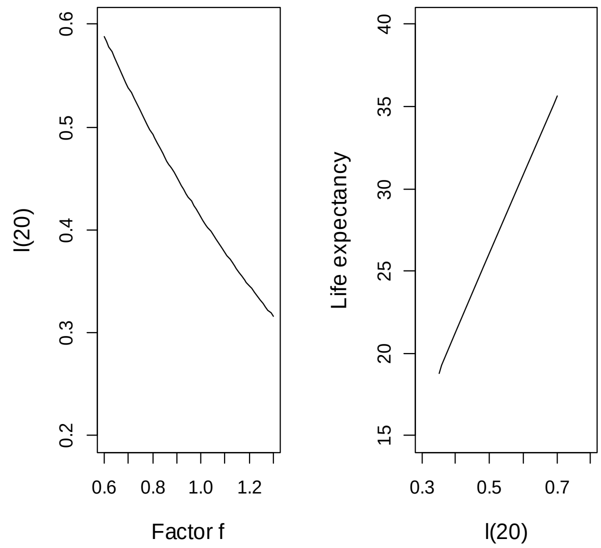

Figure 3: Effects of a proportional change in the force of mortality function

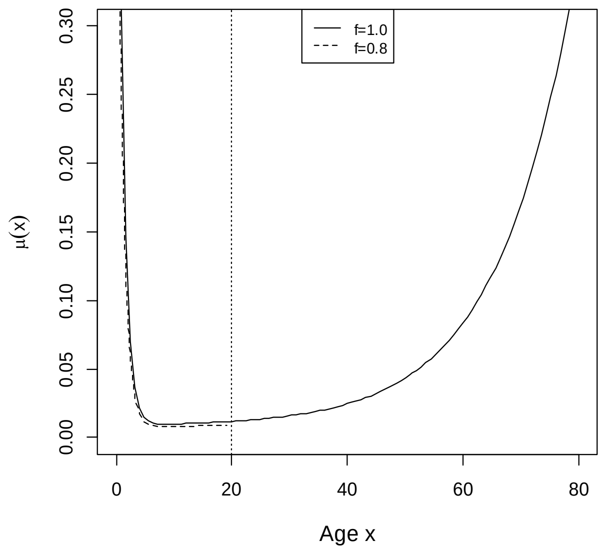

Figure 4: Proportional change in the force of mortality

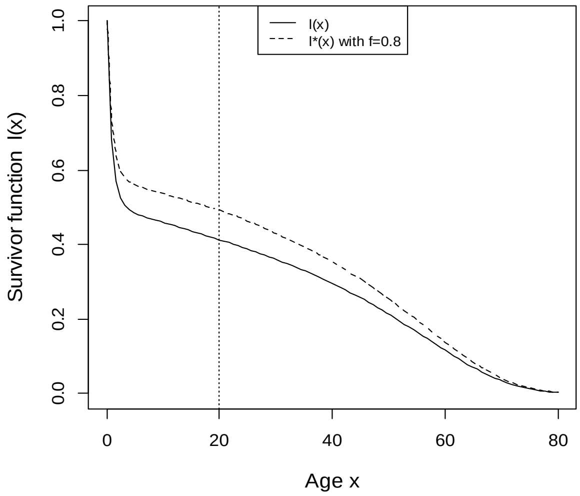

Figure 5: Effect of a decline in the force of mortality on the survivor function

Table 6: Life expectancies and survivorship rates of some historical life tables

	Complete Life Table	Süssmilch	Moser	Kerseboom
l(20)	0.4128	0.491	0.570	0.584
Mean life expectancy at birth	21.8	29.0	35.6	35.0
Simulated mean life expectancy at birth	21.8	25.6	29.4	30.1

Moser, L (1839): Die Gesetze der Lebensdauer, pp. 74 ff., pp. 324 ff. Moser´s life table is the result of his polynomial 5-parameter mortality law (cf. also Forfar (2004, Mortality Laws) based on data for female mortality of Brune´s pension fund tables of the Prussian Widows´ Annuity Society (cf. Brune, D (1837): Neue Sterblichkeits-Tabellen für Wittwen-Cassen, Journal für reine und angewandte Mathematik, 16: 58-64).

