Effect of Phase I Sample Size on the Performance of the Multivariate Sign EWMA Chart

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Abstract

The knowledge of the underlying parameters, or the existence of a sufficiently large reference sample from which they can be estimated, is required for implementing most control charts. Estimation of parameters is known to degrade charts performance. This paper studies the effects of estimation of parameters on the performance of the nonparametric multivariate sign EWMA (MSEWMA) chart by Zou and Tsung (2011). First, the effects of using their known parameter control limits are studied in case when the parameters are estimated from a relatively small Phase I (reference) sample. It is seen that in this case the chart performance is highly degraded, in that, many more false alarms are observed than what is nominally expected. Next, using simulations, correct control limits are obtained that achieve a desired in control ARL when parameters are estimated from a given size reference sample. The out-of-control performance of the MSEWMA chart with the corrected control limits is also briefly studied. The use of the proposed corrected control limits is recommended when the reference sample is not too large.

Keywords: MSEWMA chart; average run length; multivariate control chart; Statistical process control; Moving Average (MA) chart; Nonparametric procedure.

1. Introduction

Control charts are graphical tools that have become popular in process monitoring in many applications. When a change has occurred in the process which might affect product quality, the goal is to detect that quickly (see e.g. Lowry et al. 1992, Chakraborti et al. 2001, Qiu and Hawkins 2003, Dovoedo and Chakraborti 2012). Outlier (abnormal) points on the control chart signal these changes in the process. The literature on outlier detection is in itself quite vast (se e.g. Barnett and Lewis 1994, Filzmoser et al. 2008, Dovoedo 2011, Dovoedo and Chakraborti 2013, 2015). In many process monitoring applications, it is common to monitor several quality characteristics, which may be correlated, simultaneously. It is well known that when correlated variables are being monitored, multivariate control charts should be used. There are several such charts in the literature and the survey paper of Bersimis et al (2007) including references therein may be consulted for some details. Most of these multivariate control charts rely on the assumption of multivariate normality, which may not always be justifiable in practice. Some examples of these are the following popular charts: the T^2 chart of Hotelling (1947), the multivariate cumulative sum (MCUSUM) control chart of Crosier (1988), the multivariate exponentially weighted moving average (MEWMA) control chart of Lowry et al. (1992). Like in the univariate case, the performance of these charts can be significantly degraded in the presence of non-normality, especially while monitoring individual observations (Montgomery 2005). Thus, nonparametric and robust control charts are needed, as pointed out for example in Woodall and Montgomery (1999), and Chakraborti et al. (2015). The multivariate sign EWMA (MSEWMA) control chart of Zou and Tsung (2011) is one such chart that can be useful in practice.

The MSEWMA chart is appealing for several reasons: It is nonparametric; it is affine-invariant, and has a strictly distribution-free property over a larger class of population distributions, distributions with elliptical direction that includes the elliptically symmetric distributions (see Randles, 2000 for details about these families of distributions). The MSEWMA chart in-control (IC) average run length can be computed quickly via a one-dimensional Markov chain model (for elliptical direction class of distributions); it is fast to implement and it is also quite efficient in detecting process shifts, especially small to moderate shifts when the process distribution is heavy tailed or skewed.

