Bootstrap-based Confidence Intervals in Partially Accelerated Life Testing under the Generalized Exponential Distribution

Ahmed Eshebli[1], Ahmed, V.A. Samaranayake[1]
[1]Missouri University of Science and Technology, 400 West 12th St., Rolla, MO 65409

Abstract

Partially accelerated life testing (PALT) is preferable over accelerated life testing (ALT) in situations where a model linking the stress to the distribution parameters is unavailable. Under the assumption of a generalized exponential life distribution, a parametric bootstrap-based method of obtaining confidence intervals for the mean life is introduced. Its performance is studied against that of intervals obtained using the traditional delta method using Monte Carlo simulation. Results show that the bootstrap-based method performs better than the traditional approach.

Key Words: Resampling, PALT, Maximum Likelihood, Acceleration Factor, Mean Life, Life Distributions

1. Introduction

Products which under nominal use conditions last for a long period pose a problem in determining their mean life using standard life tests because only a very small fraction of them will fail under a testing period of reasonable duration. In such situations, practitioners resort to accelerated life tests (ALT). As Nelson (1980) puts it: "Accelerated life testing of a product or material is used to get information quickly on its life distribution. Test units are run under severe conditions and fail sooner than under usual conditions. A model is fitted to the accelerated failure times and then extrapolated to estimate the life distribution under usual conditions. This is quicker and cheaper than testing at usual conditions, which is usually impractical because life is so long." When the acceleration factor is known or there exists a mathematical model which specifies the life-stress relationship, the Accelerated Life Testing is the best way to get information quickly on the life distribution. However, there are some situations in which neither the acceleration factor is known nor do life-stress models exist or are very hard to assume. In such cases partially accelerated life tests (PALT) provide a better method.

Under the PALT method, a portion of the test units are placed under the nominal use (field use, design use) stress conditions and the remaining units are tested under a suitably selected higher than nominal stress level. The life distribution under the higher stress level is assumed to be the same as that under nominal use, but with the scale parameter multiplied by an acceleration factor. This factor is estimated together with the other distribution parameters. Since there is more failure data from the units that received

higher than nominal stress level, the combined data provide better estimates of the common parameters.
`
One drawback of the PALT method is that unlike in the ALT, some units have to be tested under nominal use. Thus this method is not suitable for components that are very long lasting. But items such as chemicals that have shelf-lives that are measured in months or a year or two can be tested using this method.

1.1 A Brief Review of Relevant Literature

There are a large number of publications on ALTs and a relatively smaller but an appreciable number also available for PALTs. For brevity, we will refrain from discussing all of these, but limit the discussion to a select few of these publications. An excellent coverage of Accelerated Life tests is given in Nelson (1990). Other books include Mann, Schafer, and Singapuwalla (1974), Lawless (1982), Tobias and Trindade (2011), and Meeker and Escobar (1998).

One of the more recent publications is Jayawardhana and Samaranayake (2003), that discussed obtaining lower prediction bounds for a future observation from a Weibull population at design (nominal use) stress level, using Type II censored accelerated life test data. The scale parameter of the life distribution is assumed to have an inverse power relationship with the stress level. They showed that the method works well when the low and high stresses are reasonably far apart. Alferink and Samaranayake (2011) considered accelerated degradation models and developed confidence intervals for mean life using the Delta method and the bootstrap, **a**ssuming lognormal distribution with variance dependent on stress. Another interesting paper is Kamal, Zarrin, and Islam (2013), who presented a step stress ALT plan that works well. In step stress, the components are first put at a lower stress and the unfailed components are subjected to higher stress after a specific period. More recently, Jayawardhana and Samaranayake (2014), obtained predictive density of a future observation at nominal use conditions using ALT method under lognormal life distribution and Type II censoring with non-constant variance.

Among the publications on PALTs, the following warrant mentioning. Saxena and Zarrin (2013) used the constant stress Partially Accelerated Life Test (CSPALT) and assumed Type-I censoring under the Extreme Value Type-III distribution. The Extreme Value Type-III distribution has been recommended as appropriate for high reliability components. The authors used the Maximum Likelihood (ML) method to estimate the parameters of CSPALT model and confidence intervals for the model parameters were constructed. Note that the CSPALT plan is used to minimize the Generalized Asymptotic Variance (GAV) of the ML estimators of the model parameters.

