Deleting And Annexing Data From And To The Least-Squares Linear Regression Matrix Inverse

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Abstract
Given a collection X of m observations of n variables, and m observations of a response variable Y, it is commonly of interest to calculate the least-squares regression fit of the data according to the model $Y = X\beta$. The least-squares regression estimators of β are commonly given by $\hat{\beta} = \left(X^T X\right)^{-1} X^T Y$ where the statistical inference on the values of $\hat{\beta}$ are based on the (assumed) error structures of the model.

Since $X^T X$ in is an $n \times n$ matrix (regardless of m), and $\left(X^T X\right)^{-1}$ exists only when $m \geq n$, the complexity of calculating $\left(X^T X\right)^{-1}$ becomes significantly problematical as n becomes large. Furthermore, if even one observation changes in X, $\left(X^T X\right)^{-1}$ must be recalculated to find the new values of $\hat{\beta}$.

However, since a deletion or annexation of s-many observations from/to X means $m - s$ (or m) observations are retained from the original X, the new value of $\left(X^T X\right)^{-1}$ may be calculated from the old value of $\left(X^T X\right)^{-1}$ without the need to recalculate a new $n \times n$ matrix inverse. The update of $\left(X^T X\right)^{-1}$ based on its old value and the deleted or annexed data may be accomplished through matrix addition, negation, multiplication, and the calculation of an $s \times s$ matrix inverse. When $s = 2$, there is a simple formula for such an inverse. For $s = 1$, the inversion is scalar division.

This paper presents the analytical, calculation, and programming methods for updating the least-squares regression matrix inverse after deleting or annexing any number of observations. The utility of these methods are especially manifest when s is relatively small compared to n (regardless of $m \geq n$).

Key Words: Least-squares Regression Estimators, Matrix Inversion, Matrix Numerical Methods

1. Introduction

Given a collection X of m observations of n variables, and m observations of a response variable Y, it is commonly of interest to calculate the least-squares regression fit of the data according to the model

$$Y = X\beta \tag{1}$$

The least-squares regression estimators of β are commonly given by

$$\hat{\beta} = \left(X^T X\right)^{-1} X^T Y \tag{2}$$

where the statistical inference on the values of $\hat{\beta}$ are based on the (assumed) error structures of (1).

Since $X^T X$ in (2) is an $n \times n$ matrix (regardless of m), and $\left(X^T X\right)^{-1}$ exists only when $m \geq n$, the complexity of calculating $\left(X^T X\right)^{-1}$ becomes significantly problematical as n becomes large, i.e., $n \geq 3$. Furthermore, if even *one* observation changes in X, $\left(X^T X\right)^{-1}$ must be recalculated to find the new values of $\hat{\beta}$.

However, since a deletion or annexation of s-many observations from/to X means $m - s$ (or m) observations are retained from the original X, the new value of $\left(X^T X\right)^{-1}$ may be

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calculated from the old value of $\left(X^T X\right)^{-1}$ without the need to recalculate a new $n \times n$ matrix inverse. The update of $\left(X^T X\right)^{-1}$ based on its old value and the deleted or annexed data may be accomplished through matrix addition, negation, matrix multiplication, and the calculation of an $s \times s$ matrix inverse. When $s = 2$, there is a simple formula for such an inverse. For $s = 1$, the inversion is scalar division.

This memorandum documents the analytical, calculation, and programming methods for updating the least-squares regression matrix inverse after deleting or annexing any number of observations. The utility of these methods are especially manifest when s is relatively small compared to n (regardless of $m \geq n$).

2. Calculation Methods

In the following development, the number of observations will always be assumed to be at least as large as the number of variables. Observations may only be deleted from the data if doing so would not violate this assumption.

Furthermore, it shall be understood that no null, void, or otherwise missing data, nor indicator or signaling, non-numeric data, are allowed in any position of X. The concept of deletion and annexation are otherwise not well-defined.

Given $X_{(m+s)\times n}$ data of $m + s$ observations of n variables, suppose $\left(X^T X\right)^{-1}_{n\times n}$ has already been calculated. Let

$$X = \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \end{pmatrix}_{(m+s)\times n}$$

$$W = \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+s+p)\times n}$$

$$Q = \begin{pmatrix} r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+p)\times n}$$

for $s, p \geq 1$.

Here we may view X as the original data matrix, and W as the result of appending p-many observations of n-many variables to the end of X. Similarly, we may also view W as the original matrix, and Q as the result of deleting s-many observations of n-many variables from the beginning of W.

Since $\left(X^T X\right)^{-1}$ is invariant under exchanges of rows in X, and $\left(\left(X^T\right)^T X^T\right)^{-1}$ is invariant under exchanges of columns[1] in X^T, for any matrix X, the annexation or elimination of rows in forming W and Q, respectively, need not occur at the end or at the

[1] *Proof.* Let $X_{m\times n}$ be any matrix for $n, m \geq 1$. For $i \neq j$, define e_{ij} to be the $m \times m$ identity matrix with the i^{th} and j^{th} rows interchanged.

Then

$$X_0 = e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k} X$$

is the matrix X with a rearrangement of its rows for some permutation $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_k\}$ of $\{1, 2, \ldots, m\}$.

Hence,

$$\begin{aligned}\left(X_0^T X_0\right)^{-1} &= \left((e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k} X)^T e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k} X\right)^{-1} \\ &= \left(X^T e^T_{i_k j_k} \cdots e^T_{i_2 j_2} e^T_{i_1 j_1} e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k} X\right)^{-1}\end{aligned}$$

However,

$$e^T_{i_r j_r} e_{i_r j_r} = I_{m\times m}$$

beginning of X, respectively. The rows may be appended to, or deleted from, any position in the matrix. Furthermore, the annexations or eliminations may be performed as columns as well in X^T.

Then

$$
\begin{aligned}
X^TX &= \begin{pmatrix} \alpha^T_{1\times s} & r^T_{1\times m} \\ v^T_{(n-1)\times s} & Z^T_{(n-1)\times m} \end{pmatrix} \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \end{pmatrix} \\
&= \begin{pmatrix} \left(\alpha^T\alpha + r^Tr\right)_{1\times 1} & \left(\alpha^Tv + r^TZ\right)_{1\times(n-1)} \\ \left(v^T\alpha + Z^Tr\right)_{(n-1)\times 1} & \left(v^Tv + Z^TZ\right)_{(n-1)\times(n-1)} \end{pmatrix}_{n\times n} \\
&\begin{pmatrix} A_X & (B_X)_{1\times(n-1)} \\ \left(B_X^T\right)_{(n-1)\times 1} & (D_X)_{(n-1)\times(n-1)} \end{pmatrix}_{n\times n}
\end{aligned}
$$

where

$$
\begin{aligned}
A_X &= \alpha^T\alpha + r^Tr \\
B_X &= \alpha^Tv + r^TZ \\
D_X &= v^Tv + Z^TZ
\end{aligned}
$$