There are two reasons for the uncertainty about the Roman life expectancy at birth: first, the accuracy or inaccuracy of the life expectancy figures in Ulpian's table and, second, the lack of reliable information on infant and youth mortality. If we accept Ulpian's figure as more or less adequate, then the problem reduces to the mortality at young ages. Simulations show that the life expectancy at birth strongly depends on the survivorship ratios in young ages, which are not really known. We need more information on mortality for this age class in order to make more reliable statements about the life expectancy at birth.

5. Is Ulpian's table an annuity table?

A life annuity consists of periodic (yearly) payments until the end of the recipient's life. Its value or premium depends on the life expectancy of the pensioner and the underlying interest rate. We assume that the life annuity is purchased with a one-time payment (e.g., the proceeds of a sale of real estate). For a life aged u the actuarial present value of an annual life annuity of 1 paid continuously can be determined by:

$$\bar{a}(u)=\int_u^\infty e^{-r(x-u)}\frac{l(x)}{l(u)}dx,$$

where r is the interest rate. If r=0, then the present value equals the life expectancy at age u.

$$\bar{a}(u)=\overset{0}{e}(u).$$

An approximation formula for small values of r is the annuity certain with fixed (deterministic) length of time or duration $n=\overset{0}{e}(u)$

$$\bar{a}_{\overline{n|}}=\int_0^n e^{-r\cdot x}dx=\frac{1}{r}-\frac{e^{-r\cdot n}}{r}.$$

A famous actuarial inequality states

$\bar{a}_{\overline{n|}}>\bar{a}(u)$, if $n=\overset{0}{e}(u)$.

Ciecka (2012) considers three possible interpretations of Ulpian's figures.

1. The figures are life expectancies. We considered this in the previous sections.
2. The figures are the duration of an annuity certain. He refers to Hald (2003) and Poitras (2000), who write that there was a market for annuities certain in ancient Rome. The first and the second interpretations are not necessarily inconsistent. It may be correct that Ulpian's figures are life expectancies, and that they were used as the duration term in the annuity certain (Ciecka 2012, p. 10). If this is the case, then we know the life expectancies, but not the annuities certain. We only know that the life expectancy is greater than the annuity certain, if r>0. If, on the other hand, the figures in Ulpian's table are annuities certain, then we do not know the life expectancies.
3. Greenwood (1940) argued that Ulpian´s figures were neither life expectancies nor the duration terms of annuities certain. They are simply the legal maximum valuations for any annuity. For a more detailed discussion of this point see Greenwood (1940), Parkin (1992 pp. 34ff.) or Ciecka (2012).

In the following analysis, we assume that the figures in Ulpian's table are present values of annuities.

Using the survivor function of the Gompertz distribution $l(x)=\exp\left(e^{-k\cdot m}-e^{k\cdot(x-m)}\right)$, we obtain the following formula for the value of life annuity:

$$\bar{a}(u)=\int_u^\infty e^{-r(x-u)}\frac{\exp\left(e^{-k\cdot m}-e^{k(x-m)}\right)}{\exp\left(e^{-k\cdot m}-e^{k(u-m)}\right)}dx \text{ with the modal value } m=-\frac{\ln\left(\frac{A}{k}\right)}{k}.$$

In order to estimate the parameters k, m, and r, we use approximation formulae.

