Bootstrap-based Unit Root Tests for Higher Order Autoregressive Models with GARCH (1, 1) Errors

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ABSTRACT

Bootstrap-based unit root tests are a viable alternative to asymptotic distribution-based tests and, in some cases, are preferable because of the serious size distortions the latter tests display under certain situations. While several bootstrap-based unit root tests exist for ARMA processes with homoscedastic errors, only one such test is available when the innovations are conditionally heteroskedastic. The utility of this test is limited because it is restricted to autoregressive processes of order one. We extend this test to autoregressive processes of higher orders and study the finite sample performance of the test using Monte-Carlo simulation. Results show that the proposed tests have reasonable power and size properties.

Keywords Non-stationary, Conditional volatility, Residual bootstrap, Time series

1. Introduction

Bootstrap-based unit root tests provide a good alternative to asymptotic distribution-based tests. The serious size distortions displayed by the asymptotic distribution-based tests is identified as one reason for adopting bootstrap-based tests. Several bootstrap-based unit root tests exist for ARMA processes with homoscedastic errors, but only one such test, proposed by Gospodinov and Tao (2011), is available for processes with conditional heteroskedastic errors. This test and its asymptotic properties were derived for autoregressive processes of order one, but the authors suggest that the results can be easily extended to processes of higher order. They do not, however, describe how such an extension may be carried out. For example, one may employ the type of model used in Augmented Dickey-Fuller (ADF) unit root test (Said and Dickey, 1984) or the version proposed by Phillips and Perron (1988). Moreover, the simulation results reported by the authors are limited to the first order autoregressive, AR(1), case. In this paper, we show in detail how the test proposed by Gospodinov and Tao (2011) can be extended to the general AR(p) case and obtain Monte-Carlo simulation results for higher order processes.

The most commonly used unit root tests were developed by Dickey and Fuller (1979) and are referred to as Dickey-Fuller (DF) tests. They were developed for the first order autoregressive processes. Said and Dickey (1984) generalized the Dickey-Fuller tests to be applicable to ARMA models of unknown orders. Phillips (1987) as well as Phillips and Perron (1988), provided a correction to the Dickey-Fuller tests to account for the presence of higher order terms. Leybourne and Newbold (1999), however, found that the Phillips-Perron unit root tests have serious size distortion and low power issues in finite samples, especially when the model has a moving average component.

All of the above mentioned unit root tests were developed under the assumption of homoscedastic errors. Several researchers, such as Ling and Li (1998, 2003), Ling et al. (2003), and Seo (1999), however, derived the asymptotic distributions of unit root tests for processes with GARCH errors and demonstrated power gains by incorporating the GARCH structure into the testing procedure. These are non-bootstrap unit root tests based on the asymptotic distribution of the test statistics. More recently, there has been some work on bootstrap-based unit root tests. Gospodinov (2008) derived bootstrap results when testing for nonlinearity in models with a unit root and GARCH errors. Subsequently, Gospodinov and Tao (2011) proposed a bootstrap method for approximating the finite-sample distributions of unit root tests with GARCH(1,1) errors. For the first-order autoregressive process with GARCH(1,1) errors, their published simulation results demonstrate the excellent size and power properties of the proposed bootstrap test. Up to now, however, no finite-sample results of bootstrap-based unit root tests relevant to autoregressive processes of higher orders with GARCH(1,1) errors have been published. This study aims to fill the important gap.

2. Model Formulation

Our method extends the test presented by Gospodinov and Tao (2011) to higher order autoregressive processes and the finite sample performance of the proposed test is studied using Monte-Carlo simulation.

Two formats of autoregressive models with order p are considered. Equation (1) is the classical format, and equation (3) follows the Augmented Dickey-Fuller model. The complete model formulation is:

$$y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \dots + \Phi_p y_{t-p} + \varepsilon_t, \quad t = 1, 2, \dots, T, \tag{1}$$

$$m^p - \Phi_1 m^{p-1} - \Phi_2 m^{p-2} - \dots - \Phi_{p-1} m - \Phi_p = 0, \tag{2}$$

$$\nabla y_t = r y_{t-1} + \delta_1 \nabla y_{t-1} + \delta_2 \nabla y_{t-2} + \cdots + \delta_{p-1} \nabla y_{t-p+1} + \varepsilon_t, \quad t = 1, 2, \dots, T, \tag{3}$$

$$\varepsilon_t = \sqrt{h_t}\,\eta_t, \quad \text{where } \eta_t \sim iid\, N(0,1), \quad t = 1, 2, \dots, T, \tag{4}$$

$$\text{and} \quad h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1},$$

$$\text{where } \omega > 0, \ \alpha \geq 0, \beta \geq 0, \alpha + \beta < 1, \ t = 1, 2, \dots, T. \tag{5}$$

