Lower Tolerance Bounds in Accelerated Life Testing for the Weibull Distribution

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ABSTRACT

In this paper we investigate the problem of obtaining lower tolerance bounds for a future observation from a Weibull population at field use (design) stress level, using Type II censored data from an accelerated life test having two levels of higher than design stress levels. The scale parameter of the life distribution is assumed to have an inverse power relationship with the stress level. We use the Maximum Likelihood Predictive Density method to derive a predictive density for a future observation as described by Jayawardhana and Samaranayake (2003). The use of a lower percentile point of the predictive density as a lower tolerance bound is investigated using Monte Carlo simulation. The results show that reasonable tolerance bounds can be provided using the predictive density for different levels of tolerance, tolerance content, sample sizes, and acceleration factor. We propose a percentile point of the predictive density with corrected content as the lower tolerance bound. An example using the data from Zhang et al. (2012) is demonstrated.

I. INTRODUCTION

Concept of tolerance interval was first introduced by Shewhart in 1931 in his book Economic Control of Manufactured Product and statisticians have ever since worked on tolerance limits under different scenarios. Statistical tolerance intervals have wide applications in engineering, manufacturing, occupational exposure, pharmaceutical development, and numerous other areas (Hoffman, 2010). A tolerance interval will cover a certain proportion of the population with a given confidence. For example, a manufacturer would like to know with certain confidence (95%), that at least a certain percentage (90%) of a product will operate longer than a specified time. Thoman, Bain, and Antle (1970) propose a method to find lower tolerance limits for a Weibull distribution using a method that involves finding a lower confidence limit for a function of the shape and scale parameters and the proportion (content) of the tolerance. Lawless (1975) discuss a method to construct tolerance bounds using a pivotal quantity for complete or Type II censored data from Extreme-value and Weibull distributions. Mann and Fertig (1977) discuss tolerance intervals for Extreme-Value and Weibull distributions using efficient unbiased quantile estimators. Verdemann, S.B. (1992) provides a motivational example of designing a gas tank large enough to guarantee 99% of the automobiles of a certain model will have at least 400 mile cruising range. This is a question of determining a tolerance interval for a 0.99 fraction of the mileage distribution of such automobiles. An interval $[L,U]$ is called a $p-$content, $\gamma-$confidence tolerance interval for a cumulative distribution function F if the statistics L and U, based on a random sample from F satisfy $P\left\{F(U)-F(L)\geq p\right\}\geq\gamma$ where, if possible, equality replaces the last inequality. The statistics L and

U are called lower and upper tolerance limits respectively. Fernholz & Gillespie (2001) define a corrected content tolerance interval as follows:

Definition: For given parameter values p and γ on $(0,1)$, a $\gamma-$confidence $p-$content corrected tolerance interval is an interval of the form $[L,U]$ if $P\{F(U)-F(L)\geq p^*\}\geq\gamma$ holds for some data-dependent p^* in which the sample comes from the distribution function F. The value p^* will be called the corrected content.

The objective of this paper is to provide lower tolerance intervals when the underlying distribution is Weibull and the data is obtained using accelerated life testing, with the Weibull scale parameter is related to the level of stress according to the inverse power law. In accelerated life testing products are subject to higher levels of stress than the nominal use level of stresses such as humidity, mechanical load, pressure, temperature, voltage, vibration, and use rate. Data collected under higher levels of stress are used to extrapolate the results to the design level of stress. Jayawardhana and Samaranayake (2003) derive a predictive density under the following assumptions and using the Maximum Likelihood Predictive Density method proposed by Lejeune and Faulkenberry (1982).

1. Product life has a Weibull distribution with cumulative density function $F(x)=1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}$; for $x>0$, where $\lambda>0$ and $\beta>0$.
2. The scale parameter θ is related to the stress by $\theta=\eta_0V^{-\eta_1}$, where $\eta_0>0$, $\eta_1>0$, and V is the level of stress or possibly a transformed level of stress. This model is known as the inverse power law model.
3. The shape parameter β of the Weibull distribution is a constant.
4. The lifetimes of the units are independent of each other.