Also

$$
\begin{aligned}
W^TW &= \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}^T_{n\times(m+s+p)} \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+s+p)\times n} \\
&= \begin{pmatrix} \alpha^T_{1\times s} & r^T_{1\times m} & \gamma^T_{1\times p} \\ v^T_{(n-1)\times s} & Z^T_{(n-1)\times m} & u^T_{(n-1)\times p} \end{pmatrix} \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix} \\
&= \begin{pmatrix} \left(\alpha^T\alpha + r^Tr + \gamma^T\gamma\right)_{1\times 1} & \left(\alpha^Tv + r^TZ + \gamma^Tu\right)_{1\times(n-1)} \\ \left(v^T\alpha + Z^Tr + u^T\gamma\right)_{(n-1)\times 1} & \left(v^Tv + Z^TZ + u^Tu\right)_{(n-1)\times(n-1)} \end{pmatrix}_{n\times n}
\end{aligned}
$$

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for each $1 \leq r \leq k$. This means

$$\left(X_0^TX_0\right)^{-1} = \left(X^TX\right)^{-1}$$

Similarly, for $i \neq j$, define c_{ij} to be the $n \times n$ identity matrix with the i^{th} and j^{th} columns interchanged. Then

$$X_1^T = c_{i_1j_1}c_{i_2j_2}\cdots c_{i_kj_k}X^T$$

is the matrix X^T with a rearrangement of its columns for some permutation $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_k\}$ of $\{1, 2, \ldots, n\}$.

Hence,

$$
\begin{aligned}
\left(\left(X_1^T\right)^T X_1^T\right)^{-1} &= \left(\left(c_{i_1j_1}c_{i_2j_2}\cdots c_{i_kj_k}X^T\right)^T c_{i_1j_1}c_{i_2j_2}\cdots c_{i_kj_k}X^T\right)^{-1} \\
&= \left(\left(X^T\right)^T c^T_{i_kj_k}\cdots c^T_{i_2j_2}c^T_{i_1j_1}c_{i_1j_1}c_{i_2j_2}\cdots c_{i_kj_k}X^T\right)^{-1}
\end{aligned}
$$

However,

$$c^T_{i_rj_r}c_{i_rj_r} = I_{n\times n}$$

for each $1 \leq r \leq k$. This means

$$\left(\left(X_1^T\right)^T X_1^T\right)^{-1} = \left(\left(X^T\right)^T X^T\right)^{-1}$$

■

$$= \begin{pmatrix} A & B_{1\times(n-1)} \\ B^T_{(n-1)\times 1} & D_{(n-1)\times(n-1)} \end{pmatrix}$$

where

$$\begin{aligned} A &= \alpha^T\alpha + r^T r + \gamma^T\gamma \\ B &= \alpha^T v + r^T Z + \gamma^T u \\ D &= v^T v + Z^T Z + u^T u \end{aligned}$$

To find $\left(W^T W\right)^{-1}$, Gaussian elimination may be applied to its components:

$$\left(\begin{array}{cc|cc} A & B_{1\times(n-1)} & 1 & \mathbf{0}_{1\times(n-1)} \\ B^T_{(n-1)\times 1} & D_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \end{array} \right)$$

Multiply the second row by $D^{-1}_{(n-1)\times(n-1)}$.

$$\left(\begin{array}{cc|cc} A & B_{1\times(n-1)} & 1 & \mathbf{0}_{1\times(n-1)} \\ D^{-1}B^T_{(n-1)\times 1} & I_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times 1} & D^{-1}_{(n-1)\times(n-1)} \end{array} \right)$$

Subtract the second row times $B_{1\times(n-1)}$ from the first row.

$$\left(\begin{array}{cc|cc} A - BD^{-1}B^T & \mathbf{0}_{1\times(n-1)} & 1 & -BD^{-1}_{1\times(n-1)} \\ D^{-1}B^T_{(n-1)\times 1} & I_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times 1} & D^{-1}_{(n-1)\times(n-1)} \end{array} \right)$$

Divide the first row by $k = A - BD^{-1}B^T$.

$$\left(\begin{array}{cc|cc} 1 & \mathbf{0}_{1\times(n-1)} & \frac{1}{k} & -\frac{1}{k}BD^{-1}_{1\times(n-1)} \\ D^{-1}B^T_{(n-1)\times 1} & I_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times 1} & D^{-1}_{(n-1)\times(n-1)} \end{array} \right)$$

Subtract the first row times $D^{-1}B^T_{(n-1)\times 1}$ from the second row.

$$\left(\begin{array}{cc|cc} 1 & \mathbf{0}_{1\times(n-1)} & \frac{1}{k} & -\frac{1}{k}BD^{-1}_{1\times(n-1)} \\ \mathbf{0}_{(n-1)\times 1} & I_{(n-1)\times(n-1)} & -\frac{1}{k}D^{-1}B^T_{(n-1)\times 1} & D^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k}B^T BD^{-1}\right)_{(n-1)\times(n-1)} \end{array} \right)$$