As alluded to earlier, the construction and implementation of the MSEWMA chart, like most other control charts, requires either knowing the population (location and scale) parameters or estimates of them obtained from a Phase I sample (a reference sample) from an IC process. The effects of the estimation of parameters on the performance of control charts is of great interest both from a practical and a research point of view. In the univariate case, Jensen et al. (2006) and more recently Psarakis et al. (2014) gave an overview of the effects of parameter estimation on the performance of control charts. In the multivariate case, for normal theory based charts, Champ et al. (2005) studied the performance of the Hotelling's T^2 chart when parameters are estimated. Also, Champ and Jones-Farmer (2007) studied the properties of the Hotelling's T^2 chart, the MEWMA chart, and several multivariate CUSUM charts with estimated parameters. In this paper, however, we consider nonparametric control charts and study the effects of the size of the reference sample on the performance the MSEWMA chart. This question was raised by Zou and Tsung (2011) for future work. In particular, we study the reference sample size requirements for bivariate and trivariate data, with a smoothing parameter $\lambda = 0.05$. These reference sample size requirements ensure that the performance of the MSEWMA chart is not unduly affected when parameters are estimated from the Phase I sample. Using simulations, we obtain the "corrected control limits" that should be used when parameters are estimated from relatively small reference samples, in order to maintain an in-control ARL close to the nominal value. We also briefly study the out-of-control performance (shift detection properties) of the MSEWMA chart when the corrected control limits are used.
The rest of the paper is structured as follows. In Section 2, we review the MSEWMA chart. Section 3 studies the effect of small reference sample on the performance of the MSEWMA chart. Some "corrected control limits" are obtained in Section 4. In Section 5, we briefly study the out-of-control performance of the "corrected control limits" MSEWMA chart. Conclusions are provided in Section 6.

2. Review of the Multivariate Sign EWMA (MSEWMA) chart

The MSEWMA chart incorporates into the EWMA scheme, the multivariate nonparametric test proposed by Randles (2000). Here, we first review the Randles (2000) test, followed by the MSEWMA chart itself.

2.1 The Randles (2000) Test

The test of hypotheses of interest is the one-sample problem. Let be i.i.d. from $F(\boldsymbol{X} - \boldsymbol{\theta})$, where is a continuous p-dimensional distribution "centered" at ^{T}. We want to test the hypotheses:

where is specified, or without any loss of generality, $\boldsymbol{0}$, if we replace above by (centering). The Randles (2000) test was developed in analogy with the Hotelling's test which has the test statistic given by:

where $\overline{\boldsymbol{X}}$ and are the sample mean and the sample covariance respectively of the centered data.

Now let be any non singular matrix such that $^{\boldsymbol{1}}$. Then, denoting, n, we obtain immediately ^{2}. Note then that the sample variance-covariance matrix of the transformed observations is the identity matrix and consequently the transformation makes the transformed points follow a distribution with a covariance matrix $_{\boldsymbol{p}}$. Observe also that the test statistic is times the squared length of the average of the transformed data points.

The multivariate sign test of Randles (2000) is constructed taking advantage of the preceding observations. First, the original observations are transformed (details are provided below) such that the transformed data points, denoted n, follow a distribution with the variance-covariance matrix $_{\boldsymbol{p}}$. The Randles (2000) test statistic, denoted by Q, is times the squared length of the average of the transformed observations $_{\boldsymbol{i}}$. In other words:

where the notation "ave" means an average taken over the observations $i = 1, \ldots, n$, and }. The transformed data points that satisfy the above condition are obtained as), where S is the p-dimensional spatial sign function defined by:

where is the Euclidian norm and is a data driven transformation proposed by Tyler (1987). A matrix is obtained using the *Tyler shape matrix* $_{\boldsymbol{x}}$, which is the positive definite symmetric p x p matrix with such that for any which satisfies $_{\boldsymbol{p}}$. It is then easy to see that the sample variance-covariance matrix of the transformed sample is $\boldsymbol{I_p}$ as desired. The matrix is taken to be the upper triangular Cholesky factorization of $\boldsymbol{V_x}^{-1}$ (since $_{\boldsymbol{x}}{}^{-1}$), and is called *Tyler's transformation* matrix. Note, since c)$\boldsymbol{I_p}$, that the Randles test statistic can be written as:

Randles (2000) showed that when the null hypothesis is true, under the directional symmetry assumption,

The Randles test, which is implemented in the R package “MNM”, rejects when

).

The reader is referred to Randles (2000) for details about these classes of distributions. The directionally symmetric family of distributions contains distributions such as the multivariate normal and the multivariate t distributions, and certain skewed distributions.

