The joint distribution of discrete random sets and its conncetions to subcopulas

Zheng Wei* and Tonghui Wang*

Abstract
In this paper, the characterization of the joint distribution of random set vector by the belief function is investigated. A method for constructing the joint belief function of discrete bivariate random sets through copula is given. Conversely, subcopulas can be obtained from the bivariate belief functions. For illustration of main results, several examples are given.

Key Words: Bivariate random set; Copula; Subcopula; Joint belief function; Jointly monotone of infinite order.

1. Introduction

Random sets can be used to model imprecise observations of random variables where the outcomes are assigned as set valued instead of real valued. The theory of random sets is viewed as a natural generalization of multivariate statistical analysis. Random set data can also be viewed as imprecise or incomplete observations which are frequent in today's technological societies. The distribution of the univariate random set and its properties can be found in Nguyen [5], Nguyen and Wang [6] and Shafer[10]. Recently, the characterization of joint distributions of random sets on co-product spaces was discussed by Schmelzer[8], Nguyen[7] and Wei et al [13]. In this paper, this characterization is modified for discrete random set vector.

Copulas are used to model multivariate data as they account for the dependence structure and provide a flexible representation of the multivariate distribution, as seen in Nelson [4], Harry [3] and Wei et al. [12]. The notion of copula has been introduced by Sklar [11]. Copulas are multivariate distributions with $[0, 1]$-uniform marginal, which contain the most of the multivariate dependence structure properties and do not depend on the marginals. It is known that copulas connect with marginals to obtain possible joint distributions. In order to investigate the dependence relationship between two random sets, it is necessary to built a bridge for connecting the joint belief functions of random set vector and copulas. For references, see Schmelzer [9], Nguyen [7], Joe [3] and Nelsen [4]. In this paper, a method for constructing the joint distribution of the discrete bivariate random set vector through copula is given.

This paper is organized as follows. The characterization of the joint distribution of random set vector by its joint belief functions is obtained in Section 2. A method of connecting the joint belief function of random set vector with given marginals and copula(subcopula) is given in Section 3. To illustrate our main results, several examples are given.

2. Characterization of the joint belief function of discrete random set vector

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a probability space and let E_1 and E_2 be finite sets, where Ω is sample space, $\mathcal{A}$ is a σ-algebra on subsets of Ω and P is a probability measure. Recall that a finite random set $\mathcal{S}$ with values in powerset of a finite E is a map $\mathcal{S} : \Omega \to 2^E$ such that $\mathcal{S}^{-1}(\{A\}) = \{\omega \in \Omega : \mathcal{S}(\omega) = A\} \in \mathcal{A}$ for any $A \subseteq E$. Let $f : 2^E \to [0, 1]$ be

*Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88001

$f(A) = P(\mathcal{S} = A)$, then f is a probability density function of $\mathcal{S}$ on 2^E. In the following, we will extend this definition to the cases of the random set vector.

Definition 2.1 *A random set vector $(\mathcal{S}_1, \mathcal{S}_2)$ with values in $2^{E_1} \times 2^{E_2}$ is a map $(\mathcal{S}_1, \mathcal{S}_2) : \Omega \to 2^{E_1} \times 2^{E_2}$ such that $\{\omega \in \Omega : \mathcal{S}_1(\omega) = A, \mathcal{S}_2(\omega) = B\} \in \mathcal{A}$, for any $A \subseteq E_1$ and $B \subseteq E_2$. Let $h : 2^{E_1} \times 2^{E_2} \to [0, 1]$ be a* **joint probability density function** *of $(\mathcal{S}_1, \mathcal{S}_2)$, i.e., $h \geq 0$ and $\sum\limits_{A \subseteq E_1} \sum\limits_{B \subseteq E_2} h(A, B) = 1$, where $h(A, B) = P(\mathcal{S}_1(\omega) = A, \mathcal{S}_2(\omega) = B)$, $A \subseteq E_1$ and $B \subseteq E_2$.*

Inspired by the distribution of univariate random sets, we are going to define axiomatically the concept of joint distribution functions of the random set vector $(\mathcal{S}_1, \mathcal{S}_2)$.

Theorem 2.1 *Let $(\mathcal{S}_1, \mathcal{S}_2)$ be a (nonempty) random set vector on $2^{E_1} \times 2^{E_2}$, and $H : 2^{E_1} \times 2^{E_2} \to [0, 1]$ be*

$$H(A, B) = P(\mathcal{S}_1 \subseteq A, \mathcal{S}_2 \subseteq B) = \sum_{C \subseteq A} \sum_{D \subseteq B} h(C, D), \quad A \in 2^{E_1}, \quad B \in 2^{E_2}. \tag{1}$$

Then, H satisfies the following properties:

(i) $H(\emptyset, \emptyset) = H(\emptyset, B) = H(A, \emptyset) = 0$, and $H(E_1, E_2) = 1$;