Expression (2) is the characteristic equation of the autoregressive model given in Equation (1), which is equivalent to $\prod_{i=1}^{p}(m - r_i) = 0$, where r_i, $i = 1,2,\dots,p$, are the roots of AR(p) polynomial. We assume $|r_1| \leq 1, |r_i| < 1$, for $i \geq 2$. We also let $\rho = \binom{r}{\delta_0}$, with $\delta_0 = (\delta_1, \ \delta_2, \ \dots, \ \delta_{p-1})'$, and let $\delta = (\omega, \alpha, \beta)'$. The hypothesis test we perform is $H_0: r = 0 \ vs. \ H_1: r < 0$. And the test statistic we use is

$$t(\hat{r}) = \left(-\sum_{t=1}^{T} \frac{\partial^2 l_t(\rho, \delta)}{\partial r^2}\right)^{\frac{1}{2}}_{\rho=\hat{\rho}, \ \delta=\hat{\delta}} (\hat{r} - 0), \ \text{where}$$

$$l_t(\rho, \delta) = l_t(r, \delta_0, \omega, \alpha, \beta) = -\frac{1}{2} \ln h_t - \frac{1}{2} \frac{\varepsilon_t^2}{h_t}, \ t = 1,2,\dots,T,$$

with $\hat{\rho}$ and $\hat{\delta}$ representing the maximum likelihood estimates of ρ and δ respectively.

3. Proposed Bootstrap Method

For brevity, the main steps of performing a bootstrap-based unit root test on AR(p) models with GARCH(1,1) errors are listed below for the case p=2. It is easy to see how this procedure can be extended to the general AR(p) case. The steps of the procedure are as follows:

i) Use the least-squares estimates of $\rho = \begin{pmatrix} r \\ \delta_1 \end{pmatrix}$ as initial values and maximum likelihood estimation (MLE) to obtain the estimates of both ρ and δ, and record them as $\hat{\rho}$, $\hat{\delta}$, where $\hat{\rho} = \begin{pmatrix} \hat{r} \\ \hat{\delta}_1 \end{pmatrix}$, $\hat{\delta} = (\widehat{\omega}, \hat{\alpha}, \hat{\beta})'$. Note that when p=2, $\delta_0 = \delta_1$.
ii) Compute the test statistic,

$$t(\hat{r}) = \left(-\sum_{t=1}^{T} \frac{\partial^2 l_t(\rho, \delta)}{\partial r^2} \right)^{\frac{1}{2}}_{\rho=\hat{\rho}\ ,\ \delta=\hat{\delta}} (\hat{r} - 0)\ ,\ where$$

$$l_t(\rho,\ \delta) = l_t(r, \delta_1, \omega, \alpha, \beta) = -\frac{1}{2}\ ln\, h_t - \frac{1}{2}\ \frac{\varepsilon_t^2}{h_t}\ ,\ t = 1,2, \ldots, T.$$

iii) Compute $\hat{\varepsilon}_t = \nabla y_t - \hat{r} y_{t-1} - \hat{\delta}_1 \nabla y_{t-1}$, for $t = 1,\ 2,\ \ldots, T$.
iv) Compute $\hat{h}_1 = \widehat{\omega} + \hat{\alpha}\hat{\varepsilon}_0^2 + \hat{\beta}\,\hat{h}_0$, where $\hat{\varepsilon}_0^2 = \hat{h}_0 = \frac{1}{T}\sum_{i=1}^{T} \hat{\varepsilon}_i^2$;

$$\hat{h}_t = \widehat{\omega} + \hat{\alpha}\hat{\varepsilon}_{t-1}^2 + \hat{\beta}\,\hat{h}_{t-1}\ ,\ t = 1,\ 2,\ \ldots, T.$$

