A Characterization of Burr Type III and Type XII Distributions through the Method of Percentiles

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Abstract
Burr Type III and Type XII distributions have been mainly used in statistical modeling of events in a variety of applied mathematical contexts such as fracture roughness, life testing, meteorology, modeling crop prices, forestry, reliability analysis, and in the context of Monte Carlo simulation studies. A preponderance of the applications associated with the Burr Type III and Type XII distributions are based on the method of moments (MOM). However, estimators of conventional skew and kurtosis can be substantially (a) biased, (b) dispersed, or (c) influenced by outliers. To obviate these problems, a characterization of Burr Type III and Type XII distributions based on the method of percentiles (MOP) is introduced and contrasted with the MOM in the context of estimation and fitting theoretical and empirical distributions. The methodology is based on simulating Burr Type III and Type XII distributions with specified values of medians, inter-decile ranges, left-right tail-weight ratios (a skew function), and tail-weight factors (a kurtosis function). Evaluation of the proposed procedure demonstrates that the estimates of left-right tail-weight ratios and tail-weight factors are substantially superior to their MOM-based counterparts of skew and kurtosis in terms of relative bias and relative efficiency–most notably when heavy-tailed distributions are of concern.

1. Introduction

The cumulative distribution functions (cdfs) associated with Burr Type III and Type XII distributions are given as (Burr, 1942, equations (11) and (20))

$$F(x)_{\mathrm{III}} = (1 + x^c)^{-k}, \tag{1}$$

$$F(x)_{\mathrm{XII}} = 1 - (1 + x^c)^{-k}, \tag{2}$$

where $x \in (0, \infty)$, c and k are real-valued shape parameters that are also used to compute the values of mean and standard deviation of a distribution. The parameter c is negative for Type III and positive for Type XII, whereas the parameter k is positive for both Type III and Type XII distributions (Headrick, Pant, & Sheng, 2010).

Burr Type III and Type XII distributions have received much of the attention because they include several families of non-normal distributions (e.g., the Gamma distribution) with varying degrees of skew and kurtosis (Headrick et al., 2010; Tadikamalla, 1980; Rodriguez, 1977; Burr, 1973). Burr Type III and Type XII distributions have been primarily used for statistical modeling of events arising in a variety of applied mathematical contexts. Some examples of such applications include modeling events associated with forestry (Gove, Ducey, Leak, & Zhang, 2008; Lindsay, Wood, & Woollons, 1996), fracture roughness (Nadarajah & Kotz, 2007; 2006), life testing (Wingo, 1993; 1983), operational risk (Chernobai, Fabozzi, & Rachev, 2007), option market price distributions (Sherrick, Garcia, & Tirupattur, 1996), meteorology

(Mielke, 1973), modeling crop prices (Tejeda & Goodwin, 2008), software reliability growth (Abdel-Ghaly, Al-Dayian, & Al-Kashkari, 1997), reliability analysis (Mokhlis, 2005), and in the context of Monte Carlo simulation studies (Pant & Headrick, 2013; Headrick et al., 2010).

The quantile functions associated with (1) and (2) are expressed as (Headrick et al., 2010, Equations (5) and (6))

$$q(u)_{\mathrm{III}} = \left(u^{-1/k} - 1\right)^{1/c}, \tag{3}$$

$$q(u)_{\mathrm{XII}} = \left((1-u)^{-1/k} - 1\right)^{1/c}, \tag{4}$$

where $u \sim iid$ Uniform $(0, 1)$ with cdf u and pdf 1. The shape of a Burr distribution associated with (3) or (4) is contingent on the values of the shape parameters (c and k), which can be determined based on method of moments (MOM) or method of percentiles (MOP).

To produce a valid Burr Type III or Type XII pdf, the quantile function $q(u)$ in (3) or (4) is required to be a strictly increasing monotone function of u (Headrick et al., 2010). This requirement implies that an inverse function (q^{-1}) exists such that the cdf associated with $q(u)$ in (3) or (4) can be expressed as $F\big(q(u)\big) = F(u) = u$. Differentiating both sides of this cdf with respect to u yields the parametric form of the pdf for $q(u)$ as $f\big(q(u)\big) = 1/q'(u)$. The simple closed-form expressions for the pdfs associated with (1) and (2) can be given as (Burr, 1942)

$$f(x)_{\mathrm{III}} = -ckx^{c-1}(1 + x^c)^{-(k+1)}, \tag{5}$$

$$f(x)_{\mathrm{XII}} = ckx^{c-1}(1 + x^c)^{-(k+1)}, \tag{6}$$

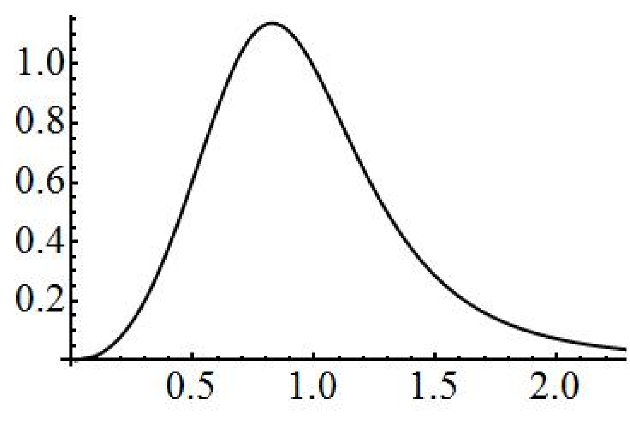

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Figure 1: The pdf of the Burr Type III distribution with skew $(\gamma_3) = 3$ and kurtosis $(\gamma_4) = 65$. The solved values of c and k used in (5) are: $c = -4.4067$ and $k = 0.7551$, which are also associated with the values of parameters and their estimates in Table 1.