This result gives the actual inverse, since

$$\begin{aligned} &\begin{pmatrix} A & B_{1\times(n-1)} \\ B^T_{(n-1)\times 1} & D_{(n-1)\times(n-1)} \end{pmatrix} \begin{pmatrix} \frac{1}{k} & -\frac{1}{k}BD^{-1}_{1\times(n-1)} \\ -\frac{1}{k}D^{-1}B^T_{(n-1)\times 1} & D^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k}B^T BD^{-1}\right)_{(n-1)\times(n-1)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{k}A - \frac{1}{k}BD^{-1}B^T & \left(-\frac{1}{k}ABD^{-1} + BD^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k}B^T BD^{-1}\right)\right)_{1\times(n-1)} \\ \left(\frac{1}{k}B^T - \frac{1}{k}DD^{-1}B^T\right)_{(n-1)\times 1} & \left(-\frac{1}{k}B^T BD^{-1} + DD^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k}B^T BD^{-1}\right)\right)_{(n-1)\times(n-1)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \left(-\frac{1}{k}ABD^{-1} + BD^{-1} + \frac{1}{k}\left(BD^{-1}B^T\right)BD^{-1}\right)_{1\times(n-1)} \\ \mathbf{0}_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \left(BD^{-1} - \frac{1}{k}\left(A - BD^{-1}B^T\right)BD^{-1}\right)_{1\times(n-1)} \\ \mathbf{0}_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathbf{0}_{1\times(n-1)} \\ \mathbf{0}_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} \frac{1}{k} & -\frac{1}{k}BD^{-1}_{1\times(n-1)} \\ -\frac{1}{k}D^{-1}B^T_{(n-1)\times 1} & D^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k}B^T BD^{-1}\right)_{(n-1)\times(n-1)} \end{pmatrix} \begin{pmatrix} A & B_{1\times(n-1)} \\ B^T_{(n-1)\times 1} & D_{(n-1)\times(n-1)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{k}A - \frac{1}{k}BD^{-1}B^T & \left(\frac{1}{k}B - \frac{1}{k}BD^{-1}D\right)_{1\times(n-1)} \\ \begin{pmatrix} -\frac{1}{k}AD^{-1}B^T \\ +D^{-1}\begin{pmatrix} I_{(n-1)\times(n-1)} \\ +\frac{1}{k}B^TBD^{-1} \end{pmatrix} B^T \end{pmatrix}_{(n-1)\times 1} & \begin{pmatrix} -\frac{1}{k}D^{-1}B^TB \\ +D^{-1}\begin{pmatrix} I_{(n-1)\times(n-1)} \\ +\frac{1}{k}B^TBD^{-1} \end{pmatrix} \end{pmatrix} D_{(n-1)\times(n-1)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{0}_{1\times(n-1)} \\ \begin{pmatrix} -\frac{1}{k}AD^{-1}B^T + D^{-1}B^T \\ +\frac{1}{k}D^{-1}B^T\left(BD^{-1}B^T\right) \end{pmatrix}_{(n-1)\times 1} & \left(I - \frac{1}{k}D^{-1}B^TB + \frac{1}{k}D^{-1}B^TB\right)_{(n-1)\times(n-1)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{0}_{1\times(n-1)} \\ \left(\left(1 - \frac{1}{k}\left(A - BD^{-1}B^T\right)\right)D^{-1}B^T\right)_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{0}_{1\times(n-1)} \\ \mathbf{0}_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \end{pmatrix}$$

Therefore,

$$\left(W^TW\right)^{-1} = \begin{pmatrix} \frac{1}{k} & -\frac{1}{k}BD^{-1}_{1\times(n-1)} \\ -\frac{1}{k}D^{-1}B^T_{(n-1)\times 1} & D^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k}B^TBD^{-1}\right)_{(n-1)\times(n-1)} \end{pmatrix}_{n\times n} \tag{3}$$

where

$$\begin{pmatrix} A & B_{1\times(n-1)} \\ B^T_{(n-1)\times 1} & D_{(n-1)\times(n-1)} \end{pmatrix} = \begin{pmatrix} \left(\alpha^T\alpha + r^Tr + \gamma^T\gamma\right)_{1\times 1} & \left(\alpha^Tv + r^TZ + \gamma^Tu\right)_{1\times(n-1)} \\ \left(v^T\alpha + Z^Tr + u^T\gamma\right)_{(n-1)\times 1} & \left(v^Tv + Z^TZ + u^Tu\right)_{(n-1)\times(n-1)} \end{pmatrix}_{n\times n}$$

and

$$k = A - BD^{-1}B^T$$

is a scalar.

Applying the same process to X^TX, we have

$$\left(X^TX\right)^{-1} = \begin{pmatrix} \frac{1}{k_X} & -\frac{1}{k_X}B_XD_X^{-1} \\ -\frac{1}{k_X}D_X^{-1}B_X^T & D_X^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k_X}B_X^TB_XD_X^{-1}\right) \end{pmatrix}_{n\times n}$$

with

$$k_X = A_X - B_XD_X^{-1}B_X^T$$

and for Q^TQ, we have

$$\left(Q^TQ\right)^{-1} = \begin{pmatrix} \frac{1}{k_Q} & -\frac{1}{k_Q}B_QD_Q^{-1} \\ -\frac{1}{k_Q}D_Q^{-1}B_Q^T & D_Q^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k_Q}B_Q^TB_QD_Q^{-1}\right) \end{pmatrix}$$

with

$$k_Q = A_Q - B_QD_Q^{-1}B_Q^T$$

and

$$A_Q = r^Tr + \gamma^T\gamma$$
$$B_Q = r^TZ + \gamma^Tu$$
$$D_Q = Z^TZ + u^Tu$$

Define

$$\left(X^TX\right)^{-1} = \begin{pmatrix} X_1 & (X_2)_{1\times(n-1)} \\ \left(X_2^T\right)_{(n-1)\times 1} & (X_3)_{(n-1)\times(n-1)} \end{pmatrix}$$

where

$$X_1 = \frac{1}{k_X}$$
$$X_2 = -\frac{1}{k_X} B_X D_X^{-1}$$
$$X_3 = D_X^{-1} \left(I_{(n-1)\times(n-1)} + \frac{1}{k_X} B_X^T B_X D_X^{-1} \right)$$

Now note that

$$D_X X_3 = I_{(n-1)\times(n-1)} + \frac{1}{k_X} B_X^T B_X D_X^{-1} \quad \text{and} \quad X_3 D_X = I_{(n-1)\times(n-1)} + \frac{1}{k_X} D_X^{-1} B_X^T B_X$$

and

$$B_X^T X_2 = -\frac{1}{k_X} B_X^T B_X D_X^{-1} \quad \text{and} \quad X_2^T B_X = -\frac{1}{k_X} D_X^{-1} B_X^T B_X$$

so that

$$D_X X_3 + B_X^T X_2 = I_{(n-1)\times(n-1)} = X_3 D_X + X_2^T B_X$$

However, since

$$X_2^T = -\frac{1}{k_X} D_X^{-1} B_X^T \quad \text{and} \quad X_2 = -\frac{1}{k_X} B_X D_X^{-1}$$

then

$$B_X^T = -\frac{1}{X_1} D_X X_2^T \quad \text{and} \quad B_X = -\frac{1}{X_1} X_2 D_X$$

which mean

$$D_X X_3 + \left(-\frac{1}{X_1} D_X X_2^T \right) X_2 = I_{(n-1)\times(n-1)} = X_3 D_X + X_2^T \left(-\frac{1}{X_1} X_2 D_X \right)$$

or

$$D_X \left(X_3 - \frac{1}{X_1} X_2^T X_2 \right) = I_{(n-1)\times(n-1)} = \left(X_3 - \frac{1}{X_1} X_2^T X_2 \right) D_X$$

Hence,

$$D_X^{-1} = X_3 - \frac{1}{X_1} X_2^T X_2 \tag{4}$$

Lemma 1 *If* $I_{s\times s} \neq v \left(Z^T Z \right)^{-1} v^T$ *and* $\left(v \left(Z^T Z \right)^{-1} v^T \right)^{-1}$ *exists, then*

$$\left(Z^T Z \pm v^T v \right)^{-1} = \left(Z^T Z \right)^{-1} \mp \left(Z^T Z \right)^{-1} v^T \left(I_{s\times s} \pm v \left(Z^T Z \right)^{-1} v^T \right)^{-1} v \left(Z^T Z \right)^{-1}$$