Ismail (2013) derived the maximum likelihood estimators (MLEs) of the parameters of the GE distribution and the acceleration factor when the data are Type-II censored under constant-stress PALT model. The likelihood ratio bounds (LRB) method was used to obtain confidence bounds of the model parameters when the sample size is small. It is also shown that the maximum likelihood estimators are consistent and their asymptotic variances decrease as the sample size increases. The numerical results reported in the paper support the theoretical findings and showed that the estimated approximate confidence intervals for the three parameters are smaller when the sample size is larger. The LRB method was used to obtain the confidence bounds of the model parameters when the sample size is small.

Abdel-Hamid (2009), **c**onsidered a constant PALT model when the observed failure times come from $\mathrm{Burr}(c, k)$ distribution under progressively Type-II right censoring. The MLEs of the parameters were obtained and their performance was studied through their mean squared errors and relative absolute biases. The paper also showed how to constructed approximate and bootstrap CIs for the parameters. The bootstrap CIs give more accurate results than the approximate intervals for small sample sizes, the Student's-*t* bootstrap CIs are better than the Percentile bootstrap CIs in the sense of having smaller widths. However, the differences between the lengths of CIs for the two methods decrease with the increase in sample size.

1.2 The Generalized Exponential Distribution

The proposed PALT method is developed for the case where the underlying life distribution is Generalized Exponential (GE). The generalized exponential distribution has been introduced and studied quite extensively by Gupta and Kundu (1999, 2001a, 2001b), and by Ragab and Ahsanullah (2001). The probability density function and the cumulative distribution function of the generalized exponential distribution function has the forms:

$$f(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1} \quad x>0,\ \alpha>0,\ \lambda>0 \tag{1}$$

$$F(x;\alpha,\lambda) = \left(1-e^{-\lambda x}\right)^{\alpha}, \tag{2}$$

where α is the shape parameter and λ the scale parameter.

The GE distribution has certain features which are distinct from the Gamma and Weibull distributions (see Gupta and Kundu (1999, 2001)). The GE model can be used as a possible alternative for analyzing skewed datasets. An interesting fact is that both Gamma and GE distributions have the likelihood ratio ordering property while Weibull does not. On the other hand, GE and Weibull distributions have the common feature of having closed form expressions for Cumulative Distribution Function (CDF) and the Hazard Function. One aspect that makes the GE distribution outperform the Weibull is the fact that the convergence of MLE's of Weibull parameters can be very slow (Bain(1976)) whereas the asymptotic confidence intervals obtained under the GE assumption maintain nominal coverage even for small sample sizes (Gupta and Kundu (2001)). Gupta and Kundu (2001) also showed that the Hazard Function of the GE distribution has proprieties similar to those of the Gamma and Weibull distributions. These properties are summarized in Table 1.

Table 1. Properties of the Hazard Function

Parameters	Gamma	Weibull	GE
$\boldsymbol{\alpha = 1}$	Constant	Constant	Constant
$\boldsymbol{\alpha > 1}$	Increasing from 0 to λ	Increasing from 0 to ∞	Increasing from 0 to λ
$\boldsymbol{\alpha < 1}$	Decreasing from ∞ to λ	Decreasing from ∞ to 0	Decreasing from ∞ to λ

2. The Proposed PALT Method and Bootstrap Intervals

The following assumptions are made regarding the proposed PALT method.

1. The total number of units under test is n .
2. π denotes the proportion of sample units allocated to accelerated condition
3. $n(1\text{-}\pi)$ of these units are allocated to nominal (field) use conditions.
4. $n\pi$ units are allocated to the high stress condition (subject to acceleration)

2.1 Likelihood Function under Type I Censoring and Asymptotic C.I.s

Under Type I censoring, the censoring time, τ, is fixed but the number of failures observed in the time τ is a random variable, say R. We assume that the number of items failing before time τ follows binomial distribution with parameters (n, p) with $p = F_X(\tau; \underline{\theta})$, where $\underline{\theta}$ is the vector of parameters of the GE distribution.