Table 1: MOM-based parameter values of skew (γ_3) and kurtosis (γ_4) and MOP-based parameter values of left-right tail-weight ratio (ξ_3) and tail-weight factor (ξ_4) with their

corresponding estimates for the pdf in Figure 1. Each bootstrapped estimate (Estimate), associated 95% bootstrap confidence interval (95% C.I.), and the standard error (SE) were based on resampling 25,000 statistics. Each statistic was based on a sample size of $n = 500$.

Skew: $\gamma_3 = 3$			Kurtosis: $\gamma_4 = 65$		
Estimate: $\hat{\gamma}_3$	95% C.I.	SE	Estimate: $\hat{\gamma}_4$	95% C.I.	SE
2.256	2.239, 2.273	0.00860	13.29	13.02, 13.57	0.14210
Left-right tail-weight ratio: $\xi_3 = 0.6464$			Tail-weight factor: $\xi_4 = 0.4888$		
Estimate: $\hat{\xi}_3$	95% C.I.	SE	Estimate: $\hat{\xi}_4$	95% C.I.	SE
0.6467	0.6459, 0.6476	0.00041	0.4879	0.4876, 0.4883	0.00017

Some of the problems associated with conventional moment-based estimators are that they can be (a) substantially biased, (b) highly dispersed, or (c) influenced by outliers (Headrick, 2011; Hosking, 1990), and thus may not be good representatives of the true parameters. To demonstrate, Figure 1 gives the graph of the pdf associated with Burr Type III distribution with skew $(\gamma_3) = 3$ and kurtosis $(\gamma_4) = 65$. Note that the pdf in Figure 1 has been used in studies such as Karian and Dudewicz (2011) and Headrick and Pant (2012). Table 1 gives the parameters and sample estimates of skew and kurtosis for the distribution in Figure 1. Inspection of Table 1 indicates that the bootstrap estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skew and kurtosis (γ_3 and γ_4) are substantially attenuated below their corresponding parameter values with greater bias and variance as the order of the estimate increases. Specifically, for sample size of $n = 500$, the values of the estimates are only 75.2%, and 20.45% of their corresponding parameters, respectively. The estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skew and kurtosis (γ_3 and γ_4) in Table 1 were calculated based on Fisher's k-statistics formulae (see, e.g., Kendall & Stuart, 1977, pp. 299-300), currently used by most commercial software packages such as SAS, SPSS, Minitab, etc., for computing the values of skew and kurtosis (where $\gamma_{3,4} = 0$ for the standard normal distribution).

Another unfavorable quality of conventional moment-based estimators of skew and kurtosis is that their values are algebraically bounded by the sample size (n) such that $|\hat{\gamma}_3| \leq \sqrt{n}$ and $\hat{\gamma}_4 \leq n$ (Headrick, 2011). This constraint implies that if a researcher wants to simulate non-normal data with kurtosis $\gamma_4 = 65$ as in Table 1, and drawing a sample of size $n = 30$ from this population, the largest possible value of the computed estimate $(\hat{\gamma}_4)$ of kurtosis (γ_4) is only 30, which is only 46.15% of the parameter.

The method of percentiles (MOP) introduced by Karian and Dudewicz (2000) in the context of generalized lambda distributions (GLDs) is an attractive alternative to the method of moments (MOM) and can be used for estimating shape parameters and fitting distributions to real-world data. The MOP-based GLDs are superior to the MOM-based GLDs for fitting theoretical and empirical distributions that cover a wide range of combinations of skew and kurtosis (Karian & Dudewicz, 2000). Some qualities of the MOP-based procedure in the context of GLDs are that (a) MOP-based procedure can be used to estimate parameters and obtain GLD fits even when the conventional moments associated with a class of GLDs do not exist, (b) the MOP-based procedure for solving equations for the GLD parameters is relatively more accurate than the MOM-based procedure, and (c) the relatively smaller variability of MOP-based sample estimators enables more accurate GLD fits than that achieved through the MOM-based approach (Karian & Dudewicz, 2000). Also, Kuo and Headrick (2014) have demonstrated that the

MOP-based characterization of Tukey's g-and-h distributions is superior to the MOM-based characterization in terms of distribution fitting, estimation, relative bias, and relative error. For example, for the Burr Type III pdf in Figure 1, the MOP-based estimates ($\hat{\xi}_3$ and $\hat{\xi}_4$) of left-right tail-weight ratio and tail-weight factor (ξ_3 and ξ_4) in Table 1 are relatively closer to their respective parameter values with much smaller variance compared to their conventional MOM-based counterparts. Inspection of Table 1 shows that for the sample size of $n = 500$, the values of the estimates are on average 100.05% and 99.82% of their corresponding parameters.

In view of the above, the main purpose of this study is to characterize the Burr Type III and Type XII distributions through the method of percentiles (MOP) in order to obviate the problems associated with method of moments (MOM)-based estimators. Specifically, the purpose of this study is to develop a methodology to simulate Burr Type III and Type XII distributions with specified values of medians (ξ_1), inter-decile ranges (ξ_2), left-right tail-weight ratios (ξ_3), and tail-weight factors (ξ_4).