Proof. We have

$$\left(Z^T Z \right)^{-1} v^T v = \left(Z^T Z \right)^{-1} v^T \left[I_{s\times s} \pm v \left(Z^T Z \right)^{-1} v^T \right]^{-1} \left[I_{s\times s} \pm v \left(Z^T Z \right)^{-1} v^T \right] v$$

and therefore

$$\begin{aligned} \left(Z^T Z \right)^{-1} v^T v &= \begin{pmatrix} \left(Z^T Z \right)^{-1} v^T \left(I_{s\times s} \pm v \left(Z^T Z \right)^{-1} v^T \right)^{-1} v \\ \pm \left(Z^T Z \right)^{-1} v^T \left(I_{s\times s} \pm v \left(Z^T Z \right)^{-1} v^T \right)^{-1} v \left(Z^T Z \right)^{-1} v^T v \end{pmatrix} \\ &= \left(\left(Z^T Z \right)^{-1} v^T \left(I_{s\times s} \pm v \left(Z^T Z \right)^{-1} v^T \right)^{-1} v \left(Z^T Z \right)^{-1} \right) \left(Z^T Z \pm v^T v \right) \end{aligned}$$

which means

$$\pm I = \pm I + \left(\left(Z^T Z\right)^{-1} v^T v - \left(\left(Z^T Z\right)^{-1} v^T \left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right)\left(Z^T Z \pm v^T v\right)\right)$$

$$= \pm\left(Z^T Z\right)^{-1}\left(Z^T Z\right) + \left(Z^T Z\right)^{-1} v^T v - \left(\left(Z^T Z\right)^{-1} v^T \left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right)\left(Z^T Z \pm v^T v\right)$$

$$= \pm\left(Z^T Z\right)^{-1}\left(\left(Z^T Z\right) \pm v^T v\right) - \left(\left(Z^T Z\right)^{-1} v^T \left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right)\left(Z^T Z \pm v^T v\right)$$

$$= \left(\pm\left(Z^T Z\right)^{-1} - \left(\left(Z^T Z\right)^{-1} v^T \left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right)\right)\left(Z^T Z \pm v^T v\right)$$

Similarly,

$$v^T v\left(Z^T Z\right)^{-1} = v^T\left[I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right]\left[I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right]^{-1} v\left(Z^T Z\right)^{-1}$$

and therefore

$$v^T v\left(Z^T Z\right)^{-1} = \begin{pmatrix} v^T\left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1} \\ \pm v^T v\left(Z^T Z\right)^{-1} v^T\left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1} \end{pmatrix}$$

$$= \left(Z^T Z \pm v^T v\right)\left(\left(Z^T Z\right)^{-1} v^T\left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right)$$

which means

$$\pm I = \pm I + \left(v^T v\left(Z^T Z\right)^{-1} - \left(Z^T Z \pm v^T v\right)\left(\left(Z^T Z\right)^{-1} v^T\left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right)\right)$$

$$= \pm\left(Z^T Z\right)\left(Z^T Z\right)^{-1} + v^T v\left(Z^T Z\right)^{-1} - \left(Z^T Z \pm v^T v\right)\left(\left(Z^T Z\right)^{-1} v^T\left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right)$$

$$= \left(\left(Z^T Z\right) \pm v^T v\right)\left(\pm Z^T Z\right)^{-1} - \left(Z^T Z \pm v^T v\right)\left(\left(Z^T Z\right)^{-1} v^T\left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right)$$

$$= \left(Z^T Z \pm v^T v\right)\left(\pm\left(Z^T Z\right)^{-1} - \left(\left(Z^T Z\right)^{-1} v^T\left(I_{s\times s} \pm v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right)\right)$$

Hence,

$$\left(Z^T Z + v^T v\right)^{-1} = \left(Z^T Z\right)^{-1} - \left(\left(Z^T Z\right)^{-1} v^T\left(I_{s\times s} + v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right) \tag{5}$$

and

$$\left(Z^T Z - v^T v\right)^{-1} = \left(Z^T Z\right)^{-1} + \left(\left(Z^T Z\right)^{-1} v^T\left(I_{s\times s} - v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z\right)^{-1}\right) \tag{6}$$

■

Note also that, using (6), we have

$$\left(Z^T Z\right)^{-1} = \left(\left(Z^T Z + v^T v\right) - v^T v\right)^{-1}$$

$$= \left(Z^T Z + v^T v\right)^{-1} + \left(\left(Z^T Z + v^T v\right)^{-1} v^T\left(I_{s\times s} - v\left(Z^T Z\right)^{-1} v^T\right)^{-1} v\left(Z^T Z + v^T v\right)^{-1}\right) \tag{7}$$

2.1 Deleting Observations

Given $W = \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+s+p)\times n}$ data of $m+s+p$ observations of n variables, and $\left(W^T W\right)^{-1}_{n\times n}$, if s-many observations are deleted from W to form

$$Q = \begin{pmatrix} r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+p)\times n}$$

then $\left(Q^T Q\right)^{-1}$ may be calculated through (3), namely

$$\left(Q^T Q\right)^{-1} = \begin{pmatrix} \frac{1}{k_Q} & -\frac{1}{k_Q} B_Q D_Q^{-1} \\ -\frac{1}{k_Q} D_Q^{-1} B_Q^T & D_Q^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k_Q} B_Q^T B_Q D_Q^{-1}\right) \end{pmatrix}_{n\times n} \tag{8}$$

with

$$k_Q = A_Q - B_Q D_Q^{-1} B_Q^T$$

and

$$\begin{aligned} A_Q &= r^T r + \gamma^T \gamma \\ B_Q &= r^T Z + \gamma^T u \\ D_Q &= Z^T Z + u^T u \end{aligned}$$

From (6) we have

$$\begin{aligned} D_Q^{-1} &= \left(Z^T Z + u^T u\right)^{-1} \\ &= \left(\left(Z^T Z + u^T u + v^T v\right) - v^T v\right)^{-1} \\ &= D^{-1} + D^{-1} v^T \left(I_{s\times s} - v D^{-1} v^T\right)^{-1} v D^{-1} \end{aligned}$$

where

$$D^{-1} = W_3 - \frac{1}{W_1} W_2^T W_2 \tag{9}$$

and

$$\left(W^T W\right)^{-1} = \begin{pmatrix} W_1 & (W_2)_{1\times(n-1)} \\ \left(W_2^T\right)_{(n-1)\times 1} & (W_3)_{(n-1)\times(n-1)} \end{pmatrix} \tag{10}$$

by analogy with (4).

Hence, $\left(Q^T Q\right)^{-1}$ is completely described by (8), (9), and (10), given $\left(W^T W\right)^{-1}$, with D invertible and $I_{s\times s} \neq v D^{-1} v^T$.