They consider the case where acceleration involves two higher than nominal stress levels L and H, with D, L, and H respectively denoting the design, low, and high stress levels. The high stress is chosen as high as possible but within limits that ensure that the assumed physical model is reasonably accurate. The low stress level is chosen to be between the design stress level and high stress level so as to provide good estimates for η_0 and η_1 but still sufficiently high enough to yield an adequate number of failures within a reasonable time (see Meeker and Hahn, 1985). Jayawardhana and Samaranayake (2003) let θ_D, θ_L, and θ_H be the values of the scale parameter at design, low, and high stress levels respectively. They let n_L and n_H items be subjected to the low and high stresses and the experiment is continued until r_L and r_H items fail under low and high levels respectively. They let x_{L1}, x_{L2}, …, x_{Lr_L} and x_{H1}, x_{H2}, …, x_{Hr_H} be the ordered failure times for the low and high stress levels respectively and let Z to be a future observation at the design level of stress.

The 100 p th percentile point of the distribution of Z is derived to be

$$\hat{z}_p=\left[\left(V_L^{\hat{\eta}_1\hat{\beta}}A_L+V_H^{\hat{\eta}_1\hat{\beta}}A_H\right)\left\{(1-p)^{-1/(r_L+r_H)}-1\right\}\right]^{1/\hat{\beta}},$$

where $A_L = x_{Lr_L}^{\hat{\beta}}\left(n_L - r_L\right) + \sum_{j=1}^{r_L} x_{Lj}^{\hat{\beta}}$ and $A_H = x_{Hr_H}^{\hat{\beta}}\left(n_H - r_H\right) + \sum_{j=1}^{r_H} x_{Hj}^{\hat{\beta}}$. All the parameters with hats are estimates. Using the data from each level of acceleration, they estimate β separately and propose a combine estimate of β as $\hat{\beta} = \frac{r_L \tilde{\beta}_L + r_H \tilde{\beta}_H}{r_L + r_H}$. The authors also propose the estimate for η_1 given by $\hat{\eta}_1 = \frac{\ln\left(r_L \hat{A}_H\right) - \ln\left(r_H \hat{A}_L\right)}{\hat{\beta}\left\{\ln\left(V_L\right) - \ln\left(V_H\right)\right\}}$. Simulation studies have shown that the percentile estimate is slightly liberal and therefore they propose an ad-hoc adjustment to replace $r_L + r_H$ by $r_L + r_H + 4$ to modify the percentile point to $\hat{z}_p \approx \left[\left(V_L^{\hat{\eta}_1 \hat{\beta}} A_L + V_H^{\hat{\eta}_1 \hat{\beta}} A_H\right)\left\{\left(1-p\right)^{-1/\left(r_L + r_H + 4\right)} - 1\right\}\right]^{1/\hat{\beta}}$.

Turning to the problem at hand, let the β-content γ-level lower tolerance bound for the future observation be given by $\hat{z}_p$ where $P\left(\int_{\hat{z}_p}^{\infty} f\left(z\right) dz \geq p\right) \geq \gamma$. Fernholz & Gillespie (2001) proposed the content corrected tolerance intervals for distributions which slightly violated normality. It is well known that a lower tolerance limit can be interpreted as a lower confidence interval of a percentile point. Our objective is to lower the percentile point estimate so that it acts as the lower confidence limit for the percentile.

A simulation study was conducted to find out what value of p^* should be used to have the desired (fixed) levels of β and γ, for a given value of p. In all the simulations we kept $n_L = n_H = r_H$. For computational convenience the ratio $\frac{n_L - r_L}{n_L}$ was defined as the Censoring Factor and $\frac{V_L - V_D}{V_H - V_D}$ was defined as the Acceleration Factor. In real life accelerated tests the items subject to higher level of stresses fail fairly quickly and some of the items subject to lower level of stress may not even fail within reasonable time period. Therefore for simulation purposes the censoring was done only at the lower level of stress. Stress levels were carefully selected so that the distance between the higher and lower stress levels are as large as possible. The physical conditions dictate the higher levels of stress; for example excessive voltage can burn an electronic component instantaneously. So the upper level of stress has physical limitations. On the other hand the lower limit should he high enough to cause some failures to collect data for the study in a timely manner. Simulation results show that the results are reasonable under certain conditions such as $V_L < 2V_D$ and $3\left(V_L - V_D\right) < V_H - V_D$.