2.2 The MSEWMA Chart

It is assumed that m_0 independent and identically distributed (iid) reference sample (Phase I) observations are available before monitoring begins in Phase II, that Phase II observations are collected over time, and that the pre-change and post-change distributions differ possibly only in their location parameters, and respectively. To be more specific let be the reference observations, where is some integer. This change-point model is given by:

where $\boldsymbol{F}_0$ is some unknown multivariate cdf, τ is some unknown change point and $_{\mathbf{1}}$. Note that in general, the location parameter $\boldsymbol{\theta}$ can be the mean, or the median or some other quantile, however, in the literature, it is common to use the median to formulate the location model and that’s what’s done hereafter.

There are more than one ways to define a multivariate median. In our case, following Hettmansperger and Randles (2002), the pre-change multivariate median $\boldsymbol{\theta_0}$ and the associated transformation matrix $\boldsymbol{A_0}$ are first defined as solutions to the following equations:

In order to implement the test and the control chart, first, estimates of the multivariate median, $\widehat{\boldsymbol{\theta_0}}$ and the transformation matrix, $\widehat{\boldsymbol{A_0}}$ are obtained from the reference sample. Following equations (2) and (3), these are obtained as the solution $(\widehat{\boldsymbol{\theta_0}},\ \widehat{\boldsymbol{A_0}})$ of where $\boldsymbol{A}$ is a $p \times p$ upper triangular positive-definite matrix with a one as the upper left-hand element. Hettmansperger and Randles (2002) showed the existence and the uniqueness of $(\widehat{\boldsymbol{\theta}_0}, \widehat{\boldsymbol{A}_0})$ under the directional symmetry assumption. They provided an algorithm for computing $\widehat{\boldsymbol{\theta}_0}$, and $\widehat{\boldsymbol{A}_0}$, which can be found in the supplemental file of Zou and Tsung (2011) as well.

Once the estimates are calculated from the Phase I reference sample, the Phase II observations are standardized and transformed to obtain unit vectors as follows:

Then the EWMA may be defined as follows:

where the initial vector $\boldsymbol{v_i} = \boldsymbol{0},$ and the smoothing parameter. Following Randles (2000) test, the charting statistic is given by:

Zou and Tsung (2011) observed that and hence proposed the final form of the charting statistic:

The chart will trigger a signal when

L,

where L is an upper control limit chosen to achieve a specific (nominal) in-control ARL value. As previously pointed out, Zou and Tsung (2011) showed that the appropriate control limits for elliptical direction distributions are the same and can be obtained using a one-dimensional Markov chain method, following Runger and Prabhu (1996) (see details in the appendix of Zou and Tsung, 2011). They also provided the control limits for various combinations of λ, p, and nominal in-control ARL values, when $(\boldsymbol{\theta_0}, \boldsymbol{A_0})$ are assumed to be known. This is essentially equivalent to having sufficiently large Phase I data to obtain reliable estimates of $\boldsymbol{\theta_0}$ and $\boldsymbol{A_0}$. This situation will be referred to as the known parameters case and these control limits will be referred to as the traditional control limits hereafter.

3. Effects of Estimated Parameters on the In-Control Performance of the MSEWMA Chart and Sample Size Requirements

In practice, when the location and scale parameters are unknown, they are estimated from a reference sample (in-control sample) consisting of m observations. When these estimated parameters are used in the monitoring statistic, the performance of control charts in general, is degraded in that the number of false alarms increases significantly, especially when the Phase I sample size, m is very small. This translates into the attained in-control ARL being much lower than the desired nominal in-control ARL. See for example Champ et al. 2005 for the case of the T^2 chart, Champ and Jones-Farmer (2007) for the cases of the MEWMA and MCUSUMs. In this section, we investigate the performance of the MSEWMA chart when the parameters are unknown and are estimated, using simulation. Note that this is a follow-up question raised by Zou and Tsung (2011). Specifically, we compute the attained in-control ARL when the estimated parameters are used in place of unknown true parameters, and the traditional (known parameter) control limits provided by Zou and Tsung (2011) are used. Several situations are investigated: We considered bivariate and trivariate data), and used the value of the smoothing parameter of the MSEWMA chart. We further consider several values of the Phase I sample size between $m = 50$ and $m = 3000$. We study the case where the nominal (desired) IC ARL is 200, as it is often the case in the multivariate literature (see e.g. Champ et al. 2005, Champ and Jones-Farmer 2007, Lowry et al. 1992).