Note that (5) could have been used if data were being deleted off the top of W to form X.

See the Appendix for more information on calculating $\left(I_{s\times s} - v D^{-1} v^T\right)^{-1}$ from known components.

2.2 Annexing Observations

Given $X = \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \end{pmatrix}_{(m+s)\times n}$ data of $m+s$ observations of n variables, and $\left(X^T X\right)^{-1}_{n\times n}$, if p-many observations are annexed to X to form

$$W = \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+s+p)\times n}$$

then $\left(W^TW\right)^{-1}$ may be calculated through (3), namely

$$\left(W^TW\right)^{-1} = \begin{pmatrix} \frac{1}{k} & -\frac{1}{k}BD^{-1}_{1\times(n-1)} \\ -\frac{1}{k}D^{-1}B^T_{(n-1)\times 1} & D^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k}B^TBD^{-1}\right)_{(n-1)\times(n-1)} \end{pmatrix}_{n\times n} \tag{11}$$

with

$$k = A - BD^{-1}B^T$$

and

$$\begin{aligned} A &= \alpha^T\alpha + r^Tr + \gamma^T\gamma \\ B &= \alpha^Tv + r^TZ + \gamma^Tu \\ D &= v^Tv + Z^TZ + u^Tu \end{aligned}$$

From (5) we then have

$$\begin{aligned} D^{-1} &= \left(v^Tv + Z^TZ + u^Tu\right)^{-1} \\ &= \left(\left(Z^TZ + v^Tv\right) + u^Tu\right)^{-1} \\ &= D_X^{-1} - D_X^{-1}u^T\left(I_{p\times p} + uD_X^{-1}u^T\right)^{-1}uD_X^{-1} \end{aligned}$$

where

$$D_X^{-1} = X_3 - \frac{1}{X_1}X_2^TX_2 \tag{12}$$

and

$$\left(X^TX\right)^{-1} = \begin{pmatrix} X_1 & (X_2)_{1\times(n-1)} \\ \left(X_2^T\right)_{(n-1)\times 1} & (X_3)_{(n-1)\times(n-1)} \end{pmatrix} \tag{13}$$

Hence, $\left(W^TW\right)^{-1}$ is completely described by (11), (12), and (13), given $\left(X^TX\right)^{-1}$, with D_X invertible and $I_{p\times p} \neq -uD_X^{-1}u^T$.

Note that (7) could have been used if data were being annexed to the top of Q to form W.

2.3 Exception Handling

Since the conditions $I_{s\times s} = vD^{-1}v^T$ and $I_{s\times s} = -vD^{-1}v^T$ cannot be true at the same time, then if $I_{s\times s} = vD^{-1}v^T$, we may choose at least one additional row for the role of $\begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \end{pmatrix}$ as a substitute in (a), and then annex the extra row(s) to the result to obtain the desired deletion.

Likewise, since the conditions $I_{p\times p} = uD_X^{-1}u^T$ and $I_{p\times p} = -uD_X^{-1}u^T$ cannot be true at the same time, then if $I_{p\times p} = -uD_X^{-1}u^T$, we may choose at least one arbitrary extra row for the role of $\begin{pmatrix} \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}$ as a substitute in (b), and then delete the extra row(s) from the result to obtain the desired annexation.

These steps may need to be taken when D^{-1} or D_X^{-1} does not exist.

3. Sequential Application

Claim 2 *Suppose s-many rows are deleted from W to form Q, followed by deleting m-many rows from Q to form R. Suppose further that $(s+m)$-many rows are deleted from W to form R^*. Then $\left(R^TR\right)^{-1} = \left(R^{*T}R^*\right)^{-1}$.*

Claim 3 *Suppose m-many rows are annexed to R to form Q, followed by annexing m-many rows to Q to form W. Suppose further that $(s+m)$-many rows are annexed to R to form W^*. Then $\left(W^TW\right)^{-1} = \left(W^{*T}W^*\right)^{-1}$.*

Proof. Both of these claims may be proven by the same method, where the "forward" of one claim's proof may be considered the "reverse" of the other claim's proof.

Given $W = \begin{pmatrix} \alpha_{s\times 1} & v_{s\times(n-1)} \\ r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+s+p)\times n}$ data of $m+s+p$ observations of n variables, and $\left(W^TW\right)^{-1}_{n\times n}$, if s-many observations are deleted from W to form $Q = \begin{pmatrix} r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+p)\times n}$, then $\left(Q^TQ\right)^{-1}$ may be calculated as

$$\left(Q^TQ\right)^{-1} = \begin{pmatrix} \frac{1}{k_Q} & -\frac{1}{k_Q}B_QD_Q^{-1} \\ -\frac{1}{k_Q}D_Q^{-1}B_Q^T & D_Q^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k_Q}B_Q^TB_QD_Q^{-1}\right) \end{pmatrix}_{n\times n}$$

with

$$Q^TQ = \begin{pmatrix} A_Q & B_Q \\ B_Q^T & D_Q \end{pmatrix}_{n\times n}$$

and

$$k_Q = A_Q - B_QD_Q^{-1}B_Q^T$$

and

$$\begin{aligned} A_Q &= r^Tr + \gamma^T\gamma \\ B_Q &= r^TZ + \gamma^Tu \\ D_Q &= Z^TZ + u^Tu \\ D_Q^{-1} &= D^{-1} \pm D^{-1}v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}vD^{-1} \\ D^{-1} &= W_3 - \frac{1}{W_1}W_2^TW_2 \end{aligned}$$

where

$$\left(W^TW\right)^{-1} = \begin{pmatrix} W_1 & (W_2)_{1\times(n-1)} \\ \left(W_2^T\right)_{(n-1)\times 1} & (W_3)_{(n-1)\times(n-1)} \end{pmatrix}$$

Now if m-many observations are deleted from Q to form $R = \begin{pmatrix} \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{p\times n}$, then $\left(R^TR\right)^{-1}$ may be calculated as

$$\left(R^TR\right)^{-1} = \begin{pmatrix} \frac{1}{k_R} & -\frac{1}{k_R}B_RD_R^{-1} \\ -\frac{1}{k_R}D_R^{-1}B_R^T & D_R^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k_R}B_R^TB_RD_R^{-1}\right) \end{pmatrix}_{n\times n}$$

with

$$R^TR = \begin{pmatrix} A_R & B_R \\ B_R^T & D_R \end{pmatrix}_{n\times n}$$

and

$$k_R = A_R - B_RD_R^{-1}B_R^T$$

and

$$A_R = \gamma^T\gamma$$

$$
\begin{aligned}
B_R &= \gamma^T u \\
D_R &= u^T u \\
D_R^{-1} &= D_Q^{-1} \pm D_Q^{-1} Z^T \left(I_{m\times m} \mp Z D_Q^{-1} Z^T \right)^{-1} Z D_Q^{-1}
\end{aligned}
$$

However, if $(s+m)$-many observations are deleted from W to form R^*, then A_{R^*} and B_{R^*} are calculated as for R, and $\left(R^{*T} R^*\right)^{-1}$ is calculated with

$$
D_R^{*-1} = D^{-1} \pm D^{-1} J^T \left(I_{(s+m)\times(s+m)} \mp J D^{-1} J^T \right)^{-1} J D^{-1}
$$

where

$$
J_{(s+m)\times(n-1)} = \begin{pmatrix} v_{s\times(n-1)} \\ Z_{m\times(n-1)} \end{pmatrix}
$$

It suffices to show that

$$
D_R^{*-1} = D_R^{-1}
$$

since $A_R = A_{R^*}$ and $B_R = B_{R^*}$.