The remainder of the paper is organized as follows. In Section 2, a brief introduction to method of moments (MOM)-based procedure for solving equations of skew (γ_3) and kurtosis (γ_4) for the shape parameters (c and k) associated with univariate Burr Type III and Type XII distributions is given. Provided in Section 3 is a brief introduction to method of percentiles (MOP) along with a procedure for characterizing the Burr Type III and Type XII distributions based on the MOP. Specifically, the MOP-based systems of equations associated with Burr Type III and Type XII distributions are derived for determining the shape parameters (c and k) for specified values of left-right tail-weight ratios (ξ_3), and tail-weight factors (ξ_4). In Section 4, a comparison between MOM- and MOP-based Burr Type III and Type XII distributions is presented in the contexts of fitting theoretical and empirical distributions and estimation of parameters. Numerical examples and the results of simulation are also provided to confirm the derivations and compare the MOP-based procedure with the conventional MOM-based procedure. In Section 5, the results of the simulation are discussed.

2. Method of Moments (MOM)-based System

The method of moments (MOM)-based values of mean (μ), standard deviation (σ), skew (γ_3), and kurtosis (γ_4) associated with a Burr Type III or Type XII distribution can be given as (Headrick et al., 2010, p. 2211, Equations 14-17)

$$\mu = (\Gamma[1 + 1/c]\Gamma[k - 1/c])/\Gamma[k] \tag{7}$$

$$\sigma = \sqrt{(\Gamma[1 + 2/c]\Gamma[k]\Gamma[k - 2/c] - \Gamma[1 + 1/c]^2\Gamma[k - 1/c]^2)}/\Gamma[k] \tag{8}$$

$$\begin{aligned}\gamma_3 = \{&\Gamma[1 + 3/c]\Gamma[k]^2\Gamma[k - 3/c] \\ &- c^{-2}(6\Gamma[1/c]\Gamma[2/c]\Gamma[k]\Gamma[k - 2/c]\Gamma[k - 1/c]) \\ &+ 2\Gamma[1 + 1/c]^3\Gamma[k - 1/c]^3\} \\ &/\{\Gamma[1 + 2/c]\Gamma[k]\Gamma[k - 2/c] - \Gamma[1 + 1/c]^2\Gamma[k - 1/c]^2\}^{3/2}\end{aligned} \tag{9}$$

$$\begin{aligned}\gamma_4 = \Big\{\{&\Gamma[1+4/c]\Gamma[k]^3\Gamma[k-4/c] \\ &- c^{-3}\big(3\Gamma[k-1/c](4c\Gamma[1/c]\Gamma[3/c]\Gamma[k]^2\Gamma[k-3/c] \\ &- 4\Gamma[1/c]^2\Gamma[2/c]\Gamma[k]\Gamma[k-2/c]\Gamma[k-1/c] \\ &+ c^3\Gamma[1+1/c]^4\Gamma[k-1/c]^3)\big)\} \\ &/\,\{\Gamma[1+2/c]\Gamma[k]\Gamma[k-2/c]-\Gamma[1+1/c]^2\Gamma[k-1/c]^2\}^2\Big\} \\ &- 3.\end{aligned} \tag{10}$$

The MOM-based procedure for characterizing Burr Type III and Type XII distributions involves a moment-matching approach in which specified values of skew and kurtosis (obtained from theoretical distributions or real-word data) are substituted on the left-hand sides of (9) and (10) for skew (γ_3) and kurtosis (γ_4), respectively. Then, (9) and (10) are simultaneously solved for shape parameters (c and k) associated with Burr Type III and Type XII distributions. The solved values of c and k can be substituted into (7) and (8) to determine the values of mean and standard deviation associated with the Burr Type III or Type XII distributions.

3. Method of Percentiles (MOP)-based System

3.1 General Definition

The MOP-based analogs of location, scale, skew function, and kurtosis function are respectively defined by median (ξ_1), inter-decile range (ξ_2), left-right tail-weight ratio (ξ_3), and tail-weight factor (ξ_4) and expressed as (Karian & Dudewicz, 2000, pp. 154-155)

$$\xi_1 = \pi_{p=0.50} \tag{11}$$

$$\xi_2 = \pi_{p=0.90} - \pi_{p=0.10} \tag{12}$$

$$\xi_3 = \frac{\pi_{p=0.50} - \pi_{p=0.10}}{\pi_{p=0.90} - \pi_{p=0.50}} \tag{13}$$

$$\xi_4 = \frac{\pi_{p=0.75} - \pi_{p=0.25}}{\xi_2} \tag{14}$$

where π_p in (11)—(14) is the $(100p)$th percentile and where $p \in (0,1)$. Note that the quantile function $q(u)$ in (3) or (4) can be considered as a substitute of π_p in (11)—(14) in the context of Burr Type III and Type XII distributions as $u \in (0,1)$.

3.2 MOP-based Burr Type III Distribution

Substituting $q(u)$ from (3) into (11)—(14) and simplifying the resulting expressions, the MOP-based system of Burr Type III distribution can be given as follows.

$$\xi_1 = \left(2^{1/k} - 1\right)^{1/c} \tag{15}$$

$$\xi_2 = \left((10/9)^{1/k} - 1\right)^{1/c} - \left(10^{1/k} - 1\right)^{1/c} \tag{16}$$

$$\xi_3 = \frac{\left(2^{1/k}-1\right)^{1/c} - \left(10^{1/k}-1\right)^{1/c}}{((10/9)^{1/k}-1)^{1/c} - (2^{1/k}-1)^{1/c}} \tag{17}$$

$$\xi_4 = \frac{\left((4/3)^{1/k}-1\right)^{1/c} - \left(4^{1/k}-1\right)^{1/c}}{((10/9)^{1/k}-1)^{1/c} - (10^{1/k}-1)^{1/c}} \tag{18}$$