In our simulation, we consider distributions from the following families, which are among the families of distributions considered in Zou and Tsung (2011):
(i) -dimensional standard multivariate normal distribution, denoted by $_p$; (ii) p-dimensional standard multivariate t distribution with ζ degrees of freedom, denoted by $t_{p,\zeta}$; and (iii) p-dimensional multivariate distribution with independent marginal chi-square distributions with degrees of freedom each, denoted $\chi^2_{p,\zeta}$. We consider two values of ζ, for the distributions above mentioned. For distributions cases and), the covariance matrix used in our simulation is the identity matrix $\boldsymbol{I_p}$. To estimate the in-control (IC) ARLs reported in Table 1, we simulate 10,000 runs, and average the run lengths.

Here are the steps for obtaining the IC ARLs:

Step 1: Generate reference observations from the multivariate IC distribution in question and estimate the location and scale parameters and $\widehat{\boldsymbol{A_0}}$ described in section 2.2.

Step 2: Initially Generate a new observation from the IC distribution, then set RL and compute the first MSEWMA charting statistic using the procedure described in Section 2.2.

Step 3: If no signal, increase by 1 and generate a new IC observation then go to the next step. If a signal is observed, then go to step 5.

Step 4: Compute the MSEWMA charting statistic corresponding to the new observation, then go to step 3.

Step 5: Record the run length (in a vector).

Step 6: Repeat steps 1-5, the desired number of times.

Step 7: Average the run lengths recorded in the vector in step 5 after step 6 is completed.

The results of the simulations are reported in Table 1. It is observed that using a limited number of Phase I observations (small m) to estimate the location and scale parameters, while using the traditional control limits by Zou and Tsung (2011), can seriously affect the in-control performance of the MSEWMA chart . For example, with 05, in case of the bivariate normal distribution $_2$, when 50, the attained IC ARL is 107.4; that is a decrease of about 46.3% compared to the nominal value of 200. For 05, and the trivariate distribution, $_3$, when 50, using the traditional control limit results in an attained IC ARL of 90.95; that is a decrease of about 54.5% compared to the nominal value of 200. These decreases in IC ARL are clearly not acceptable, thus, the need for "corrected control limits". Using the results reported in Table 1, along with Figure 1, we recommend, for $\lambda = 0.05$, when using the traditional control limits, a reference sample size of at least $m = 2000$ when $p = 2$ or 3. Using these reference sample sizes for parameter estimation insures that the attained IC ARL will be within 5% of the desired ARL of 200, unless the distribution of data is extremely skewed, in which case more reference observations may be needed.

4. Corrected Control Limits for the MSEWMA Chart when Parameters are Estimated

In some applications today, data are plenty and obtaining a reference dataset as large as may not be difficult. In other applications however, it may take a long period of time to obtain "good" reference data of that size. Waiting until the required sample size for the reference sample is achieved may not be a good or a viable option, since certain assignable changes in the process may go undetected during that period. Using the traditional control limits results in smaller than desired IC ARLs, which translates into a significant increase in the false alarm rate. So this is not a good option either. In situations where the reference sample size is small, corrected control limits should be used in place of the traditional control limits for the specific reference sample size. The idea is to slightly widen the upper control limit in such a way that the desired IC ARL is attained. These corrected control limits are obtained using a bisection search, the steps of which are given below. All ARLs are computed using the procedure described in Section 3. Here, multivariate normal reference sample observations are used to compute and . This since the test is distribution-free for any elliptical direction distribution.

First, let's denote $_h$, the in-control average run length of the MSEWMA chart corresponding to a control limit h based on a large number of runs (10,000 runs). Also let be the desired nominal) IC ARL.