Notation

x_i: Observed lifetime of item *i* tested at the nominal (field) use condition.

y_j: Observed lifetime of item *j* tested at high stress condition.

δ_{ui}: Indicator function denoting the censoring state of i^{th} observation under nominal use condition, with $\delta_{ui} = 1$ if the observation is uncensored.

δ_{aj}: Indicator function denoting the censoring state of j^{th} observation under high stress condition, with $\delta_{ai} = 1$ if the observation is uncensored.

n_u: Number of items that failed at nominal use condition.

n_a: Number of items that failed at high stress condition.

τ: The censoring time of the life test (for all units).

$x_{(1)} \leq \cdots \leq x_{(n_u)} \leq \tau$: Ordered failure times at nominal use condition.

$y_{(1)} \leq \cdots \leq y_{(n_a)} \leq \tau$: Ordered failure times at high stress condition.

β: Denotes the acceleration factor ($\beta > 1$).

Note that under Assumption 8 the life distribution for the units under high stress is given by $f(x;\alpha,\lambda) = \alpha\beta\lambda e^{-\beta\lambda x}\left(1-e^{-\beta\lambda x}\right)^{\alpha-1} \quad x>0,\ \alpha>0,\ \lambda>0,\ \beta>1$.

It can be shown that the total likelihood function, $L\left(\underline{x},\underline{y}\middle|\beta,\alpha,\lambda\right)$, for the parameters, given the observed data, is proportional to the expression:

$$(\alpha\lambda)^{\mathrm{n_u}}\left(1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right)^{n(1-\pi)-\mathrm{n_u}}e^{-\lambda\sum_{i=1}^{\mathrm{n_u}}x_i}\prod_{i=1}^{\mathrm{n_u}}\left[\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\right]\times$$
$$(\alpha\lambda\beta)^{\mathrm{n_a}}\left(1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right)^{n\pi-\mathrm{n_a}}e^{-\lambda\beta\sum_{j=1}^{\mathrm{n_a}}y_j}\prod_{j=1}^{\mathrm{n_a}}\left[\left(1-e^{-\lambda\beta y_j}\right)^{\alpha-1}\right].$$

The MLE's of the parameters can estimated numerically by minimizing the log likelihood function.

The Asymptotic confidence intervals for the parameters α, λ, and β can be obtained using the convergence in distribution result:

$$\sqrt{n}\left((\hat{\alpha}-\alpha),(\hat{\lambda}-\lambda),(\hat{\beta}-\beta)\right)\to N\left(\underline{0},\,I^{-1}(\alpha,\lambda,\beta)\right),$$

where the $I(\alpha,\lambda,\beta)$ is the fisher information matrix given by

$$I(\alpha,\lambda,\beta)=\begin{bmatrix} I_{11}(\alpha) & I_{12}(\alpha\lambda) & I_{13}(\alpha\beta) \\ I_{21}(\lambda\alpha) & I_{22}(\lambda) & I_{23}(\lambda\beta) \\ I_{31}(\beta\alpha) & I_{32}(\beta\lambda) & I_{33}(\beta) \end{bmatrix}=\begin{bmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\alpha\partial\beta} \\ \frac{\partial^2 l}{\partial\lambda\partial\alpha} & \frac{\partial^2 l}{\partial\lambda^2} & \frac{\partial^2 l}{\partial\lambda\partial\beta} \\ \frac{\partial^2 l}{\partial\beta\partial\alpha} & \frac{\partial^2 l}{\partial\beta\partial\lambda} & \frac{\partial^2 l}{\partial\beta^2} \end{bmatrix},$$

and employing the standard z-based confidence interval formulations,

$$\hat{\alpha}\mp Z_{\gamma/2}\sqrt{I_{11}^{-1}(\hat{\alpha})}\ ,\hat{\lambda}\mp Z_{\gamma/2}\sqrt{I_{22}^{-1}(\hat{\lambda})}\ ,\hat{\beta}\mp Z_{\gamma/2}\sqrt{I_{33}^{-1}(\hat{\beta})}\,.$$

The asymptotic confidence interval for the mean life at nominal use condition is given by

$$\hat{\mu}\mp Z_{\gamma/2}\sqrt{Var(\hat{\mu})}\,,$$

where $Var(\hat{\mu})$ is obtained using the standard delta method.