3.3 MOP-based Burr Type XII Distribution

Substituting $q(u)$ from (4) into (11)—(14) and simplifying the resulting expressions, the MOP-based system of Burr Type XII distribution can be given as follows.

$$\xi_1 = \left(2^{1/k}-1\right)^{1/c} \tag{19}$$

$$\xi_2 = \left(10^{1/k}-1\right)^{1/c} - \left((10/9)^{1/k}-1\right)^{1/c} \tag{20}$$

$$\xi_3 = \frac{\left(2^{1/k}-1\right)^{1/c} - \left((10/9)^{1/k}-1\right)^{1/c}}{(10^{1/k}-1)^{1/c} - (2^{1/k}-1)^{1/c}} \tag{21}$$

$$\xi_4 = \frac{\left(4^{1/k}-1\right)^{1/c} - \left((4/3)^{1/k}-1\right)^{1/c}}{(10^{1/k}-1)^{1/c} - ((10/9)^{1/k}-1)^{1/c}} \tag{22}$$

In the context of Burr Type III and Type XII distributions, the parameter values of median (ξ_1), inter-decile range (ξ_2), left-right tail-weight ratio (ξ_3), and tail-weight factor (ξ_4) in (15)—(18) or (19)—(22) have the following restrictions

$$\xi_1 \in (0, \infty), \quad \xi_2 \geq 0, \quad \xi_3 \geq 0, \quad 0 \leq \xi_4 \leq 1 \tag{23}$$

where a symmetric distribution will have $\xi_1 = \text{Median} = \text{Mean}$ and $\xi_3 = 1$.

For the specified values of left-right tail-weight ratio (ξ_3) and tail-weight factor (ξ_4) the systems of equations (17)−(18) and (21)−(22) can be simultaneously solved for real values of c and k. The solved values of c and k can be substituted in (3) and (4), respectively, for generating the Burr Type III and Type XII distributions. Further, the solved values of c and k can be substituted in (15)−(16) and (19)−(20) to determine the values of median (ξ_1) and inter-decile range (ξ_2) associated with the Type III and Type XII distributions, respectively.

Let $X_1 < X_2 < X_3 < \cdots < X_i < X_{i+1} < \cdots < X_n$ be the order statistics of a sample $(Y_1, Y_2, Y_3, \ldots, Y_n)$ of size n. Let $\hat{\pi}_p$ be the (100p)th percentile from this sample, where $p \in (0,1)$. Let $(n+1)p = i + (a/b)$, where i is a positive integer and a/b is a proper fraction. Then, $\hat{\pi}_p$ can be computed as (Karian & Dudewicz, 2000, p. 154)

$$\hat{\pi}_p = X_i + (a/b)(X_{i+1} - X_i)\,. \tag{24}$$

For a sample of size n, the MOP-based estimates of ξ_1-ξ_4 in (15)−(18) or (19)−(22), can be computed in two steps as: (a) compute the values of 10th, 25th, 50th, 75th, and 90th percentiles using (24) and (b) substitute these percentiles into (11)–(14) to obtain the sample estimates of ξ_1-ξ_4.

In the next section, examples are provided to demonstrate the aforementioned methodology and the advantages of MOP procedure over the conventional MOM procedure in the contexts of distribution fitting and estimation.

4. Comparison of L-Moments with Conventional Moments

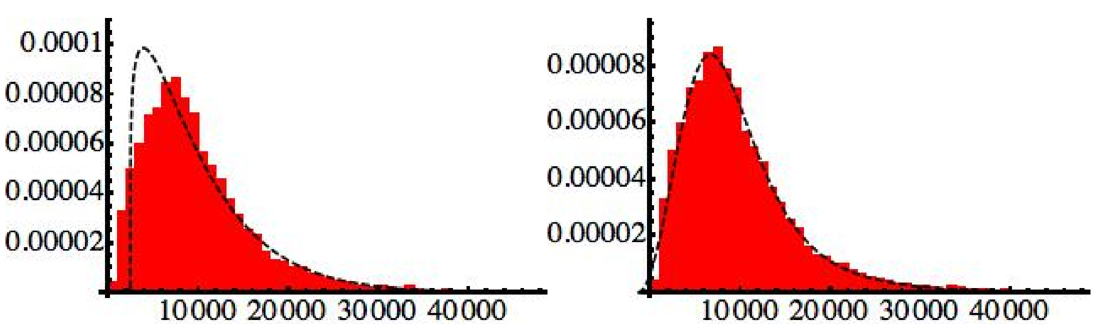
4.1. Distribution Fitting

Figure 2 shows the conventional MOM- and the MOP-based Burr Type XII pdfs superimposed on the histogram of total hospital charges (in U.S. dollars) data of 12,145 heart attack patients discharged from all of the hospitals in New York State in 1993. There were 12,844 cases with 699 missing values for the total hospital charges. See, the website:
http://wiki.stat.ucla.edu/socr/index.php/SOCR_Data_AMI_NY_1993_HeartAttacks

The conventional MOM-based estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skew and kurtosis (γ_3 and γ_4) and the MOP-based estimates ($\hat{\xi}_3$ and $\hat{\xi}_4$) of left-right tail-weight ratio and tail-weight factor (ξ_3 and ξ_4) were computed for the sample of size $n = 12{,}145$ patients. The estimates of γ_3 and γ_4 were computed based on Fisher's k-statistics formulae (Kendall & Stuart, 1977, pp. 47-48), whereas the estimates of ξ_3 and ξ_4 were computed using (11)−(14) and (24), respectively. These sample estimates were then used to solve for the values of shape parameters (c and k) using (A) (9) and (10) and (B) (21) and (22), respectively, for the MOM- and MOP-based fits. The solved values of c and k were subsequently used in (6) to superimpose the Burr Type XII pdfs shown in Figure 2.