Step 1. Obtain two numbers and h_2 (h $_2$) such that and $_0$.

Step 2. Find $/2$ and compute .

Step 3. If then assign h $_3$. If then assign $_3$.

Step 4. Repeat steps 2 and 3 until is sufficiently close to (within 0.5% of $_0$).

Step 5. Use as the desired control limit h.

We considered trivariate data), and the values 025, and 0.05 of the smoothing parameter. Several values of the reference sample size between and

2000, are studied in the simulation. The steps above are performed for each combination of p, λ, and m. The obtained corrected control limits for the MSEWMA chart are reported in Table 2. The last column of Table 2 shows the traditional control limits. The corrected control limits are of interest to practitioners, who may not have large enough reference sample. These results are displayed in Figure 2. It is seen that as the reference sample size increases, the corrected control limit decreases, in general.

5. Out-of Control ARL Performance of the Estimated Parameter MSEWMA Chart

Here, we briefly compare the out-of-control performance of the estimated parameter MSEWMA chart (using the corrected control limits) with that of the known parameter MSEWMA chart (using the traditional control limits from Zou and Tsung (2011). We have previously shown that when the known parameter control limits are used with the estimated parameter MSEWMA chart, the attained IC ARL can deviate significantly from the desired nominal value and can thus degrade the in-control performance of the MSEWMA chart. Using the corrected control limits however should make the attained IC ARL closer to the nominal value. Thus in a simulation study, we compare the known parameter MSEWMA chart (using the traditional control limits) to the estimated parameter MSEWMA chart (using corrected control limits).

In the study, we include underlying distributions from each category mentioned above. These are ζ, where the previous notation are used. We considered trivariate data (p 3), and two values of the size of the reference sample $m = 100$, and 1000, in addition to the known parameter case (m ∞). The out-of-control scenario considered here is same as the one used in Zou and Tsung (2011), a shift of size b occurs in the first component and we study the cases where takes values: and 2.5, where b corresponds to the "no change" situation. The results for the smoothing parameter are reported in Table 3. These results are displayed in Figure 3.

We observe from Table 3 and Figure 3 that, in general, when the location shift is small, the known parameter MSEWMA chart performs better than the estimated parameter MSEWMA chart, when the Phase I sample size is small. The difference in performance between the two charts lessens as the size of the Phase I sample increases. However, for moderate to large location shifts, the two charts have very similar out-of-control performance. For example (from Table 3), for a shift of size in the first component of the distribution χ_p $_3$, the out of control ARLs for reference samples of size $m = 100$, $m = 1000$, and (known parameter case) are 93.54, 69.45, and 66.47 respectively. However for the same situation as just described with a shift size of 5, the out of control ARL are 10.14, 9.17, 9.02 for reference samples of size 100, $m = 1000$, and $m = \infty$ respectively.

Note that while the known parameter MSEWMA chart out-of-control performance seems attractive, it requires much more Phase I data to maintain the nominal in-control ARL. This makes the proposed estimated parameter MSEWMA chart a good option for the practitioner when a large reference sample is not available.

6. Conclusion

The MSEWMA chart proposed by Zou and Tsung (2011) is an appealing multivariate nonparametric chart. Here is a summary of our findings:

(1) It is observed that using the traditional (known parameter) control limits provided by Zou and Tsung (2011), when parameters are estimated from a small reference sample, results in an increased number of false alarms.

(2) We provided the practitioner with some guidance as to how many reference observations are needed to construct the MSEWMA chart with the traditional control limits, for bivariate and trivariate data ($p = 2$ and $p = 3$) when using a smoothing parameter of $\lambda = 0.05$.

(3) We provided tables of corrected control limits that produce an in-control ARL of 200, for use when the necessary Phase I sample size is not available for the application of traditional control limits with trivariate data ($p = 3$), for smoothing parameter and 0.05.