2.2 The Proposed Bootstrap Method and the Monte-Carlo Procedure

The Monte-Carlo procedure used for the simulation study is given below. The steps for the bootstrap method that can be utilized to obtain confidence bounds for the mean life is imbedded in this procedure and are given in *italics*.

- Random samples were generated from the GE distribution by using the transformation $x_i=\left(\frac{-1}{\lambda}\right)\ln\left[1-u_i^{(1/\alpha)}\right]$, $i=1,2,\ldots,n$ where $u_i's$ are random sample from a uniform (0, 1) distribution. Similarly, data for the high stress condition was also generated.

- Distribution parameters were varied in the study but results for only one set: $(\alpha = 2.5, \lambda = 2.4, \beta = 1.5, 2.0)$ and $\mu = 1/\lambda\,[\psi(\alpha+1) - \psi(1)] = 0.7002$ where $\psi(.)$ Digamma function is presented here. The censoring time was set at τ=0.6 , 0.8, and 1.0.
- The n test items were divided into equal sample proportions by setting $\pi = 0.5$, such that 1/2 the items are allocated at accelerated condition and the remaining 1/2 are allocated to the nominal use condition.
- *Maximum likelihood method was used to estimate the parameters with the same censoring time τ used for both samples.*
- *The nonlinear equations of the maximum likelihood estimates were solved iteratively using Newton Raphson method.*
- The resulting estimates of the parameters and acceleration factor were used to construct asymptotic confidence limits with confidence level at $\gamma = 0.95$ and also the asymptotic variance and covariance matrix of the estimators (for use in the delta method based intervals).
- *Used the estimated parameters $\hat{\alpha}, \hat{\lambda}$ to generate data from the estimated nominal use GE distribution using the transformation x_i=(-1/$\hat{\lambda}$) ln[1-$u_i^{(1/\hat{\alpha})}$].*
- *This was repeated for 1,000 bootstrap samples.*
- *The GE parameters were estimated using each bootstrap sample and the estimated parameters were used to obtain a bootstrap estimate $\hat{\mu}^*$ of μ.*
- *Using the empirical distribution of the mean $\hat{\mu}^*$ obtained from bootstrap estimates, confidence interval for mean is constructed using quantile at $\left(\frac{1-\gamma}{2}\right)$100% and $1 - \left(\frac{1-\gamma}{2}\right)$100%.*
- Coverage probabilities were computed based on 1,000 simulation runs.

3. Results and Discussion

Only select results from the simulation experiments are reported below for brevity. All simulations results reported here are for α =2.5 and λ =2.4, but the acceleration factor β is set at 1.5 and 2.0. The censoring parameter τ was set at values 0.6, 0.8 and 1.0. Additional results are available upon request. Note that under the parameter combinations given above, the mean life at nominal use condition is 0.7002 units.

Tables 2a and 2b give the results for the parameter combination α =2.5, λ =2.4, β =1.5, π=.5, and τ=.6, with the former giving MLEs and the asymptotic bounds and the coverages for the asymptotic intervals and the latter table giving results for the bootstrap intervals. Tables 3a and 3b provide results for the parameter combination α =2.5, λ =2.4, β =1.5, π=.5, and τ=.8. Tables 4a and 4b provide results for the parameter combination α =2.5, λ =2.4, β =1.5, π=.5, and τ=1.0. Tables 5a and 5b gives the results for the parameter combination α =2.5, λ =2.4, β =2.0, π=.5, and τ=.8.

Table 2a. Simulations Results for Asymptotic C.I.s and MLEs
α =2.5, λ =2.4, β =1.5, π=.5, τ=.6

Parameter	*n*	MLE	MSE	Asymp. Lower Bound 95%	Asymp. Upper Bound 95%	Asymp. C.I. Coverage
μ	50	0.718224	0.031466	0.571449	0.834240	0.921000
	75	0.715944	0.023177	0.595597	0.811668	0.944000
	100	0.710524	0.012728	0.608993	0.795522	0.935000

Table 2b. Simulations Results for Bootstrap C.I.s
α =2.5, λ =2.4, β =1.5, π=.5, τ=.6