We have

$$
\begin{aligned}
I_{(s+m)\times(s+m)} \mp J D^{-1} J^T &= I_{(s+m)\times(s+m)} \mp \begin{pmatrix} v_{s\times(n-1)} \\ Z_{m\times(n-1)} \end{pmatrix} D^{-1} \begin{pmatrix} v_{s\times(n-1)} \\ Z_{m\times(n-1)} \end{pmatrix}^T \\
&= \begin{pmatrix} I_{s\times s} \mp v D^{-1} v^T & \mp v D^{-1} Z^T \\ \mp Z D^{-1} v^T & I_{m\times m} \mp Z D^{-1} Z^T \end{pmatrix}_{(s+m)\times(s+m)}
\end{aligned}
$$

Hence,

$$
\left(I_{(s+m)\times(s+m)} \mp J D^{-1} J^T \right)^{-1} = \begin{pmatrix} N_{s\times s} & M_{s\times m} \\ P_{m\times s} & S_{m\times m} \end{pmatrix}_{(s+m)\times(s+m)}
$$

where

$$
\begin{aligned}
N &= \left(I_{s\times s} \mp v D^{-1} v^T \right)^{-1} \left(I_{s\times s} \mp v D^{-1} v^T + v D^{-1} Z^T \left(I_{m\times m} \mp Z D_Q^{-1} Z^T \right)^{-1} Z D^{-1} v^T \right) \left(I_{s\times s} \mp v D^{-1} v^T \right)^{-1} \\
M &= \pm \left(I_{s\times s} \mp v D^{-1} v^T \right)^{-1} v D^{-1} Z^T \left(I_{m\times m} \mp Z D_Q^{-1} Z^T \right)^{-1} \\
P &= \pm \left(I_{m\times m} \mp Z D_Q^{-1} Z^T \right)^{-1} Z D^{-1} v^T \left(I_{s\times s} \mp v D^{-1} v^T \right)^{-1} \\
S &= \left(I_{m\times m} \mp Z D_Q^{-1} Z^T \right)^{-1}
\end{aligned}
$$

Then $J^T \left(I_{(s+m)\times(s+m)} \mp J D^{-1} J^T \right)^{-1} J$ may be expressed as follows.

$$
\begin{aligned}
&\begin{pmatrix} v_{s\times(n-1)} \\ Z_{m\times(n-1)} \end{pmatrix}^T \begin{pmatrix} N_{s\times s} & M_{s\times m} \\ P_{m\times s} & S_{m\times m} \end{pmatrix} \begin{pmatrix} v_{s\times(n-1)} \\ Z_{m\times(n-1)} \end{pmatrix} \\
&= \begin{pmatrix} \left(v^T N + Z^T P \right)_{(n-1)\times s} & \left(v^T M + Z^T S \right)_{(n-1)\times m} \end{pmatrix} \begin{pmatrix} v_{s\times(n-1)} \\ Z_{m\times(n-1)} \end{pmatrix} \\
&= \left(v^T N v + Z^T P v + v^T M Z + Z^T S Z \right)_{(n-1)\times(n-1)} \\
&= \begin{pmatrix} v^T \left(I_{s\times s} \mp v D^{-1} v^T \right)^{-1} \begin{pmatrix} I_{s\times s} \mp v D^{-1} v^T \\ + v D^{-1} Z^T \left(I_{m\times m} \mp Z D_Q^{-1} Z^T \right)^{-1} Z D^{-1} v^T \end{pmatrix} \left(I_{s\times s} \mp v D^{-1} v^T \right)^{-1} v \\
\pm Z^T \left(I_{m\times m} \mp Z D_Q^{-1} Z^T \right)^{-1} Z D^{-1} v^T \left(I_{s\times s} \mp v D^{-1} v^T \right)^{-1} v \\
\pm v^T \left(I_{s\times s} \mp v D^{-1} v^T \right)^{-1} v D^{-1} Z^T \left(I_{m\times m} \mp Z D_Q^{-1} Z^T \right)^{-1} Z \\
+ Z^T \left(I_{m\times m} \mp Z D_Q^{-1} Z^T \right)^{-1} Z \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} Z^T\left(I_{m\times m} \mp ZD_Q^{-1}Z^T\right)^{-1}Z\left(I_{(n-1)\times(n-1)} \pm D^{-1}v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}v\right) \\ +v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}v\left(I_{(n-1)\times(n-1)} \pm D^{-1}Z^T\left(I_{m\times m} \mp ZD_Q^{-1}Z^T\right)^{-1}Z\right) \\ +v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}vD^{-1}Z^T\left(I_{m\times m} \mp ZD_Q^{-1}Z^T\right)^{-1}ZD^{-1}v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}v \end{pmatrix}$$

$$= \begin{pmatrix} Z^T\left(I_{m\times m} \mp ZD_Q^{-1}Z^T\right)^{-1}Z\left(D^{-1} \pm D^{-1}v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}vD^{-1}\right)D \\ +v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}v\begin{pmatrix} I_{(n-1)\times(n-1)} \\ \pm D^{-1}Z^T\left(I_{m\times m} \mp ZD_Q^{-1}Z^T\right)^{-1}Z\left(D^{-1} \pm D^{-1}v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}vD^{-1}\right)D \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}v \\ +\left(I_{(n-1)\times(n-1)} \pm v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}vD^{-1}\right)Z^T\left(I_{m\times m} \mp ZD_Q^{-1}Z^T\right)^{-1}ZD_Q^{-1}D \end{pmatrix}$$

Therefore,

$$\begin{aligned} D_R^{*-1} &= D^{-1} \pm D^{-1}J^T\left(I_{(s+m)\times(s+m)} \mp JD^{-1}J^T\right)^{-1}JD^{-1} \\ &= \begin{pmatrix} D^{-1} \pm D^{-1}v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}vD^{-1} \\ \pm\left(D^{-1} \pm D^{-1}v^T\left(I_{s\times s} \mp vD^{-1}v^T\right)^{-1}vD^{-1}\right)Z^T\left(I_{m\times m} \mp ZD_Q^{-1}Z^T\right)^{-1}ZD_Q^{-1} \end{pmatrix} \\ &= D_Q^{-1} \pm D_Q^{-1}Z^T\left(I_{m\times m} \mp ZD_Q^{-1}Z^T\right)^{-1}ZD_Q^{-1} \\ &= D_R^{-1} \end{aligned}$$