v) Let $\hat{\eta}_t = \frac{\hat{\varepsilon}_t}{\sqrt{\hat{h}_t}}$, and $\widetilde{\eta}_t$ be centered $\hat{\eta}_t$, for $t = 1,\ 2,\ \ldots, T$.
vi) Resample η_t^*, $t = 1,\ 2, \ldots, 2T$, from $\{\pm\widetilde{\eta}_t\}_{t=1}^{T}$. Note that $\{\pm\widetilde{\eta}_t\}_{t=1}^{T}$ contain both the $\widetilde{\eta}_t$ and the values $\widetilde{\eta}_t$ multiplied by -1. This ensures the symmetry of the underlying distribution that will be resampled.
vii) Compute $h_t^* = \widehat{\omega} + (\widehat{\alpha}\eta_{t-1}^{*2} + \widehat{\beta})h_{t-1}^*$, and let $h_0^* = \hat{\varepsilon}_0^2$ or $\hat{h}_0$, for $t = 1, 2,\ \ldots,\ 2T$.
viii) Compute $\nabla y_t^* = 0 * y_{t-1}^* + \hat{\delta}_1 \nabla y_{t-1}^* + \sqrt{h_t^*}\eta_t^*$, $t = 1,\ 2,\ \ldots,\ 2T$. That is, under $H_0{:}\ r = 0$, we have

$$y_t^* - y_{t-1}^* = \hat{\delta}_1(y_{t-1}^* - y_{t-2}^*) + \sqrt{h_t^*}\eta_t^*\ ,$$
$$y_t^* = (1 + \hat{\delta}_1)y_{t-1}^* + (-\hat{\delta}_1)y_{t-2}^* + \sqrt{h_t^*}\eta_t^*\ ,\quad \text{where}$$
$$t = 2,\ 3,\ \ldots, 2T,\ \text{and}\ y_0^* = y_1^* = 0\ .$$

ix) Drop the first T values of y_t^*. Fit ∇y_t^* against y_{t-1}^* and ∇y_{t-1}^* and estimate r^* and δ_1^* using least squares.
x) Use the least-squares estimates as initial values and obtain MLEs of $\rho^* = \begin{pmatrix} r^* \\ \delta_1^* \end{pmatrix}$, $\delta^* = (\omega^*, \alpha^*, \beta^*)'$, and denote these estimates as $\hat{\rho}^*$ and $\hat{\delta}^*$.
xi) Compute the bootstrap test statistic,

$$t^*(\hat{r}^*) = \left(-\sum_{t=1}^{T} \frac{\partial^2 l_t^*(\rho^*, \delta^*)}{\partial r^{*2}} \right)^{\frac{1}{2}}_{\rho^*=\hat{\rho}^*,\ \delta^*=\hat{\delta}^*} (\hat{r}^* - 0)\,,$$

$$\text{where } l_t^*(\rho^*,\ \delta^*) = l_t^*(r^*, \delta_1^*, \omega^*, \alpha^*, \beta^*) = -\frac{1}{2}\ln h_t^* - \frac{1}{2}\frac{\varepsilon_t^{*2}}{h_t^*}\ , \quad t = 1,2,\dots,T.$$

xii) Repeat Step vi) ~ xi) B times, say $B = 1{,}000$, and calculate the lower 5^{th} percentile of $t^*(\hat{r}^*)$, $t^*_{0.05}$, then compare $t^*_{0.05}$ with $t(\hat{r})$. If $t(\hat{r}) < t^*_{0.05}$, reject H_0 and let $\text{rej} = 1$; otherwise, do not reject and let $\text{rej} = 0$.
xiii) Repeat Step i) ~ xii) M times, say $M = 1{,}000$, and calculate the significance level (empirical size) or the power of the test as: $\text{sig or power} = \frac{\sum \text{rej}}{M}$.

4. Simulation Results

To carry out the simulations, we use expression (1) together with (4) and (5) to generate the raw time series $\{y_t\}_{t=1}^{2T}$, and then throw away the first T values of the series. Fit model (3) to the remaining series of length T and calculate the least-squares estimates of r and other coefficients. The same goes for Step ix) under Proposed Bootstrap Method.

MATLAB was used to perform Monte-Carlo simulations and bootstrap procedures. The simulation results on AR(2) models with GARCH(1,1) errors show that the proposed tests have reasonable power and size properties.