To superimpose the Burr Type XII pdf (dashed curves), the quantile function $q(u)$ from (4) was transformed as (A) $\bar{X} + S(q(u) - \mu) / \sigma$, and (B) $m_1 + m_2(q(u) - \xi_1)/\xi_2$, respectively, where ($\bar{X}$, S) and (μ, σ) are the values of (mean, standard deviation), whereas (m_1, m_2) and (ξ_1, ξ_2) are the values of (median, inter-decile range) obtained from the actual data and the Burr Type XII pdf, respectively.

Inspection of the two panels in Figure 2 illustrates that the MOP-based Burr Type XII pdf provides a better fit to the total hospital charges data. The Chi-Square goodness of fit statistics along with their corresponding p-values in Table 2 provide evidence that the MOM-based Burr Type XII pdf does not provide a good fit to these real-world data, whereas, the MOP-based Burr Type XII pdf fits very well. The degrees of freedom for the Chi-Square goodness of fit tests were computed as df = 5 = 10 (class intervals) – 4 (estimates of the parameters) – 1 (sample size).



Estimates	Shape parameters
$\bar{X} = 9879.1$	$c = 1.1665$
$S = 6558.4$	$k = 40.9856$
$\hat{\gamma}_3 = 1.7035$	
$\hat{\gamma}_4 = 4.34$	

(A)

Estimates	Shape parameters
$m_1 = 8445$	$c = 2.7793$
$m_2 = 14684.8$	$k = 1.3833$
$\hat{\xi}_3 = 0.5477$	
$\hat{\xi}_4 = 0.4868$	

(B)

Figure 2: Histograms of the total hospital charges (in U.S. dollars) data of 12,145 heart attack patients superimposed by the (A) MOM- and (B) MOP-based Burr Type XII pdfs.

Table 2: Chi-square goodness of fit statistics for the conventional MOM- and the MOP-based Burr Type XII approximations for the total hospital charges ($n = 12{,}145$) data in Figure 2.

Percent	Exp.	Obs. (MOM)	Obs. (MOP)	total hospital charges (MOM)	total hospital charges (MOP)
10	1214.5	1444	1214	< 3537.98	< 3248.40
20	1214.5	809	1214	3537.98 – 4554.41	3248.40 – 4763.57
30	1214.5	968	1147	4554.41 – 5617.68	4763.57 – 6015.52
40	1214.5	1127	1236	5617.68 – 6779.64	6015.52 – 7207.91
50	1214.5	1372	1262	6779.64 – 8095.82	7207.91 – 8445.00
60	1214.5	1452	1243	8095.82 – 9648.28	8445.00 – 9827.63
70	1214.5	1344	1164	9648.28 – 11583.63	9827.63 – 11510.03
80	1214.5	1353	1225	11583.63 – 14224.23	11510.03 – 13828.69
90	1214.5	1175	1226	14224.23 – 18585.93	13828.69 – 17933.20
100	1214.5	1101	1214	18585.93 or more	17933.20 or more
				$\chi^2 = 343.46$	$\chi^2 = 8.96$
				$p < 0.0001$	$p = 0.1107$

4.2. Estimation

An example to demonstrate the advantages of MOP-based estimation over the conventional MOM-based estimation is provided in Figure 3 and Tables 3−6. Given in Figure 3 are the pdfs of the F (3, 10), Chi-square (df = 1), Extreme Value (0, 1), and Logistic (0, 1) distributions superimposed, respectively, by the Burr Type XII, Type III, Type XII, and Type III pdfs (dashed curves) in both (A) conventional MOM- and (B) MOP-based systems. The conventional MOM-based parameters of skew (γ_3) and kurtosis (γ_4) associated with these four distributions, given in Table 3, were computed by using equations (11)−(13) from (Headrick et al., 2010, page 2211). The values of shape parameters (c and k) given in Table 3 were determined by simultaneously solving (9) and (10). The values of c and k were used in (5) and (6) to superimpose the conventional MOM-based Burr Type XII, Type III, Type XII, and Type III distributions, respectively, as shown in Figure 3 (Panel A).

The MOP-based parameters of left-right tail-weight ratio (ξ_3) and tail-weight factor (ξ_4) associated with the four distributions in Figure 3, given in Table 4, were obtained in two steps as: (a) compute the values of 10th, 25th, 50th, 75th, and 90th percentiles from the cdfs of the four distributions and (b) substitute these five percentiles into (11)−(14) to compute the values of ξ_3 and ξ_4. The values of shape parameters (c and k) given in Table 4 were determined by solving the systems of equations (17)−(18) and (21)−(22), respectively. These values of c and k were used in (5) and (6) to superimpose the MOP-based Burr Type XII, Type III, Type XII, and Type III distributions, respectively, as shown in Figure 3 (Panel B).