One can derive the lower tolerance limit as a lower confidence limit for a percentile point. Using the definition of the $p-$ content $\gamma-$confidence tolerance interval $P\left[\int_{L\left(x_1, x_2, .., x_n\right)}^{\infty} f\left(x;\theta\right) dx \geq p\right] \geq \gamma$, a simple derivation will produce the result $P\left[L\left(x_1, x_2, .., x_n\right) \leq Z_{1-p}\right] \geq \gamma$. If one can find out the $(1-\gamma)$ th percentile of the distribution of the $\left(1-p\right)$th percentile of Z , it will be equal to the required tolerance limit. An estimate of the $(1-p)$th percentile of Z is given by $\hat{Z}_{1-p} \approx \left\{\left(V_L^{\hat{\eta}\hat{\beta}}\hat{A}_L + V_H^{\hat{\eta}\hat{\beta}}\hat{A}_H\right)\left[(p)^{-1/(r_L+r_H+4)} - 1\right]\right\}^{1/\hat{\beta}}$.

Calculation of the lower confidence limit for this percentile point is difficult due to the complexity of the estimates. In this study we use the content corrected tolerance interval proposed by Fernholz & Gillespie (2001). For parameter values p and γ on (0, 1), a γ-confidence p-content corrected lower tolerance interval is an interval of the form $[L,\infty]$ if $P\left\{1-F(L) \geq p^*\right\} \geq \gamma$ holds for some data dependent p^* in which the sample coming from the distribution function F. Our approach attempts be to find this p^*.

II. CALCULATING p^*

According to our simulations, we propose the value of the $\left(1-p^*\right)$ to be the positive root of the following quadratic equation:

$$a\left(1-p^*\right)^2 + b\ \left(1-p^*\right) + c\ \text{AccFac} + d\ \text{CenFac} + \text{Constant-}(1\text{-}\gamma) = 0\,.$$

For convenience, let us use $1-p^* = q^*$. Tables at the end of the paper provides the coefficients of the terms $\left(q^*\right)^2$, q^*, AccFac, and CenFac. Tables also provide the corresponding values for the Constant term. For a given value of γ, one can find the appropriate p^* using this method.

III. EXAMPLE

Zhang et al. (2012) reports an accelerated life testing data set in which the stress is the electric current, nominal level of stress is $I_0 = 3.20\ mA$, and four accelerated levels of stresses $I_1 = 9.64\ mA$, $I_2 = 12.36\ mA$, $I_3 = 17.09\ mA$ and $I_4 = 22.58\ mA$. For this example we use the accelerated stress levels I_1 and I_3 since these two were kept as constant stress levels.

Estimates of β and η_1 are invariant of the units and therefore we converted the failure time to weeks for computational ease.

Stress	Failure time /hours									
	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
I_1	1691.5	2084.67	2100.32	2374.5	2421.5	2586	2621.5	2680.5	2868	2879.5
I_3	601.50	689.67	697.33	716.50	785.50	854.50	889.50	1115.67	1131.33	1251.50

Let V_D=1, $V_L = \dfrac{9.64}{3.2} = 3.0125$, $V_H = \dfrac{17.09}{3.2} = 5.3406$, n_L=10, n_H=10, r_L=10, r_H=10, p=0.90, and $\gamma = 0.90$. Then Censoring Factor (CenFac) = $\dfrac{n_L - r_L}{n_L} = \dfrac{10-10}{10} = 0$ and Acceleration Factor (AccFac) = $\dfrac{V_L - V_D}{V_H - V_D} = \dfrac{3.0125-1}{5.3406-1} = 0.4636$. From Table 1, a= 6.743, b= 3.2665, c= 0.3782, d=0.0125, and Constant= -0.1572. Though this data violates the best conditions learned form the simulation study, namely $V_L < 2V_D$ and $3(V_L - V_D) < V_H - V_D$, we continue the example as a demonstration of the method. Now,

$$a\left(q^*\right)^2 + b\, q^* + c\ \text{AccFac} + d\ \text{CenFac} + \text{Constant-}(1\text{-}\gamma) = 0.$$

Plugging the values to the quadratic equation we get, $6.743\left(q^*\right)^2 + 3.2665\, q^* + 0.3782\ (0.4636) + 0.0125\ (0)\text{-}0.1572\text{-}(1\text{-}0.90) = 0$ and