■

4. Commutative Application

Claim 4 *Suppose m-many rows are deleted from Q to form R, followed by annexing c-many rows to R to form Y. Suppose further that c-many rows are annexed to Q to form S, followed by deleting m-many rows from S to form Y^*. Then $\left(Y^TY\right)^{-1} = \left(Y^{*T}Y^*\right)^{-1}$.*

Proof. Suppose m-many observations are deleted from $Q = \begin{pmatrix} r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+p)\times n}$ to form $R = \begin{pmatrix} \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{p\times n}$, then $\left(R^TR\right)^{-1}$ may be calculated as

$$\left(R^TR\right)^{-1} = \begin{pmatrix} \frac{1}{k_R} & -\frac{1}{k_R}B_RD_R^{-1} \\ -\frac{1}{k_R}D_R^{-1}B_R^T & D_R^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k_R}B_R^TB_RD_R^{-1}\right) \end{pmatrix}_{n\times n}$$

with

$$R^TR = \begin{pmatrix} A_R & B_R \\ B_R^T & D_R \end{pmatrix}_{n\times n}$$

and

$$k_R = A_R - B_RD_R^{-1}B_R^T$$

and

$$\begin{aligned} A_R &= \gamma^T\gamma \\ B_R &= \gamma^Tu \\ D_R &= u^Tu \\ D_R^{-1} &= D_Q^{-1} + D_Q^{-1}Z^T\left(I_{m\times m} - ZD_Q^{-1}Z^T\right)^{-1}ZD_Q^{-1} \end{aligned}$$

Now suppose c-many observations are annexed to R to form $Y = \begin{pmatrix} \gamma_{p\times 1} & u_{p\times(n-1)} \\ l_{c\times 1} & g_{c\times(n-1)} \end{pmatrix}_{(p+c)\times n}$, then $\left(Y^TY\right)^{-1}$ may be calculated as

$$\left(Y^TY\right)^{-1} = \begin{pmatrix} \frac{1}{k_Y} & -\frac{1}{k_Y}B_YD_Y^{-1} \\ -\frac{1}{k_Y}D_Y^{-1}B_Y^T & D_Y^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k_Y}B_Y^TB_YD_Y^{-1}\right) \end{pmatrix}_{n\times n}$$

with

$$Y^TY = \begin{pmatrix} A_Y & B_Y \\ B_Y^T & D_Y \end{pmatrix}_{n\times n}$$

and

$$k_Y = A_Y - B_YD_Y^{-1}B_Y^T$$

and

$$\begin{aligned} A_Y &= \gamma^T\gamma + l^Tl \\ B_Y &= \gamma^Tu + l^Tg \\ D_Y &= u^Tu + g^Tg \\ D_Y^{-1} &= D_R^{-1} - D_R^{-1}g^T\left(I_{c\times c} + gD_R^{-1}g^T\right)^{-1}gD_R^{-1} \end{aligned}$$

However, if c-many observations are annexed to $Q = \begin{pmatrix} r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+p)\times n}$ to form $S = \begin{pmatrix} r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \\ l_{c\times 1} & g_{c\times(n-1)} \end{pmatrix}_{(m+p+c)\times n}$, then $\left(S^TS\right)^{-1}$ may be calculated as

$$\left(S^TS\right)^{-1} = \begin{pmatrix} \frac{1}{k_S} & -\frac{1}{k_S}B_SD_S^{-1} \\ -\frac{1}{k_S}D_S^{-1}B_S^T & D_S^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k_S}B_S^TB_SD_S^{-1}\right) \end{pmatrix}_{n\times n}$$

with

$$S^TS = \begin{pmatrix} A_S & B_S \\ B_S^T & D_S \end{pmatrix}_{n\times n}$$

and

$$k_Y = A_S - B_SD_S^{-1}B_S^T$$

and

$$\begin{aligned} A_S &= r^Tr + \gamma^T\gamma + l^Tl \\ B_S &= r^TZ + \gamma^Tu + l^Tg \\ D_S &= Z^TZ + u^Tu + g^Tg \\ D_S^{-1} &= D_Q^{-1} - D_Q^{-1}g^T\left(I_{c\times c} + gD_Q^{-1}g^T\right)^{-1}gD_Q^{-1} \end{aligned}$$

And if m-many observations are deleted from $S = \begin{pmatrix} r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \\ l_{c\times 1} & g_{c\times(n-1)} \end{pmatrix}_{(m+p+c)\times n}$ to form $Y^* = \begin{pmatrix} \gamma_{p\times 1} & u_{p\times(n-1)} \\ l_{c\times 1} & g_{c\times(n-1)} \end{pmatrix}_{(p+c)\times n}$, then $\left(Y^{*T}Y^*\right)^{-1}$ may be calculated as

$$\left(Y^{*T}Y^*\right)^{-1} = \begin{pmatrix} \frac{1}{k_{Y^*}} & -\frac{1}{k_{Y^*}}B_{Y^*}D_{Y^*}^{-1} \\ -\frac{1}{k_{Y^*}}D_{Y^*}^{-1}B_{Y^*}^T & D_{Y^*}^{-1}\left(I_{(n-1)\times(n-1)} + \frac{1}{k_{Y^*}}B_{Y^*}^TB_{Y^*}D_{Y^*}^{-1}\right) \end{pmatrix}_{n\times n}$$

with

$$Y^{*T}Y^* = \begin{pmatrix} A_{Y^*} & B_{Y^*} \\ B^T_{Y^*} & D_{Y^*} \end{pmatrix}_{n\times n}$$

and

$$k_{Y^*} = A_{Y^*} - B_{Y^*} D^{-1}_{Y^*} B^T_{Y^*}$$

and

$$\begin{aligned} A_{Y^*} &= \gamma^T\gamma + l^T l \\ B_{Y^*} &= \gamma^T u + l^T g \\ D_{Y^*} &= u^T u + g^T g \\ D^{-1}_{Y^*} &= D^{-1}_S + D^{-1}_S Z^T \left(I_{m\times m} - ZD^{-1}_S Z^T\right)^{-1} ZD^{-1}_S \end{aligned}$$

However, since $D_{Y^*} = D_Y$, then $D^{-1}_{Y^*} = D^{-1}_Y$, which means $\left(Y^{*T}Y^*\right)^{-1} = \left(Y^TY\right)^{-1}$, since $A_{Y^*} = A_Y$ and $B_{Y^*} = B_Y$. ■

Corollary 5 *Suppose m-many rows are deleted from* $Q = \begin{pmatrix} r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{(m+p)\times n}$ *to form* $R = \begin{pmatrix} \gamma_{p\times 1} & u_{p\times(n-1)} \end{pmatrix}_{p\times n}$.