(4) We found that the estimated parameter MSEWMA chart (with the corrected control limits) has good shift detection properties

This work is ongoing and further results will be reported elsewhere.

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TABLES AND FIGURES

Table 1: IC ARL of the MSEWMA chart when the traditional UCLs given by Zou and Tsung (2011) are used with m Phase I p-variate observations from various distributions. The nominal IC ARL=200 and λ=0.05

Distribution	50	100	150	200	300	500
	107.8	132.3	148.6	154.2	164.4	175.9
	106.9	131.1	144.5	157.1	166.9	177.5
2	105.0	130.5	143.2	156.0	166.0	176.3
	104.6	133.0	147.3	154.6	164.7	178.1
	107.6	132.6	144.1	153.6	164.5	177.6
	91.73	120.0	131.5	146.0	155.6	171.4
	90.20	119.1	135.7	143.1	154.8	171.3
3	89.81	118.7	133.0	148.9	159.6	171.9
	90.95	119.7	133.8	144.4	157.3	170.3
	91.33	118.2	136.0	145.0	158.0	172.7

Distribution	750	1000	1500	2000	2500	3000
	186.9	187.4	195.3	194.9	194.7	195.7
	186.1	185.0	189.2	194.1	194.3	193.8
2	185.3	186.9	190.9	195.1	196.6	193.3
	185.7	187.6	187.6	193.6	194.7	195.1
	184.9	183.7	191.4	196.5	194.8	197.6
	179.7	181.4	192.4	190.0	193.0	194.1
	180.8	186.0	187.9	192.8	191.7	195.4
3	180.5	182.4	186.8	192.8	191.4	193.3
	179.5	181.5	188.7	191.9	192.2	198.0
	180.3	180.5	191.0	191.7	194.0	193.2

Table 2: The corrected upper control limit values for a nominal IC ARL = 200, when m Phase I trivariate ($p = 3$) observations from elliptical direction distributions are used to estimate the parameters, for two values of the smoothing parameter λ of the MSEWMA chart. m corresponds to the known parameters case.

Λ	50	100	150	200	300	500
0.025	11.441	9.964	9.383	8.984	8.586	8.237
0.05	11.973	10.852	10.379	10.113	9.781	9.582

Λ	750	1000	1500	2000	∞
0.025	8.121	8.005	7.889	7.922	7.689
0.05	9.449	9.383	9.31	9.279	9.176

Table 3: Out of control ARL of MSEWMA chart for various distributions when a location shift of size b occurs in the first component of three-dimensional data. The corrected control limits are used for $m = 100$ and $m = 1000$. corresponds to the known parameters case. The smoothing parameter is λ=0.05.

	100	1000	∞	100	1000	∞	100	1000	∞
0	201.6	199.9	198.2	204.3	196.6	203.0	199.7	196.3	192.6
0.1	180.8	164.2	161.5	152.9	126.9	122.2	169.4	155.1	149.8
0.25	110.5	83.44	81.96	62.61	47.75	46.57	93.54	69.45	66.47
0.5	44.01	35.07	33.93	23.86	20.71	20.26	33.63	26.47	25.78
1	17.66	15.65	15.33	12.51	11.36	11.13	13.45	11.86	11.66
1.5	12.16	10.95	10.80	10.01	9.133	9.006	10.14	9.172	9.02
2	10.04	9.153	9.018	9.036	8.299	8.132	9.046	8.27	8.152
2.5	9.052	8.286	8.181	8.551	7.807	7.683	8.526	7.843	7.729
UCL	10.852	9.383	9.176	10.852	9.383	9.176	10.852	9.383	9.176

