Rejection Based on Runs in Process Control with Misclassifications and Multiple Quality Levels

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Abstract
Items are produced in one of three quality levels with the percentages of each level dependent on whether the process is in control or out of control. Misclassification of an item's quality level is allowed in the model and so multiple inspections are done. A decision as to whether the process is in or out of control is based on runs.

Keywords: Quality, Inspection, Runs

1. Introduction

Taguchi, Elsayed, and Hsiang(1989), Taguchi, Chowdhury, and Wu (2004), and Nayebpur and Woodall (1993) consider on-line process control by attributes involving an inspection of every h^{th} item produced. Initially the process is assumed to be in control and have some high fraction of conforming items. At some random time the process goes out of control and the fraction of conforming items decreases. When an inspected item is considered nonconforming, the process is stopped for adjustment.

In Borges, Ho, and Turnes (2001) the inspection process itself may be subject to possible diagnostic errors. In a single classification, a conforming item might be mistakenly classified as nonconforming and a nonconforming item might mistakenly be judged as conforming. Thus repeated classifications of each inspected item are made before making the final determination as to whether to judge the item as conforming or nonconforming. If the item has been judged in this final determination to be nonconforming, the process is judged out of control and is stopped for adjustment. If not, the process is considered in control and is not stopped for adjustment. Since there are possible diagnostics errors in the repeated classifications, an item may be judged to be nonconforming and thus the process is judged out of control, when it actually is not. In this case the process is stopped for adjustment, however no cause can be found and the process then is restarted. On the other hand, it is also possible that the process goes out of control, but is not detected. In that case, it remains out of control until this is detected at a later time at which time it will be adjusted and be put back in control.

Trindade, Ho, and Quinino (2007) study the rule in which the final determination of whether the inspected item is conforming and the process is in control uses majority rule in a pre-specified number of repeated classifications. Quinino, Colin, and Ho (2009) investigate a rule in which the item is determined to be conforming and the process to be in control if and only if there are k classifications as conforming before f classifications as nonconforming, where k and f are some pre-specified positive integers. The acronym TCTN is used to describe this rule since the decision is based on the total number of classifications as conforming and nonconforming. Smith and Griffith (2009) studied this rule and in later papers (2011, 2012) consider alternative rules called CCTN and CCCN.

In Griffith and Smith (2013), they extended this rule to a $TC_1TC_{12}TN(\nu,\rho)$ multistate model in which items produced are superior, acceptable, or unacceptable rather than simply conforming or nonconforming. When the process is in control, the percentages are p_1, p_2, and $1 - p_1 - p_2$ respectively and when the process is out of control, the percentages are p_1^*, p_2^*, and $1 - p_1^* - p_2^*$ ($> 1 - p_1 - p_2$). The process of inspecting an item is imperfect and is subject to possible misclassification. An item which is actually superior can mistakenly be judged to be only acceptable or even possibly unacceptable. Similarly, misclassifications can occur for an item which is actually acceptable or unacceptable. They let p_{ss}, p_{sa}, and p_{su} be the respective probabilities that an item which is superior is judged to be superior, acceptable, and unacceptable respectively. For acceptable items, p_{as}, p_{aa}, and p_{au} they denoted these probabilities and for unacceptable items, p_{us}, p_{ua}, and p_{uu} will represent these probabilities. Every h^{th} item is inspected repeatedly and the process is judged to be in control if either l items are classified as superior or k items are classified as either superior or acceptable prior to f being classified as unacceptable. This rule is denoted by the acronym $TC_1TC_{12}TN$ and the notation $TC_1TC_{12}TN(\nu,\rho)$ is used to represent the probability that there are l classified as superior or k as superior or acceptable prior to f classified as unacceptable where ν is the probability that the inspected item is judged superior and ρ is the probability that the inspected item is judged as acceptable.