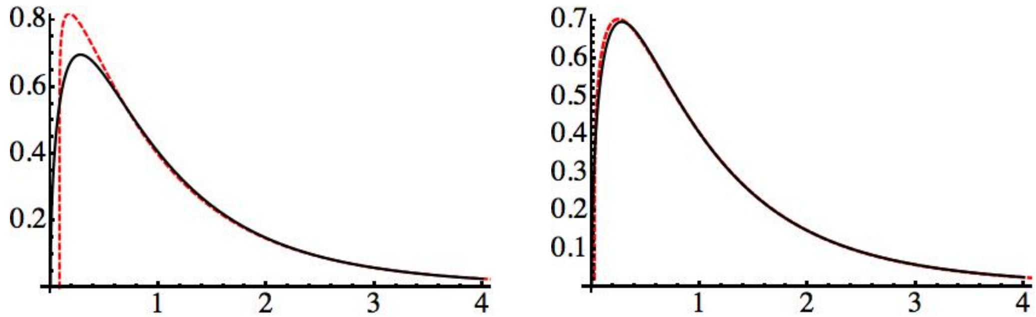
To superimpose the Type III or Type XII distribution, the quantile function $q(u)$ in (3) or (4) was transformed into: (A) $\bar{X} + S(q(u) - \mu) / \sigma$, and (B) $m_1 + m_2(q(u) - \xi_1)/\xi_2$, respectively, where ($\bar{X}$, S) and (μ, σ) are the values of (mean, standard deviation), whereas (m_1, m_2) and (ξ_1, ξ_2) are the values of (median, inter-decile range) obtained from the original distribution and the respective Burr Type III or Type XII approximation, respectively.

The advantages of MOP-based estimators over the MOM-based estimators can also be demonstrated in the context of Burr Type III and Type XII distributions by considering the Monte Carlo simulation results associated with the indices for the percentage of relative bias (RB%) and standard error (St. Error) reported in Tables 5 and 6.

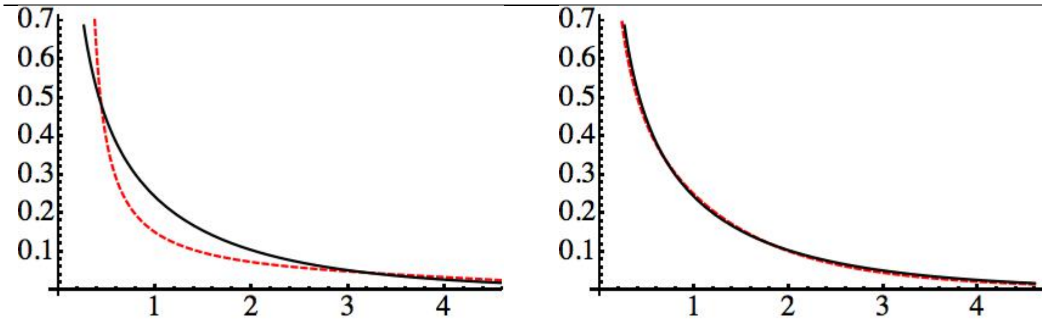
Specifically, a Fortran (Microsoft, 1994) algorithm was written to simulate 25,000 independent samples of sizes $n = 25$ and $n = 1000$, and the conventional MOM-based estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skew and kurtosis (γ_3 and γ_4) and the MOP-based estimates ($\hat{\xi}_3$ and $\hat{\xi}_4$) of left-right tail-weight ratio and tail-weight factor (ξ_3 and ξ_4) were computed for each of the ($2 \times 25{,}000$) samples based on the parameters and the values of c and k listed in Tables 3 and 4. The estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of γ_3 and γ_4 were computed based on Fisher's k-statistics formulae (Kendall & Stuart, 1977, pp. 47-48), whereas the estimates ($\hat{\xi}_3$ and $\hat{\xi}_4$) of ξ_3 and ξ_4 were computed using (11)−(14) and (24). Bias-corrected accelerated bootstrapped average estimates (Estimate), associated 95% confidence intervals (95% Bootstrap C.I.), and standard errors (St. Error) were obtained for each type of estimates using 10,000 resamples via the commercial software package Spotfire S+ (TIBCO, 2008). Further, if a parameter was outside its associated 95% bootstrap C.I., then the percentage of relative bias (RB%) was computed for the estimate as

$$\mathrm{RB\%} = 100 \times (\mathrm{Estimate} - \mathrm{Parameter})/\mathrm{Parameter} \qquad (25)$$

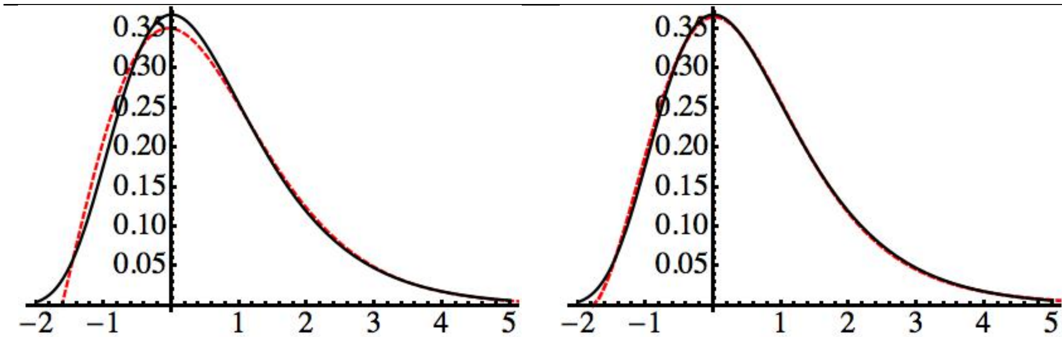
In order to demonstrate the advantages of MOP-based procedure over the MOM-based procedure, the results of simulation are discussed in the next section. Also discussed in the next section are the advantages that MOP-based procedure has over the MOM-based procedure in terms of distribution fitting.