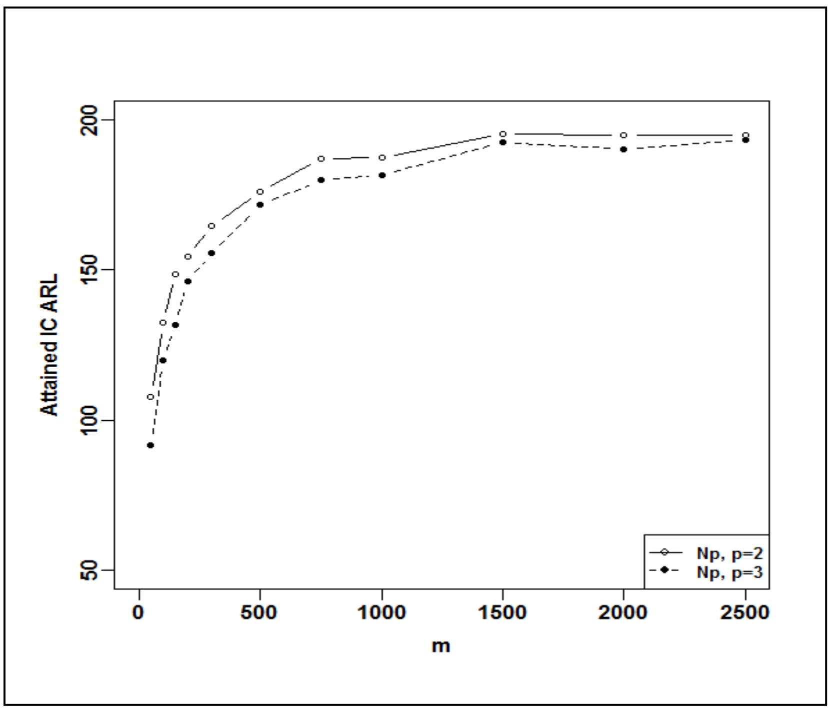

Figure 1: Attained IC ARL of the MSEWMA chart, as a function of m, the size of the reference sample, where the data are p-dimensional normal and the smoothing parameter 05.

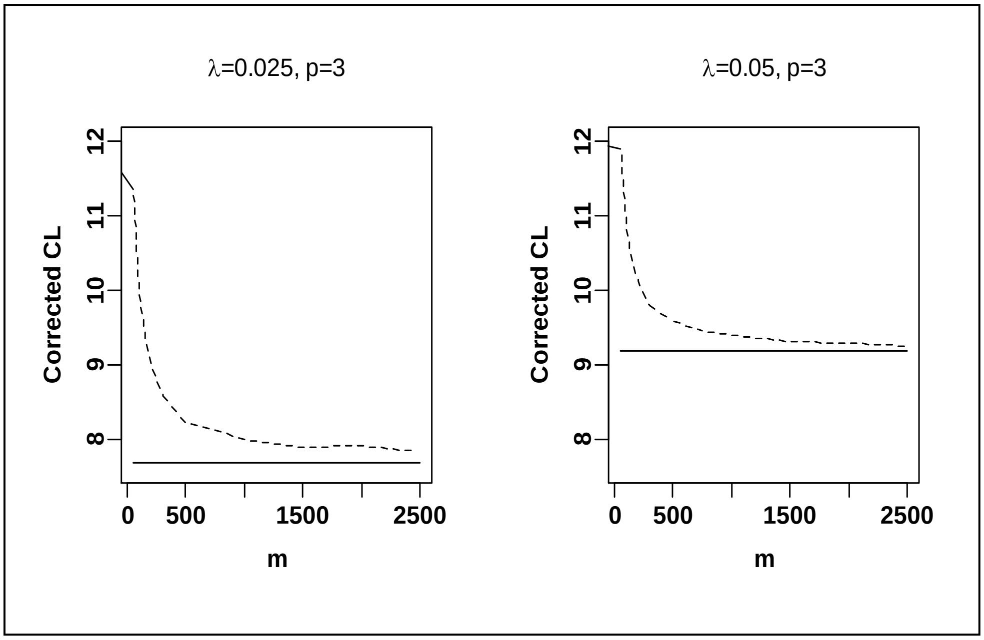

Figure 2: Estimated parameter MSEWMA chart corrected control limits as a function of the size of the reference sample m for trivariate (p = 3) data from elliptical direction distributions. The solid line corresponds to the traditional control limit.