The simulation results for $n = 200$ are given in Table 4.1 ~ Table 4.10. We did the same amount of simulations for both $n = 200$ and $n = 400$ cases and obtained reasonable results. Due to the page limit, the results for the case with $n = 400$ will not be included, but will be available upon request.

For the simulation we considered AR(2) models with roots $r_1 \in \{0.5, 0.9, 1.0\}$, $r_2 \in \{0.2, 0.5, 0.9\}$. The (α, β) combinations considered are (0,0), (0.5,0.4), (0.25,0.7), (0.399,0.6), (0.199,0.8), (0.7,0.25), (0.6,0.399), (0.8,0.199), (0.2,0.4) and (0.4,0.2). To save space, not all combinations are reported. Simulation results show that the size of the tests ranges from 0.04 to 0.058. The latter case occurs when the second root, r_2, is close to the unity (Table 4.3). In fact, a size above 0.05 is obtained whenever $\alpha + \beta$ is close to one and $r_2 = 0.9$, except in one case (Table 4.7) when the size equals 0.05.

The power of the test increases with decreases in r_1 and r_2. For example, in Table 4.2, one sees that the power is 0.915 when $r_1 = r_2 = 0.9$ but increases to 0.996 when $r_1 = 0.9$ but $r_2 = 0.2$. The power is practically one when $r_1 = r_2 = 0.5$ or lower. A more interesting result can be observed by comparing the power when $r_1 = r_2 = 0.9$ under the non-heteroskedastic case (Table 4.1) to the power under $\alpha = 0.5, \beta = 0.4$. Under homoscedastic errors the power is 0.757, which climbs to 0.915 under heteroskedasticity. A similar phenomenon is also observed when comparing results in Table 4.4 to those in Table 4.9. Table 4.9 looks at the case where $\alpha = 0.2, \beta = 0.4$ (so $\alpha + \beta = 0.6$) in contrast to Table 4.4 where $\alpha = 0.399,\ \beta = 0.6$ (so $\alpha + \beta =$

0.999). Increasing $\alpha + \beta$ seems to increase the power, especially for the case with $r_1 = r_2 = 0.9$. The power for this case given in Table 4.9 is 0.802 whereas the power reported for this case in Table 4.4 is 0.921. Overall, the proposed method seems to work well for all cases, maintaining its size and producing good power.

Table 4.1: Monte-Carlo Simulation Results for the case T=200, α=0 and β=0

Nominal level = 0.05
Number of Monte-Carlo simulations M = 1,000
Number of bootstraps within each simulation B = 1,000

$T = 200$, $\alpha = 0$, $\beta = 0$	r_1	r_2	**Size**	**Power**
	1	0.2	0.04	
	1	0.5	0.04	
	1	0.9	0.044	
	0.9	0.2		0.993
	0.9	0.5		0.988
	0.9	0.9		0.757
	0.5	0.5		1
	0.5	0.2		1

Table 4.2: Monte-Carlo Simulation Results for the case T=200, α=0.5 and β=0.4

Nominal level = 0.05
Number of Monte-Carlo simulations M = 1,000
Number of bootstraps within each simulation B = 1,000

$T = 200$, $\alpha = 0.5$, $\beta = 0.4$	r_1	r_2	**size**	**power**
	1	0.2	0.041	
	1	0.5	0.04	
	1	0.9	0.053	
	0.9	0.2		0.996
	0.9	0.5		0.992
	0.9	0.9		0.915
	0.5	0.5		1
	0.5	0.2		1

Table 4.3: Monte-Carlo Simulation Results for the case T=200, α=0.25 and β=0.7

Nominal level = 0.05
Number of Monte-Carlo simulations M = 1,000
Number of bootstraps within each simulation B = 1,000

$T = 200$, $\alpha = 0.25$, $\beta = 0.7$	r_1	r_2	size	power
	1	0.2	0.05	
	1	0.5	0.054	
	1	0.9	0.058	
	0.9	0.2		0.99
	0.9	0.5		0.987
	0.9	0.9		0.853
	0.5	0.5		1
	0.5	0.2		1

Table 4.4: Monte-Carlo Simulation Results for the case T=200, α=0.399 and β=0.6

Nominal level = 0.05
Number of Monte-Carlo simulations M = 1,000
Number of bootstraps within each simulation B = 1,000