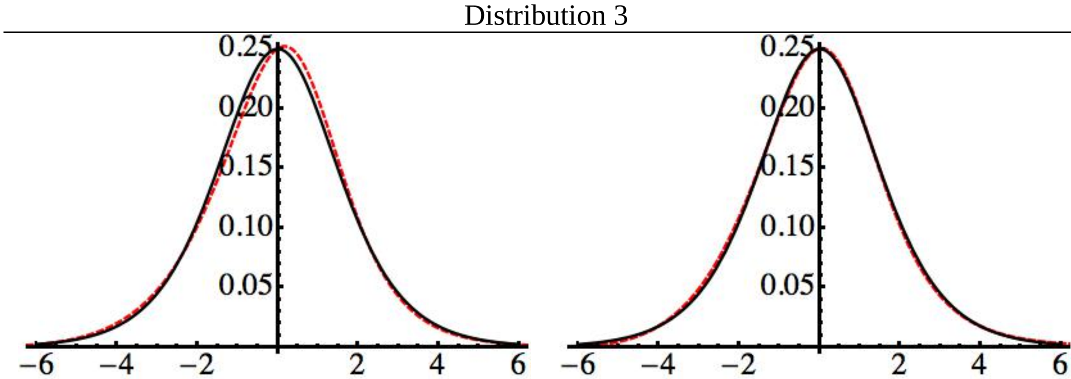
Distribution 1



Distribution 2



Distribution 3



Distribution 4

(A) (B)

Figure 3: The pdfs (dashed curves) of the four distributions: Distribution 1 ≔ Burr Type XII ≈ F (3, 10), Distribution 2 ≔Burr Type III ≈ Chi square (df= 1), Distribution 3 ≔ Burr Type XII ≈ Extreme Value (0, 1), and Distribution 4 ≔ Burr Type III ≈ Logistic (0, 1) superimposed by (A) MOM- and (B) MOP-based Burr Type III and Type XII pdfs, respectively.

Table 3: MOM-based parameters of the mean (μ), standard deviation (σ), skew (γ_3), and kurtosis (γ_4) along with values of shape parameters (c and k) for the four distributions (dashed curves) in Figure 3 (Panel A).

Dist	μ	σ	γ_3	γ_4	c	k
1	0.3062	0.3628	4.2212	59.4545	1.0977	4.4817
2	0.1394	0.2636	2.8284	12.0	−5.5421	0.0272
3	0.4155	0.2421	1.1395	2.4	2.0251	5.3744
4	0.9375	0.1392	0.0	1.2	−15.7764	0.5402

Table 4: MOP-based parameters of median (ξ_1), inter-decile range (ξ_2), left-right tail-weight ratio (ξ_3), and tail-weight factor (ξ_4) along with values of shape parameters (c and k) for the four distributions (dashed curves) in Figure 3 (Panel B).

Dist	ξ_1	ξ_2	ξ_3	ξ_4	c	k
1	0.3261	1.0086	0.3473	0.4706	1.2404	3.1166
2	0.1983	1.1878	0.1951	0.4542	−2.1131	0.2009
3	0.6160	0.8837	0.6373	0.5098	2.4468	2.5995
4	0.9071	0.5081	1.0	0.5	−10.1486	0.5308

5. Discussion and Conclusion

One of the advantages of MOP-based procedure over the conventional MOM-based procedure can be expressed in the context of estimation. Inspection of Tables 5 and 6 indicates that the MOP-based estimators of left-right tail-weight ratio (ξ_3) and tail-weight factor (ξ_4) are much less biased than the MOM-based estimators of skew (γ_3) and kurtosis (γ_4) when samples are drawn from the distributions with more severe departures from normality. For example, for samples of size $n = 25$, the estimates of γ_3 and γ_4 for Distribution 1 (skewed and heavy-tailed) were, on average, 58.54% and 94.38% below their associated parameters, whereas the estimates of ξ_3 and ξ_4 were 4.98% above and 9.46% below their associated parameters. This advantage of MOP-based estimators can also be expressed by comparing their relative standard errors (RSEs), where RSE $= \{(\mathrm{St.Error}/\mathrm{Estimate}) \times 100\}$. Comparing Tables 5 and 6, it is evident that the estimators of ξ_3 and ξ_4 are more efficient as their RSEs are considerably smaller than the RSEs associated with the conventional MOM-based estimators of γ_3 and γ_4. For example, in terms of Distribution 1 in Figure 3, inspection of Tables 5 and 6 (for $n = 1000$), indicates that RSE measures of: RSE $(\hat{\xi}_3) = 0.05\%$ and RSE $(\hat{\xi}_4) = 0.03\%$ are considerably smaller than the RSE measures of: RSE $(\hat{\gamma}_3) = 0.27\%$ and RSE $(\hat{\gamma}_4) = 0.85\%$. This demonstrates that the estimators of ξ_3 and ξ_4 have more precision because they have less variance around their bootstrapped estimates.

Another advantage of MOP-based procedure can be highlighted in the context of distribution fitting. In the context of fitting real-world data, the MOP-based Burr Type XII in Figure 2 (Panel B) provides a better fit to the total hospital charges data than the conventional MOM-based Burr Type XII in Figure 2 (Panel A). Comparison of the four distributions in Figure 3 (Panels A and B) clearly indicates that MOP-based Burr Type III and Type XII distributions provide a better fit to the theoretical distributions compared with their conventional MOM-based counterparts. This advantage is most pronounced in the context of the first two distributions: Distribution 1 and Distribution 2, where MOP-based Burr Type XII and Type III (Panel B) provide a better fit to the F (3, 10) and Chi-square (df = 1) distributions than their conventional MOM-based counterparts (Panel A).