Griffith and Smith (2014) studied the case where the process is judged to be in control if there are l consecutive items classified as superior or k consecutive items as superior or acceptable prior to f classified as unacceptable and call this rule $CC_1CC_{12}TN$. Letting ν be the probability that the inspected item is judged superior and ρ be the probability that the inspected item is judged as acceptable the notation $CC_1CC_{12}TN(\nu,\rho)$ is used to represent the probability that there are l consecutive items classified as superior or k consecutive items as superior or acceptable prior to f classified as unacceptable. The method for calculation of this probability for a given ν and ρ will be given in matrix form using a Markov chain approach in section 3.

In this current paper the case where the process is judged to be in control if there are l consecutive items classified as superior or k consecutive items as superior or acceptable prior to f consecutive classified as unacceptable and call this rule $CC_1CC_{12}CN$. If we let ν be the probability that the inspected item is judged superior and ρ be the probability that the inspected item is judged as acceptable we will use the notation $CC_1CC_{12}CN(\nu,\rho)$ to represent the probability that there are l consecutive items classified as superior or k consecutive items as superior or acceptable prior to f consecutive classified as unacceptable. The method for calculation of this probability for a given ν and ρ will be given in matrix form using a Markov chain approach in section 3.

2. Probabilistic Results

Using the notation of the preceding section, the geometric distribution as a waiting time distribution, and basic probability results such as the law of total probability, we can obtain a number of results.

Proposition 1

A) Given that the item being inspected is superior, the probability that process is judged to be control is $\mathrm{CC_1CC_{12}CN}(p_{ss}, p_{sa})$.

B) Given that the item being inspected is acceptable, the probability that process is judged to be control is $\mathrm{CC}_1\mathrm{CC}_{12}\mathrm{CN}(p_{as}, p_{aa})$.

C) Given that the item being inspected is unacceptable, the probability that process is judged to be control $\mathrm{CC}_1\mathrm{CC}_{12}\mathrm{CN}(p_{us}, p_{ua})$.

Proposition 2
A) Given that the process is in control, the probability that the process is judged to be in control and the inspected item is superior with probability p_1, acceptable with probability p_2, and unacceptable with probability $1 - p_1 - p_2$, then

$$P_{II} = P(judged\ in\ control \mid in\ control)$$
$$= p_1\mathrm{CC}_1\mathrm{CC}_{12}\mathrm{CN}(p_{ss}, p_{sa}) + p_2\mathrm{CC}_1\mathrm{CC}_{12}\mathrm{CN}(p_{as}, p_{aa}) + (1 - p_1 - p_2)\mathrm{CC}_1\mathrm{CC}_{12}\mathrm{CN}(p_{us}, p_{ua})$$

B) Given that the process is out of control, the probability that the process is judged to be in control and the inspected item is superior with probability p_1^*, acceptable with probability p_2^*, and unacceptable with probability $1 - p_1^* - p_2^*$, then

$$P_{OI} = P(judged\ in\ control \mid out\ of\ control)$$
$$= p_1^*\mathrm{CC}_1\mathrm{CC}_{12}\mathrm{CN}(p_{ss}, p_{sa}) + p_2^*\mathrm{CC}_1\mathrm{CC}_{12}\mathrm{CN}(p_{as}, p_{aa}) + (1 - p_1^* - p_2^*)\mathrm{CC}_1\mathrm{CC}_{12}\mathrm{CN}(p_{us}, p_{ua})$$

Proposition 3
Once it goes out of control, the distribution of the number of inspections needed to determine it is out of control is the geometric distribution with success parameter $1 - P_{OI}$.

Proposition 4
Let Y = time measured in decision time until the process is actually out of control and π is the probability of a shift on any item produced then the

$$P(Y = y) = [(1-\pi)^h]^{y-1}[1-(1-\pi)^h] = \theta(1-\theta)^{y-1}, \quad y = 1, 2, 3, \ldots \quad .$$

So Y has a geometric distribution with parameter $\theta = 1 - (1-\pi)^h$.