$T = 200$, $\alpha = 0.399$, $\beta = 0.6$	r_1	r_2	size	power
	1	0.2	0.045	
	1	0.5	0.047	
	1	0.9	0.055	
	0.9	0.2		0.995
	0.9	0.5		0.992
	0.9	0.9		0.921
	0.5	0.5		1
	0.5	0.2		1

Table 4.5: Monte-Carlo Simulation Results for the case T=200, α=0.199 and β=0.8

Nominal level = 0.05 **Number of Monte-Carlo simulations M = 1,000** **Number of bootstraps within each simulation B = 1,000**				
$T = 200$ $\alpha = 0.199$ $\beta = 0.8$	r_1	r_2	**size**	**power**
	1	0.2	0.045	
	1	0.5	0.048	
	1	0.9	0.055	
	0.9	0.2		0.989
	0.9	0.5		0.983
	0.9	0.9		0.827
	0.5	0.5		1
	0.5	0.2		1

Table 4.6: Monte-Carlo Simulation Results for the case T=200, α=0.7 and β=0.25

Nominal level = 0.05 **Number of Monte-Carlo simulations M = 1,000** **Number of bootstraps within each simulation B = 1,000**				
$T = 200$ $\alpha = 0.7$ $\beta = 0.25$	r_1	r_2	**size**	**power**
	1	0.2	0.051	
	1	0.5	0.044	
	1	0.9	0.056	
	0.9	0.2		0.997
	0.9	0.5		0.997
	0.9	0.9		0.955
	0.5	0.5		1
	0.5	0.2		1

Table 4.7: Monte-Carlo Simulation Results for the case T=200, α=0.6 and β=0.399

Nominal level = 0.05
Number of Monte-Carlo simulations M = 1,000
Number of bootstraps within each simulation B = 1,000

$T = 200$, $\alpha = 0.6$, $\beta = 0.399$	r_1	r_2	size	power
	1	0.2	0.054	
	1	0.5	0.045	
	1	0.9	0.05	
	0.9	0.2		0.999
	0.9	0.5		0.999
	0.9	0.9		0.955
	0.5	0.5		1
	0.5	0.2		0.999

Table 4.8: Monte-Carlo Simulation Results for the case T=200, α=0.8 and β=0.199

Nominal level = 0.05
Number of Monte-Carlo simulations M = 1,000
Number of bootstraps within each simulation B = 1,000

$T = 200$, $\alpha = 0.8$, $\beta = 0.199$	r_1	r_2	size	power
	1	0.2	0.045	
	1	0.5	0.043	
	1	0.9	0.05	
	0.9	0.2		0.998
	0.9	0.5		0.999
	0.9	0.9		0.971
	0.5	0.5		1
	0.5	0.2		1

Table 4.9: Monte-Carlo Simulation Results for the case T=200, α=0.2 and β=0.4

Nominal level = 0.05
Number of Monte-Carlo simulations M = 1,000
Number of bootstraps within each simulation B = 1,000

$T = 200$, $\alpha = 0.2$, $\beta = 0.4$	r_1	r_2	**size**	**power**
	1	0.2	0.044	
	1	0.5	0.04	
	1	0.9	0.04	
	0.9	0.2		0.993
	0.9	0.5		0.989
	0.9	0.9		0.802
	0.5	0.5		1
	0.5	0.2		1

Table 4.10: Monte-Carlo Simulation Results for the case T=200, α=0.4 and β=0.2

Nominal level = 0.05
Number of Monte-Carlo simulations M = 1,000
Number of bootstraps within each simulation B = 1,000

$T = 200$, $\alpha = 0.4$, $\beta = 0.2$	r_1	r_2	**size**	**power**
	1	0.2	0.042	
	1	0.5	0.043	
	1	0.9	0.041	
	0.9	0.2		0.994
	0.9	0.5		0.991
	0.9	0.9		0.86
	0.5	0.5		1
	0.5	0.2		1

5. Conclusion and Future Work

An existing method for conducting bootstrap-based unit root tests in first order autoregressive models with GARCH errors was extended to the general autoregressive case. Simulation results indicate that the proposed method mitigates the size distortion issue significantly, and achieves high powers at different combinations of the autoregressive roots and GARCH coefficients. This method has been further extended to AR(p) models with EGARCH error structure and a simulation study is currently underway. A future task may be to extend the methodology to ARIMA processes with unknown order.

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