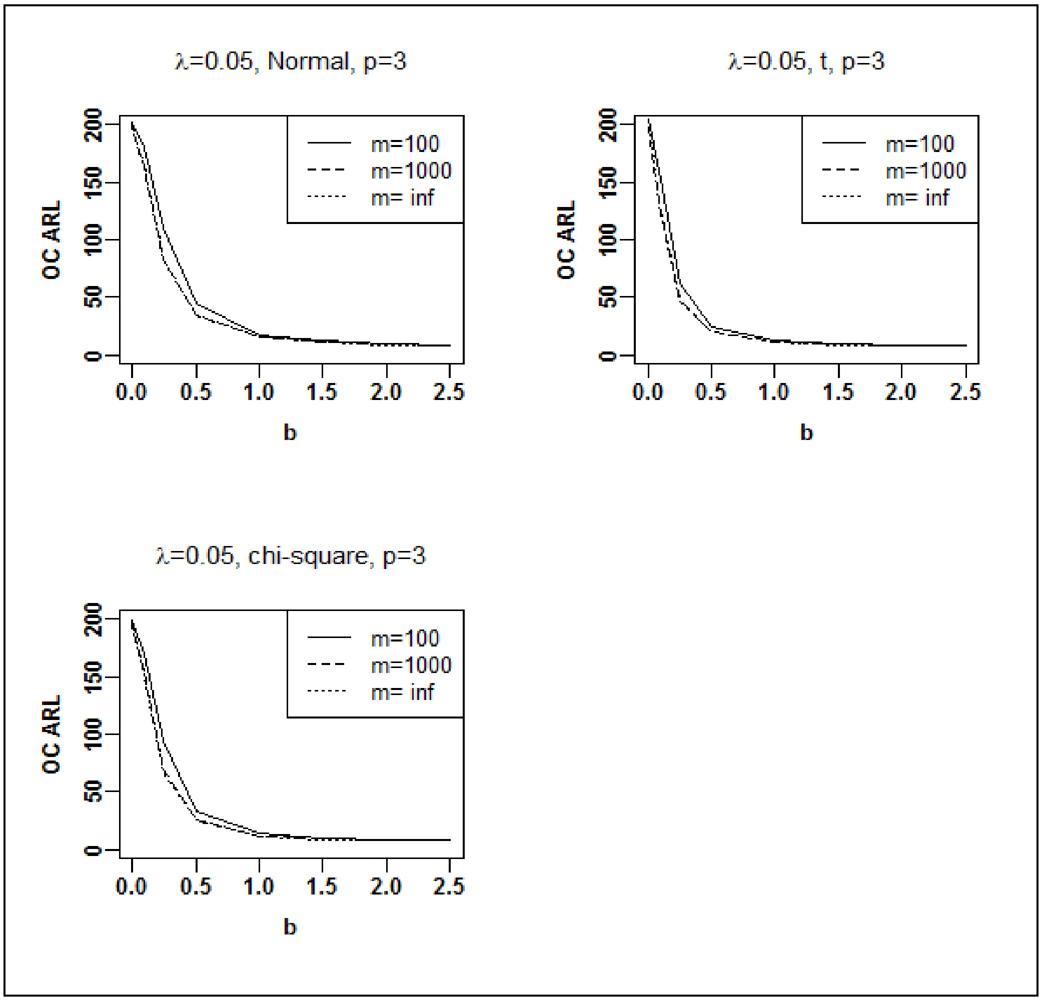

Figure 3: Out-of control ARL comparison between the known parameters MSEWMA chart with the traditional control limits and the estimated parameter MSEWMA chart with the corrected control limits, when a shift of size b occurs in the first component of three-dimensional observations from $N_{p=3}$, $t_{p=3,\zeta=3}$, and $\chi^2_{p=3,\zeta=3,}$ respectively, with 05 as the smoothing parameter, and m is the Phase I sample size used; corresponds to the known parameters case.