Proposition 5
Let X = time measured in decision time until the process is judged out of control then $P(X = x) = \sum_{y=1}^{\infty} P(X = x|Y = y)P(Y = y)$ where

$$\mathrm{P}(X = x|Y = y) = \begin{cases} [P_{II}]^{x-1}[1 - P_{II}], & x < y \\ [P_{II}]^{x-1}[1 - P_{OI}], & x = y \\ [P_{II}]^{y-1}[P_{OI}]^{x-y}[1 - P_{OI}], & x > y \end{cases}$$

$$\mathrm{P}(X = x) = \sum_{y=1}^{x-1}[P_{II}]^{y-1}[P_{OI}]^{x-y}[1 - P_{OI}]\,(\theta(1-\theta)^{y-1}) + [P_{II}]^{x-1}[1 - P_{OI}](\theta(1-\theta)^{y-1}) + \sum_{y=x+1}^{\infty}[P_{II}]^{x-1}[1 - P_{II}](\theta(1-\theta)^{y-1})$$

3. Markov Chain Analysis

Consider the Markov Chain $\{X_n\}$ with state space

$$\{(r,s,t):\ 0 \le r \le l,\ 0 \le s \le k,\ 0 \le t \le f\} \cap \{(r,s,t):(r > s) \cup (t \neq 0 \cap (r \neq 0 \text{ or } s \neq 0))\}^C$$

where $X_{n=}(r,s,t)$ means that after the n^{th} classification there are a r consecutive superior successes, s consecutive superior or acceptable successes, and t consecutive unacceptable. The transition probabilities are of the form

$$P(X_n = (r + 1, s + 1, 0) \mid X_{(n-1)} = (r,s,t)) = \nu,$$
$$P(X_n = (0, s + 1, 0) \mid X_{(n-1)} = (r,s,t)) = \rho,$$
$$P(X_n = (0,0,t + 1) \mid X_{(n-1)} = (r,s,t)) = 1 - (\nu + \rho)$$

Using this information and the recognition that the absorbing states are the states such that $r = l$, $s = k$, or $t = f$ we can easily determine the one-step probability matrix **P** which is described as follows.

For each Markov chain there are absorbing (recurrent) states, which correspond to the end of the repetitive classifications for a single item and the consequent decision. Let A denote the set of absorbing states and a denote the number of absorbing states. The decision that the process is in control we will call acceptance and the decision the process is out of control we will call rejection. In fact, the singleton sets consisting of each of these absorbing states are recurrent classes. The remaining states are transient which we will denote by T and likewise the number of transient states by t. Written in canonical form, the one-step transition probability matrix **P** for the Markov chain is $\begin{bmatrix} \mathbf{P_1} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}$, where $\mathbf{P_1}$ is the $a \times a$ identity matrix for the absorbing states, **R** is a $t \times a$ matrix containing the one-step probabilities of the transient states to the recurrent (absorbing) states, **Q** is a $t \times t$ matrix containing the one-step probabilities among the transient states, and **0** is the $a \times t$ zero matrix. The one-step probabilities of **R** and **Q** are determined by the transition probabilities given for each test. The first row of **Q** contains the one step transition probabilities from state (0,0,0).

To compute the moments of the decision time, we will define the following notation. Since elements of T appear as subscripts, we will use i and j as typical elements of T. However, it should be noted that when we do so, each of i and j refer to an ordered triple such as (r,s,t). Let,