Suppose further that c-many rows are annexed to Q to form $S = \begin{pmatrix} r_{m\times 1} & Z_{m\times(n-1)} \\ \gamma_{p\times 1} & u_{p\times(n-1)} \\ l_{c\times 1} & g_{c\times(n-1)} \end{pmatrix}_{(m+p+c)\times n}$.

Then

$$D^{-1}_S + D^{-1}_S g^T \left(I_{c\times c} - gD^{-1}_S g^T\right)^{-1} gD^{-1}_S = D^{-1}_R - D^{-1}_R Z^T \left(I_{m\times m} + ZD^{-1}_R Z^T\right)^{-1} ZD^{-1}_R \tag{14}$$

Note the special case in this corollary where $m = 1$ or $c = 1$. When $m = 1$, we have $D^{-1}_R - \frac{\left(ZD^{-1}_R\right)^T ZD^{-1}_R}{1+ZD^{-1}_R Z^T}$ on the right hand side of (14), which might be significantly easier to calculate than the left hand side. A similar result holds for $c = 1$.

5. Initialization

At the beginning of an analysis there may be no data to populate X. As an initialization, let $X = I_{n\times n}$, where n is the number of variables for which data is available. In this case, $\left(X^TX\right)^{-1} = I_{n\times n}$.

When the first observation $\begin{pmatrix} \gamma_{1\times 1} & u_{1\times(n-1)} \end{pmatrix}$ is received, then define

$$X_1 = \begin{pmatrix} \gamma_{1\times 1} & u_{1\times(n-1)} \\ \mathbf{0}_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \end{pmatrix}_{n\times n}$$

and update $\left(X^TX\right)^{-1}$ based on X_1, which is X with the first observation deleted, then $\begin{pmatrix} \gamma_{1\times 1} & u_{1\times(n-1)} \end{pmatrix}$ annexed in the (new) first row. This gives $\left(X_1^TX_1\right)^{-1}$.

When the next observation $\begin{pmatrix} \alpha_{1\times 1} & v_{1\times(n-1)} \end{pmatrix}$ is received, then define

$$X_2 = \begin{pmatrix} \alpha_{1\times 1} & v_{1\times(n-1)} \\ \gamma_{1\times 1} & u_{1\times(n-1)} \\ \mathbf{0}_{(n-2)\times 2} & I_{(n-2)\times(n-2)} \end{pmatrix}_{n\times n}$$

and update $\left(X_1^T X_1\right)^{-1}$ based on X_2, which is X_1 with the second observation deleted, then $\begin{pmatrix} \alpha_{1\times 1} & v_{1\times(n-1)} \end{pmatrix}$ annexed in the (new) first row. This gives $\left(X_2^T X_2\right)^{-1}$.

Continue this process until n observations have been received, which would be the first moment (in this process) where the data matrix X contains all observations (rather than the "filler" rows of the form $(0,0,\ldots,0,1,0,\ldots,0)_{1\times n}$).

6. Additional Calculation Details

Additional calculation details, including a step-by-step algorithm for annexing and deleting an arbitrary set of data, an implementation example in MAPLE, and several exact numerical examples are available by request from the author.

7. Appendix: The Elementary Calculation Of $\left(I \pm xM^{-1}x^T\right)^{-1}$

Claim 6 *For $x_{a\times b}$ and $M_{b\times b}$, we have*

$$\left(I_{a\times a} \pm xM^{-1}x^T\right)^{-1} = I_{a\times a} \mp x\left(M \pm x^T x\right)^{-1} x^T$$

Corollary 7 *For $x_{a\times b}$, we have*

$$\left(I_{a\times a} \pm xx^T\right)^{-1} = I_{a\times a} \mp x\left(I_{b\times b} \pm x^T x\right)^{-1} x^T$$

The size of s is usually significantly less than n, which facilitates the calculations of $\left(I_{s\times s} - vD^{-1}v^T\right)^{-1}$ and $\left(I_{p\times p} + uD_X^{-1}u^T\right)^{-1}$. Indeed, if $s=1$, then $I_{s\times s} - vD^{-1}v^T$ is a scalar, and if $s=2$ there is a simple formula for the inverse. Since all deletions from W and all annexations to X may be accomplished by sequential application of the cases where $s=1$ or $s=2$ (and larger values of s where specialized algorithms are available), these special cases of the update methods are sufficient to implement all values of s.

However, for vector $q_{s\times 1}$, since

$$\begin{aligned}
\left(I_{s\times s} \mp \frac{qq^T}{1 \pm q^T q}\right)\left(I_{s\times s} \pm qq^T\right) &= I_{s\times s} \pm qq^T \mp \frac{qq^T}{1 \pm q^T q} - \frac{\left(qq^T\right)_{s\times s}\left(qq^T\right)_{s\times s}}{1 \pm q^T q} \\
&= I_{s\times s} \pm qq^T \mp \frac{qq^T \pm q\left(q^T q\right)q^T}{1 \pm q^T q} \\
&= I_{s\times s} \pm qq^T \mp \frac{\left(1 \pm q^T q\right)qq^T}{1 \pm q^T q} \\
&= I_{s\times s}
\end{aligned}$$

and

$$\begin{aligned}
\left(I_{s\times s} \pm qq^T\right)\left(I_{s\times s} \mp \frac{qq^T}{1 \pm q^T q}\right) &= I_{s\times s} \pm qq^T \mp \frac{qq^T}{1 \pm q^T q} - \frac{\left(qq^T\right)_{s\times s}\left(qq^T\right)_{s\times s}}{1 \pm q^T q} \\
&= I_{s\times s}
\end{aligned}$$

then if $\left(vD^{-1}v^T\right)_{s\times s}$ or $\left(uD_X^{-1}u^T\right)_{p\times p}$ may be written as qq^T where $q_{s\times 1} = v_{s\times(n-1)}E_{(n-1)\times 1}$ and $q^T q \neq 1$, or $q_{p\times 1} = u_{p\times(n-1)}F_{(n-1)\times 1}$ and $q^T q \neq -1$, we would have

$$\left(I_{s\times s} - vD^{-1}v^T\right)^{-1} = I_{s\times s} + \frac{vD^{-1}v^T}{1 - E^T v^T vE} \tag{15}$$

or

$$\left(I_{p\times p} + uD_X^{-1}u^T\right)^{-1} = I_{p\times p} - \frac{uD_X^{-1}u^T}{1 + F^T u^T uF} \tag{16}$$