a) Pollard (1991, p. 12) provides an approximation formula, which is, with our symbols:

$$\bar{a}_1(u)=\frac{1-\left(\frac{\left(k\cdot e^{k(u-m)}\right)}{k+k\cdot e^{k(u-m)}}\right)^{\frac{r}{k}}\cdot\left(1+\frac{r}{2(r+k)}\right)\cdot\left(k+\left(k\cdot(u-m)\right)\right)^{-2}}{r}.$$

b) Since only an approximation formula of the mean life expectancy based on a Taylor series exists, we approximate the mean by the median life expectancy for the annuity certain. The difference between mean and median life expectancy is not substantial for adult ages, especially when the survivor rates from a certain age follow a more or less linear pattern (see the Achard-Moivre mortality law in Pflaumer (2014), section 3).
Therefore, we propose as a second approximation formula for small interest rates r the use of the median instead of the mean, that is:

$$\bar{a}_{\overline{\tilde{e}(u)}|} = \int_0^{\tilde{e}(u)} e^{-r\cdot x}dx = \frac{1}{r} - \frac{e^{-r\cdot \tilde{e}(u)}}{r}.$$

Considering that:

$$\tilde{e}(u) = \frac{\ln\left(e^{k\cdot u}+e^{k\cdot m}\ln 2\right)}{k} - u,$$

we finally get as an approximation for the value of the annuity:

$$\bar{a}_2(u) = \bar{a}_{\overline{\tilde{e}(u)}|} = \frac{1}{r} - \frac{e^{-r\cdot n}}{r} = \frac{1}{r} - \frac{e^{r\cdot u}\left(e^{k\cdot u}+e^{k\cdot m}\cdot\ln 2\right)^{-\frac{r}{k}}}{r}$$

with $n=\tilde{e}(u)$ and $\lim_{r\to 0} \bar{a}_2(u)=\tilde{e}(u)$.

We used non-linear least squares to estimate the parameters. The regressands are the figures from Ulpian's table, and the regressors are the ages from 20 to 70. The results are given in Tables 7 and 8.

Table 7: Regression results based on Pollard's approximation formula

MODEL: $a_1(x)$ = (1-((k*exp(k*x-k*m))/(k+(k*exp(k*x-k*m))))^(r/k)*(1+r/2*(r+k)*(k+(k *exp(k*x-k*m)))^-2))/r for x=20,21,…70

		Lower	Upper	
Parameter	Estimate	Std Error	95% C.I.	95% C.I.
m	63.86365	0.727126	62.40166	65.32563
k	0.111613	9.6667E-03	0.092176	0.131049
r	0.016020	1.7440E-03	0.012513	0.019526

Convergence criterion met after 34 iterations.

Residual SS (SSE)	55.477
Residual MS (MSE)	1.1558
Standard Deviation	1.0751
Degrees of Freedom	48
AICc	13.161
Pseudo R^2	0.9837

Cases Included 51 Missing Cases 0

The estimation results of the two models are similar. They do not exclude the possibility that Ulpian's figures are indeed annuity premiums, which are calculated with an interest rate of 1.5% or 1.6%. Figure 6 shows the approximation of the annuity value as a function of age x. The two methods yield nearly the same prediction values. The values of k and m are higher than in the pure life expectancy models. This result is plausible, since the life expectancy is higher than the annuity certain. The modal value m is, as a consequence, higher, and because of the lower force of mortality the slope of k must be greater with respect to the age limit of about 70.

Table 8: Regression results based on the median annuity certain approximation formula

MODEL: $a_2(x)$ = 1/r-(exp(r*x)*(exp(k*x)+exp(k*m)*ln(2))^(-r/k))/r for x=20,21,…70

Parameter	Estimate	Std Error	Lower 95% C.I.	Upper 95% C.I.
m	61.94272	0.972953	59.98647	63.89897
k	0.093136	9.0325E-03	0.074975	0.111297
r	0.014982	2.0022E-03	0.010956	0.019008

Convergence criterion met after 30 iterations.