- $\mathbf{I}_{t \times t}$ = identity matrix of dimension $t \times t$
- $\mathbf{M}_{t \times t} = (\mathbf{I}_{t \times t} - \mathbf{Q}_{t \times t})^{-1}$ - the fundamental matrix of dimension $t \times t$
- $\mathbf{e}_m$ = column vector of length t where the m^{th} element is one and the remaining elements are zero.
- $\boldsymbol{e_m}'$ is defined to be the transpose of $\mathbf{e}_m$
- $\mathbf{u}_{\{RS\}}$ = column vector where all the elements corresponding to the rejection states are one, and the remainder of the elements are zero.
- $\mathbf{1}_z$ = column vector of ones of length z
- N_{ij} = random variable that represents the number of times the process visits state j before it eventually enters a recurrent state, having initially started from state i $(i,j \in \mathrm{T})$.
- $\mu_{ij} = E(N_{ij})$ for $i,j \in$ T.
- $\mathbf{M}_\rho = \left[\sum_{j \in T} \mu_{ij}\right] = \mathbf{M1}_t$ - column vector such that the m^{th} element is the sum of the m^{th} row of **M**
- $\mathbf{M}_{\rho^2} = \left[\left(\sum_{j \in T} \mu_{ij}\right)^2\right] = diag(\mathbf{M}_{\boldsymbol{\rho}})\mathbf{M}_{\boldsymbol{\rho}}$ - column vector such that the m^{th} element is the square of the sum of the m^{th} row of **M.** Note: diag $(\mathbf{M}_\rho)$ is a diagonal matrix whose entries are the corresponding entries of $\mathbf{M}_{\rho}$.

The results below are given without proof and based on formulas in Bhat (1984).

Given that the item being inspected is superior, what is the probability the process is judged to be in control, the mean, the variance, and probability mass function of the time until a decision is reached? What if the item is acceptable? What if the item is unacceptable?

Consider the decision time for a single item for i.i.d. classifications according as stated in section 1.

1) $CC_1CC_{12}CN$ (ν,ρ) = 1 - $\mathbf{e}_1'\,\mathbf{MR}\,\mathbf{u}_{\{RS\}}$

2) Expected decision times

   E(Decision time | superior) = $\mathbf{e}_1'\mathbf{M1}_t$ where (ν,ρ) = (p_{ss}, p_{sa})

   E(Decision time | acceptable) = $\mathbf{e}_1'\mathbf{M1}_t$ where (ν,ρ) = (p_{as}, p_{aa})

   E(Decision time | unacceptable) = $\mathbf{e}_1'\mathbf{M1}_t$ where (ν,ρ) = (p_{us}, p_{ua})

   E(Decision time | in control) = p_1E(Decision time | superior) + p_2E(Decision time | acceptable) + (1-p_1 - p_2) E(Decision time | unacceptable)

   E(Decision time | out of control) = p_1^*E(Decision time | superior) + p_2^*E(Decision time | acceptable) + (1-$p_1^* - p_2^*$) E(Decision time | unacceptable)

3) The variances of decision time are

   Var(Decision time | superior) =$e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}}-\mathbf{M}_{\rho^2}\right]$ where (ν,ρ) = (p_{ss}, p_{sa})

   Var(Decision time | acceptable) =$e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}}-\mathbf{M}_{\rho^2}\right]$ where (ν,ρ) = (p_{as}, p_{aa})

   Var(Decision time | unacceptable) = $e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}}-\mathbf{M}_{\rho^2}\right]$ where (ν,ρ) = (p_{us}, p_{ua})

4) The probability mass function of the decision time

   P(decision time = m | superior) = $\mathbf{e}_1'\,\mathbf{Q}^{m-1}\mathbf{R}\,\mathbf{1}_a$ where (ν,ρ) = (p_{ss}, p_{sa})

   P(decision time = m | acceptable) = $\mathbf{e}_1'\,\mathbf{Q}^{m-1}\mathbf{R}\,\mathbf{1}_a$ where (ν,ρ) = (p_{as}, p_{aa})

   P(decision time = m | unacceptable) = $\mathbf{e}_1'\,\mathbf{Q}^{m-1}\mathbf{R}\,\mathbf{1}_a$ where (ν,ρ) = (p_{us}, p_{ua})

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