Parameter	n	Bootstrap Lower Bound 95%	Bootstrap Upper Bound 95%	Bootstrap C.I. Coverage
μ	50	0.405068	1.367601	0.936700
	75	0.453311	1.176897	0.939500
	100	0.483147	1.051275	0.943900

Table 3a. Simulations Results for Asymptotic C.I.s and MLEs
α =2.5, λ =2.4, β =1.5, π=.5, τ=.8

Parameter	*n*	MLE	MSE	Asymp. Lower Bound 95%	Asymp. Upper Bound 95%	Asymp. C.I. Coverage
μ	50	0.711110	0.021889	0.571449	0.834240	0.921000
	75	0.702451	0.014470	0.595050	0.810705	0.933000
	100	0.709317	0.009428	0.605980	0.791605	0.936000

Table 3b. Simulations Results for Bootstrap C.I.s
α =2.5, λ =2.4, β =1.5, π=.5, τ=.8

Parameter	n	Bootstrap Lower Bound 95%	Bootstrap Upper Bound 95%	Bootstrap C.I. Coverage
μ	50	0.404121	1.264267	0.938800
	75	0.457659	1.106968	0.949800
	100	0.500748	1.053099	0.954430

Table 4a. Simulations Results for Asymptotic C.I.s and MLEs
α =2.5, λ =2.4, β =1.5, π=.5, τ=1.0

Parameter	*n*	MLE	MSE	Asymp. Lower Bound 95%	Asymp. Upper Bound 95%	Asymp. C.I. Coverage
μ	50	0.708061	0.016594	0.567329	0.829897	0.931000
	75	0 .687499	0 .010054	0.591657	0.807097	0.942000
	100	0 .708817	0.008913	0 .606182	0 .792432	0.946000

Table 4b. Simulations Results for Bootstrap C.I.s
α =2.5, λ =2.4, β =1.5, π=.5, τ=1.0

Parameter	n	Bootstrap Lower Bound 95%	Bootstrap Upper Bound 95%	Bootstrap Coverage
μ	50	0.407540	1.163585	0.945500
	75	0.445111	0.995156	0.956800
	100	0.486591	1.016866	0.968900

Table 5a. Simulations Results for Asymptotic C.I.s and MLEs
α =2.5, λ =2.4, β =2.0, π=.5, τ=0.8

Parameter	*n*	MLE	MSE	Asymp. Lower Bound 95%	Asymp. Upper Bound 95%	Asymp. C.I. Coverage
μ	50	0.706543	0.020173	0.565926	0.826414	0.931000
	75	0.688825	0.012045	0.591563	0.806085	0.929000
	100	0.709002	0.009291	0.605980	0.791605	0.936000

Table 5b. Simulations Results for Bootstrap C.I.s
α =2.5, λ =2.4, β =2.0, π=.5, τ=0.8

Parameter	n	Bootstrap Lower Bound 95%	Bootstrap Upper Bound 95%	Bootstrap C.I. Coverage
μ	50	0.399118	1.189022	0.939500
	75	0.436927	1.041010	0.948500
	100	0.500594	1.052936	0.954430

The results show that the MLE of the mean is close to the true value at all sample sizes and parameter combinations. Simulation results not reported here also show that the maximum likelihood estimates of the other parameters are very good. Results also show that the parametric bootstrap-based intervals provide coverages closer to the nominal value than the large sample intervals based on the asymptotic variance covariance matrix. Both intervals did better for moderate to large sample sizes when compared to results for the case when sample size under acceleration and nominal used was kept at 25 components each. The smaller censoring time also seems to reduce the coverage probability slightly. Overall, the bootstrap method did very well under most situations. Results for additional parameter combinations not reported herein also show similar results. Preliminary results, not reported in here, also show reasonable coverage for bootstrap-based confidence intervals constructed for other model parameters.

The parameter combinations we have considered so far, however, do not look at results for censoring parameter values less than 0.6 and acceleration levels below 1.5 and above 2.0. These combinations may yield unsatisfactory results. Work on these additional combinations are currently under way.