In summary, the proposed MOP-based procedure is an attractive alternative to the more traditional MOM-based procedure in the context of Burr Type III and Type XII distributions. In particular, MOP-based procedure has distinct advantages when distributions with large departures from normality are used. Finally, we note that Mathematica (Wolfram, 2012) source codes are available from the authors for implementing both the conventional MOM- and MOP-based procedures.

Table 5. Skew (γ_3) and Kurtosis (γ_4) results for the conventional MOM-based procedure.

Dist.	Parameter	Estimate	95% Bootstrap C.I.	St. Error	RB%
		$n = 25$			
1	$\gamma_3 = 4.2212$	$\hat{\gamma}_3 = 1.75$	$1.7397, 1.7597$	0.00510	-58.54
	$\gamma_4 = 59.4545$	$\hat{\gamma}_4 = 3.34$	$3.2911, 3.3871$	0.02430	-94.38
2	$\gamma_3 = 2.8284$	$\hat{\gamma}_3 = 2.151$	$2.1428, 2.1596$	0.00429	-23.95
	$\gamma_4 = 12.0$	$\hat{\gamma}_4 = 4.215$	$4.1715, 4.2614$	0.02306	-64.88
3	$\gamma_3 = 1.1395$	$\hat{\gamma}_3 = 0.7929$	$0.7859, 0.7994$	0.00346	-30.42
	$\gamma_4 = 2.4$	$\hat{\gamma}_4 = 0.5014$	$0.4786, 0.5220$	0.01111	-79.11
4	$\gamma_3 = 0.0$	$\hat{\gamma}_3 = -0.051$	$-0.0585, -0.0430$	0.00391	-----
	$\gamma_4 = 1.2$	$\hat{\gamma}_4 = 0.3396$	$0.3243, 0.3552$	0.00791	-71.70
		$n = 1000$			
1	$\gamma_3 = 4.2212$	$\hat{\gamma}_3 = 3.606$	$3.5888, 3.6275$	0.00972	-14.57
	$\gamma_4 = 59.4545$	$\hat{\gamma}_4 = 26.29$	$25.8657, 26.7366$	0.22370	-55.78
2	$\gamma_3 = 2.8284$	$\hat{\gamma}_3 = 2.761$	$2.7548, 2.7681$	0.00335	-2.38
	$\gamma_4 = 12.0$	$\hat{\gamma}_4 = 10.17$	$10.0615, 10.3107$	0.06265	-15.25
3	$\gamma_3 = 1.1395$	$\hat{\gamma}_3 = 1.124$	$1.1219, 1.1266$	0.00122	-1.36
	$\gamma_4 = 2.4$	$\hat{\gamma}_4 = 2.266$	$2.2451, 2.2861$	0.01048	-5.58
4	$\gamma_3 = 0.0$	$\hat{\gamma}_3 = -0.0045$	$-0.0064, -0.0025$	0.00101	-----
	$\gamma_4 = 1.2$	$\hat{\gamma}_4 = 1.158$	$1.1487, 1.1683$	0.00498	-3.5

Table 6. Left-right tail-weight ratio (ξ_3) and tail-weight factor (ξ_4) results for the MOP-based procedure.

Dist.	Parameter	Estimate	95% Bootstrap C.I.	St. Error	RB%
		$n = 25$			
1	$\xi_3 = 0.3473$	$\hat{\xi}_3 = 0.3646$	$0.3625, 0.3671$	0.00120	4.98
	$\xi_4 = 0.4706$	$\hat{\xi}_4 = 0.4261$	$0.4245, 0.4277$	0.00080	-9.46
2	$\xi_3 = 0.1951$	$\hat{\xi}_3 = 0.2161$	$0.2143, 0.2178$	0.00089	10.76
	$\xi_4 = 0.4542$	$\hat{\xi}_4 = 0.4026$	$0.4008, 0.4044$	0.00090	-11.36
3	$\xi_3 = 0.6373$	$\hat{\xi}_3 = 0.6887$	$0.6850, 0.6928$	0.00197	8.07
	$\xi_4 = 0.5098$	$\hat{\xi}_4 = 0.4702$	$0.4688, 0.4715$	0.00066	-7.77
4	$\xi_3 = 1.0$	$\hat{\xi}_3 = 1.0960$	$1.0901, 1.1024$	0.00313	9.6
	$\xi_4 = 0.5$	$\hat{\xi}_4 = 0.4623$	$0.4610, 0.4635$	0.00063	-7.54
		$n = 1000$			
1	$\xi_3 = 0.3473$	$\hat{\xi}_3 = 0.3481$	$0.3478, 0.3485$	0.00017	0.23
	$\xi_4 = 0.4706$	$\hat{\xi}_4 = 0.4703$	$0.4700, 0.4705$	0.00014	-0.06
2	$\xi_3 = 0.1951$	$\hat{\xi}_3 = 0.1961$	$0.1959, 0.1963$	0.00012	0.51
	$\xi_4 = 0.4542$	$\hat{\xi}_4 = 0.4541$	$0.4538, 0.4544$	0.00016	-----
3	$\xi_3 = 0.6373$	$\hat{\xi}_3 = 0.6379$	$0.6373, 0.6384$	0.00030	-----
	$\xi_4 = 0.5098$	$\hat{\xi}_4 = 0.5094$	$0.5092, 0.5096$	0.00012	-0.08
4	$\xi_3 = 1.0$	$\hat{\xi}_3 = 1.001$	$0.9997, 1.0015$	0.00044	-----
	$\xi_4 = 0.5$	$\hat{\xi}_4 = 0.4996$	$0.4993, 0.4998$	0.00011	-0.84

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