Residual SS (SSE)	58.414
Residual MS (MSE)	1.2170
Standard Deviation	1.1032
Degrees of Freedom	48
AICc	15.792
Pseudo R^2	0.9828

Cases Included 51   Missing Cases 0

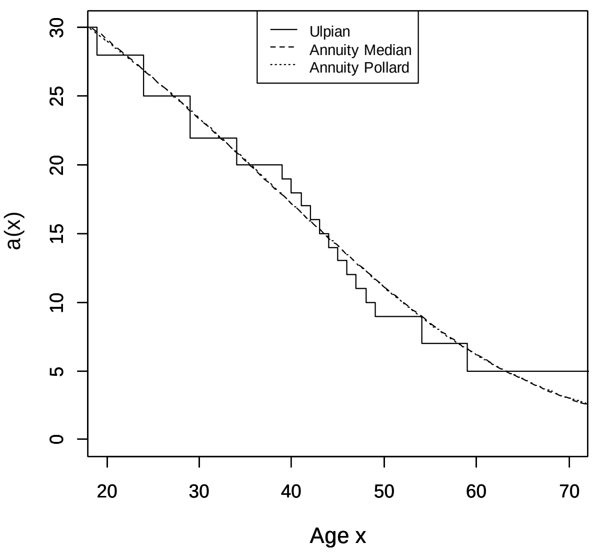

Figure 6: Annuity table functions

If the figures in Ulpian's table represented annuity premiums calculated with an interest rate of about 1.5%, then the median life expectancy at age 20 would be 40 years (see Figure 7). This seems very high, if it is compared with life tables from the eighteenth and early nineteenth century. On the other hand, Duncan-Jones (1990, p. 94) reports mean life expectancies at age 25 ranging between 32 and 34. These life expectancies were estimated from a register of town councilors in Canusium in Southern Italy in 223 AD (Album of Canusium).

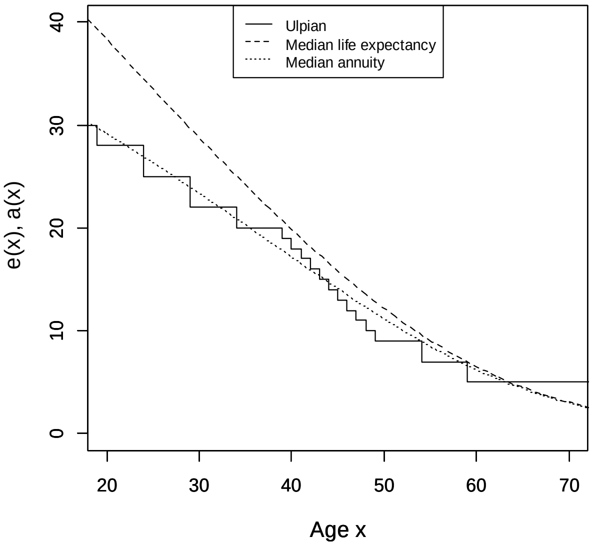

Figure 7: Comparison of life expectancy and annuity (k=0.093, m=61.9, r=0.015)

Table 9: Life expectancies and survivorship rates of some historical life tables

	Suessmilch	Kerseboom	Moser
l(20)	0.491	0.584	0.570
Life expectancy at birth	29.0	35.0	35.6
Median Life expectancy at age 20	36	37.9	43.2
Mean Life expectancy at age 20	35.5	36.8	39.3

Moser, L (1839): Die Gesetze der Lebensdauer, pp. 74 ff., pp. 324 ff.

But could the creators of Ulpian's table have had the profound knowledge to calculate the value of annuities? Certainly they did not use our approximation formulae, and they did not know how to calculate the value for life annuities with actuarial methods. It was not until the seventeenth century that Jan de Witt (1625-1672) and Edmond Halley (1656-1742) calculated the correct premiums for life annuities for the first time. However, according to Kopf (1926, pp. 231ff.) there is much evidence that the Romans had the statistical material and the required arithmetical skill to construct crude annuity tables. Maybe the creators could calculate a discrete version of the annuity certain. It is more probable that they would use an approximation formula if they calculated the value of an annuity. The present value of a payment of 1 in m years and an interest rate of r, using simple interest, can be calculated by $\frac{1}{(1+r\cdot m)} \approx (1-r\cdot m)$. If $r \cdot m$ is small