4.0 Conclusions and Future Work

A parametric bootstrap-based method for constructing confidence intervals for the mean life of a component based on data from a partially accelerated life test under the assumption of a generalized exponential life distribution is introduced. The generalized exponential distribution combines several of the useful features of Gamma and Weibull Distributions and thus is a valuable tool in modeling lifespans of products. The results of a Monte-Carlo simulation study show that the proposed intervals provide coverage close to the nominal level, especially for moderate to large sample sizes. Preliminary results, not reported in here, also show reasonable coverage for bootstrap-based confidence intervals constructed for model parameters. Future extensions include the use of the non-parametric rather than the parametric bootstrap and extending the procedure to other types of life distributions under Type I censoring.

Acknowledgements

The first author wishes to thank the Missouri University of Science and Technology for providing resources to conduct this research. He also wishes to thank the Quality and Productivity Section of the American Statistical Association for providing a Travel Award to attend 2015 JSM.

References

Abdel-Hamid, A. (2009). "Constant-partially accelerated life tests for Burr type-XII distribution with progressive type-II censoring," Computational Statistics & Data Analysis 53(7), 2511-2523.

Alferink, A. and Samaranayake, V. A. (2011). "Lifetime Predictive Density Estimation in Accelerated Degradation Testing for Lognormal Response Distributions with Arrhenius Rate Relationship". *JSM Proceedings, Quality and Productivity Section*. Alexandria, VA: American Statistical Association, 4373-4385.

Bain, L. J. (1976*). Statistical Analysis of Reliability and Life Testing Model*. Marcel and Dekker Inc., New York.

Gupta, R. D. and Kundu, D. (1999). Generalized exponential distribution. *Australian and New Zealand Journal of Statistics*, 41(2), 173-188.

Gupta, R. D. and Kundu, D. (2001a). Exponentiated exponential distribution, an alternative to Gamma and Weibull distributions. *Biometrical Journal*, 43(1), 117-130.

Gupta, R. D. and Kundu, D. (2001b). Generalized exponential distributions: different methods of estimation. *Journal of Statistical Computation and Simulation*, 69(4), 315-338.

Mann, N. R., Schafer, R. E., and Singapurwalla, N. D. (1974). *Methods for Statistical Analysis of Reliability and Life Data*, Wiley, New York.

Meeker, W. Q. and Escobar, L. A. (1998). *Statistical Methods for Reliability Data*. Wiley, New York.

Nelson, W. (1980) "Accelerated Life Testing-Step-Stress Models and Data Analyses," *IEEE Transactions on Reliability*, R-29(2), 103-108.

Nelson, W. (1990). *Accelerated Testing: Statistical Models, Test Plans and Data Analyses.* Wiley, New York.

Ismail, A. A. (2013). "Estimating The Generalized Exponential Distribution Parameters and The Acceleration Factor Under Constant-Stress Partially Accelerated Life Testing With Type-II Censoring," *Strength of Materials*, 45(6), 693-702.

Jayawardhana, A. A .and Samaranayake, V. A. (2003). "Prediction Bounds in Accelerated Life Testing: Weibull Models with Inverse Power Relationship" *Journal of Quality Technology*, 35(1), 89-103.

Jayawardhana, A. A. and Samaranayake, V. A. (2014). "Predictive Density Estimation in Accelerated Life Testing for Lognormal Life Distributions," in *JSM Proceedings, Quality and Productivity Section*. Alexandria, VA: American Statistical Association. 2325-2338.

Lawless, J.F. (1982). *Statistical models and methods for lifetime data,* Wiley, New York

Ragab, M. Z. and Ahsanullah, M. (2001). "Estimation of the location and parameters of the generalized exponential distribution based on order statistics," *Journal of Statistical Computation and Simulation*, 69, 109-124.

Tobias, P. A. and Trindade, D. (2011). *Applied Reliability*, Third Edition, CRC Press, New York.

M. Kamal, S. Z. and Islam, A. (2013). "Constant Stress Partially Accelerated Life Test Design for Inverted Weibull Distribution with Type-I Censoring" *Algorithms Research* 2(2): 43-49.

Saxena, S. and Zarrin, S. (2013). "Estimation of Partially Accelerated Life Tests for the Extreme Value Type-III Distribution with Type-I Censoring," *International Journal of Probability and Statistics*, 2(1): 1-8.