$$6.743\left(q^*\right)^2 + 3.2665\, q^* - 0.081866 = 0.$$

Positive root of this equation is 0.024, and therefore q^*=0.024 and p^*=0.976. Following the method described in Jayawardhana and Samaranayake (2003), the simple estimators of β using stress level I_1 and I_3 respectively are: $\tilde{\beta}_L = \dfrac{10(1.262)}{-20.9212 + \left(\dfrac{8}{10-8}\right)5.6788} = 7.034$ and $\tilde{\beta}_L = 3.744$. In our method we assumed that β is a constant regardless of the stress level. Even though the available data violates this assumption we continue the example for the demonstration purposes. Using the weighted average of the two estimates we get $\hat{\beta} = \dfrac{10(7.034) + 10(3.744)}{20} = 5.389$. Zhang et al. (2012) estimated the shape parameter at the accelerated stress levels I_1 and I_3 as 6.5764 and 4.2754 respectively. An average of their estimates is 5.4259. Other quantities can be estimated as $\hat{A}_L = 22264140$, $\hat{A}_H = 132210$, and $\hat{\eta}_1 = 1.6614$. Then the 90% confident, 90% content lower tolerance limit can be calculated as

$$\begin{aligned}
\hat{Z}_{1-p} &\approx \left\{\left(V_L^{\hat{\eta}\hat{\beta}}\hat{A}_L + V_H^{\hat{\eta}\hat{\beta}}\hat{A}_H\right)\left[(p)^{-1/(r_L+r_H+4)} - 1\right]\right\}^{1/\hat{\beta}} \\
&= \left\{\left(3.0125^{1.6614(5.389)}(22264140) + 5.3406^{1.6614(5.389)}(132210)\right)\left[(0.976)^{-1/(24)} - 1\right]\right\}^{1/5.389} \\
&= 45.6377\,weeks \\
&= 7667\ hours
\end{aligned}$$

IV. CONCLUSIONS

We propose an easy to use method to find lower tolerance interval for Weibull life distributions under accelerated life testing scenario. Under certain conditions the proposed method works well. Accurate estimation of β and η_1 is critical to the accuracy of the result. Jayawardhana and Samaranayake (2003) prescribe the estimation method for the two parameters. There are some limitations to this method. Jayawardhana and Samaranayake reports that when the difference between the high level of stress and the low level of stress relatively smaller respect to the difference between the low level of stress and the design level of stress, the estimation of the parameter η_1 is not very reliable. When the Acceleration Factor is greater than 0.3, proposed method does not work well in finding a reasonably accurate lower tolerance bound. This is mostly due to the estimation errors of the parameters of the predictive distribution. The current study is limited to the content values (p) 0.90, 0.95, and 0.99. Another limitation on this method is that this method works well only for $0.80 \leq \gamma \leq 0.975$. The values of are not very useful in practice but it would have been better if the method is accurate up for $0.80 \leq \gamma \leq 0.99$.

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Table 1

Coefficients of the Quadratic Equation for 90% Content					
$n_L = n_H$	a	b	c	d	Constant
10	6.743	3.2665	0.3782	0.0125	-0.1572
20	29.376	1.6137	0.3462	0.0211	-0.1630
30	51.875	-0.8771	0.3412	0.0484	-0.1281
40	76.198	-3.7683	0.3163	0.0347	-0.0505
50	93.152	-5.9698	0.3153	0.0572	-0.0035
60	114.570	-8.7772	0.2874	0.0393	0.0876
70	125.650	-10.2035	0.2909	0.0600	0.1180
80	141.422	-12.4653	0.2664	0.0431	0.1965

Table 2

Coefficients of the Quadratic Equation for 95% Content					
$n_L = n_H$	a	b	c	d	Constant
10	6.679	7.2132	0.3024	-0.0275	-0.1100
20	70.512	5.9575	0.2843	-0.0069	-0.1578
30	149.284	1.9177	0.2909	0.0200	-0.1443
40	232.790	-2.7148	0.2697	0.0112	-0.0910
50	300.417	-6.9999	0.2748	0.0333	-0.0505
60	380.170	-12.0250	0.2502	0.0186	0.0261
70	425.050	-15.1557	0.2582	0.0384	0.0582
80	491.020	-19.4557	0.2367	0.0241	0.1288

Table 3

Coefficients of the Quadratic Equation for 99% Content					
$n_L = n_H$	a	b	c	d	Constant
10	-436.140	38.0123	0.1339	-0.0611	-0.0111
20	1375.430	29.2763	0.1350	-0.0310	-0.0554
30	3186.310	12.6143	0.1390	-0.0141	-0.0539
40	4577.110	0.1260	0.1260	-0.0414	-0.0407
50	5786.470	-13.6466	0.1306	-0.0029	-0.0277
60	6803.100	-23.9968	0.1148	-0.0007	-0.0083
70	7372.090	-31.6350	0.1207	0.0031	-0.0021
80	8044.790	-38.7033	0.1087	-0.0018	0.0133