$\sum_{m=1}^{n} (1-r\cdot m)$ is therefore an approximation for the annuity certain under the condition that the interest rates are low, and is easy to calculate. Since:

$$e^{-r\cdot m} = 1 - r\cdot m + \frac{r^2\cdot m^2}{2} .... \approx (1-r\cdot m)$$

the simple interest rate model yields similar results to the annuity certain model, when the interest rate is low. Using $\sum_{m=1}^{n}(1-r\cdot m) = n - n\cdot\frac{r}{2} - n^2\cdot\frac{r}{2}$ and $n = \tilde{e}(u)$

it follows for the annuity value using simple interest

$$\bar{a}_3(u) = \frac{1}{r} - \frac{e^{-r\cdot n}}{r} = -\frac{\left(\frac{\ln\left(e^{k\cdot u}+e^{k\cdot m}\ln 2\right)}{k}-x\right)\left(\left(\frac{\ln\left(e^{k\cdot u}+e^{k\cdot m}\ln 2\right)}{k}-u\right)\cdot r+r-2\right)}{2}$$

$$\lim_{r\to 0}\bar{a}_3(u) = \tilde{e}(u).$$

The regression results are shown in Table 10. The results do not differ much from the previous results (Tables 7 and 8).

Table 10: Regression results based on simple interest rate approximation

MODEL: a_3(x)= -(LN(exp(k*x)+exp(k*m)*LN(2))/k-x)*((LN(exp(k*x)+exp(k*m)*LN(2))/k-x)*r+r-2)/2 for x=20,21,…70

Parameter	Estimate	Std Error	Lower 95% C.I.	Upper 95% C.I.
m	61.75774	0.845832	60.05708	63.45840
k	0.092273	8.1099E-03	0.075967	0.108579
r	0.012235	1.2199E-03	9.78170E-03	0.014687

Convergence criterion met after 34 iterations.

Residual SS (SSE)	57.036
Residual MS (MSE)	1.1883
Standard Deviation	1.0901
Degrees of Freedom	48
AICc	14.575
Pseudo R²	0.9832

Cases Included 51 Missing Cases 0

6. Conclusion

When constructing a life table using Ulpian’s table we mainly have to deal with three types of uncertainty:

1. Are the figures mean or median life expectancies?
2. Are the figures life expectancies or annuity premiums?
3. How high is infant and youth mortality?

Since for adult ages the difference between median and mean life expectancy is small, the two assumptions yield similar results: the modal adult age or normal age of death is around 57 years. After that age the death probability sharply increases. If the values of Ulpian’s table represented annuities calculated with an interest rate of about 1.5%, then mortality would be much lower than supposed, with a modal value of about 60 years. There is controversy in the literature about whether the Romans had the data and the methods to calculate even approximate values for annuities. Not only the relatively high median life expectancies at younger ages but also the low interest rate of only 1.5%, which results from the proposed models, speak against the annuity hypothesis. Why did they use only 1.5% and not a higher rate? In the Roman Empire the rate of interest varied in the range of 4% to 12% (see, e.g., Homer & Sylla 2005 p. 52). Ulpian’s table is not suitable for calculating life expectancy at birth. We have to make additional assumptions on infant and youth mortality. Frier (1982) sees these last assumptions as the weakest point in the life table construction. Our simulations show that life expectancy at birth strongly depends on infant and youth mortality. A range between slightly less than 20 years and slightly more than 30 years is quite possible, if only 40% to 60% of the newborns survived to the age of 20 years. An alternative to simulation is the usage of other high mortality model life tables. Further research is needed, as suggested by Woods (2007, p. 395), who states that more experimentation with low life expectancy models is required. However, these models should reflect the steep increase of the force of mortality, or rather the steep decline in the life expectancy function, at older ages in order to be suitable for a Roman life table which is based on Ulpian's table.

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