(ii) H is **monotone of infinite order on each component***, i.e., for any B in 2^{E_2} and any distinct sets $A_1, A_2, \cdots, A_k$ in 2^{E_1}, $k \geq 1$,*

$$H\left(\bigcup_{i=1}^{k} A_i, B\right) \geq \sum_{\emptyset \neq I \subseteq \{1,2,\cdots,k\}} (-1)^{|I|+1} H\left(\bigcap_{i \in I} A_i, B\right), \tag{2}$$

and for any $A \in 2^{E_1}$ and any distinct sets $B_1, B_2, \cdots, B_\ell$ in 2^{E_2}, $\ell \geq 1$,

$$H\left(A, \bigcup_{j=1}^{\ell} B_j\right) \geq \sum_{\emptyset \neq J \subseteq \{1,2,\cdots,\ell\}} (-1)^{|J|+1} H\left(A, \bigcap_{j \in J} B_j\right); \tag{3}$$

and

(iii) $H(.,.)$ is **jointly monotone of infinite order***, i.e., for distinct sets $A_1, A_2, \cdots, A_k$ in 2^{E_1} and distinct $B_1, B_2, \cdots, B_\ell$ in 2^{E_2}, where k, ℓ are positive integers,*

$$\begin{aligned} H\left(\bigcup_{i=1}^{k} A_i, \bigcup_{j=1}^{\ell} B_j\right) \geq & \sum_{\emptyset \neq I \subseteq \{1,2,\cdots,k\}} (-1)^{|I|+1} H\left(\bigcap_{i \in I} A_i, \bigcup_{j=1}^{\ell} B_j\right) \\ & + \sum_{\emptyset \neq J \subseteq \{1,2,\cdots,\ell\}} (-1)^{|J|+1} H\left(\bigcup_{i=1}^{k} A_i, \bigcap_{j \in J} B_j\right) \\ & - \sum_{\emptyset \neq I \subseteq \{1,2,\cdots,k\}} \sum_{\emptyset \neq J \subseteq \{1,2,\cdots,\ell\}} (-1)^{|I|+|J|} H\left(\bigcap_{i \in I} A_i, \bigcap_{j \in J} B_j\right) \end{aligned} \tag{4}$$

Proof. The Property (i) is obvious. For Property (ii), it is sufficient to show that (2) holds for any fixed $B \in 2^{E_2}$. Indeed, we can treat $H(A, B)$ as a univariate belief function of random set $(\mathcal{S}_1, B)$ so that (2) holds. (3) can be proved similarly.

Now, for Property (iii), let$\mathcal{J}(C) = \{i = 1, 2, \cdots, k$ such that $C \subset A_i\}$ and $\mathcal{K}(D) = \{j = 1, 2, \cdots, \ell$ such that $D \subset B_j\}$. Then $C \subseteq \bigcap_{i \in \mathcal{J}(C)} A_i$ if $\mathcal{J}(C) \neq \emptyset$ and $D \subseteq \bigcap_{j \in \mathcal{K}(D)} B_j$ if $\mathcal{K}(D) \neq \emptyset$. Clearly,

$$
\begin{aligned}
H\left(\bigcup_{i=1}^{k} A_i, \bigcup_{j=1}^{\ell} B_j\right) &= \sum_{C \subseteq \bigcup_{i=1}^{k} A_i} \sum_{D \subseteq \bigcup_{j=1}^{\ell} B_j} h(C, D) \geq \sum_{\substack{C \subseteq E_1 \\ \mathcal{J}(C) \neq \emptyset}} \sum_{\substack{D \subseteq E_2 \\ \mathcal{K}(D) \neq \emptyset}} h(C, D) \\
&+ \sum_{\substack{C \subseteq \bigcup_{i=1}^{k} A_i \\ \mathcal{J}(C) = \emptyset}} \sum_{\substack{D \subseteq E_2 \\ \mathcal{K}(D) \neq \emptyset}} h(C, D) + \sum_{\substack{C \subseteq E_1 \\ \mathcal{J}(C) \neq \emptyset}} \sum_{\substack{D \subseteq \bigcup_{j=1}^{\ell} B_j \\ \mathcal{K}(D) = \emptyset}} h(C, D) \\
&\equiv \text{(I)} + \text{(II)} + \text{(III)} \\
&= -(I) + \{(II) + (I)\} + \{(III) + (I)\}.
\end{aligned}
$$

Note that for any nonempty finite sets A and B, the following identities always hold:

$$
\sum_{\emptyset \neq C \subseteq A} (-1)^{|C|+1} = 1 \quad \text{and} \quad \sum_{\emptyset \neq C \subseteq A} \sum_{\emptyset \neq D \subseteq B} (-1)^{|C|+|D|} = 1.
$$

By using these identities we can rewrite (I), (I)+(II) and (I)+(III) given above as follows,

$$
\begin{aligned}
\text{(I)} &= \sum_{\substack{C \subseteq E_1 \\ \mathcal{J}(C) \neq \emptyset}} \sum_{\substack{D \subseteq E_2 \\ \mathcal{K}(D) \neq \emptyset}} h(C, D) \\
&= \sum_{\substack{C \subseteq E_1 \\ \mathcal{J}(C) \neq \emptyset}} \sum_{\substack{D \subseteq E_2 \\ \mathcal{K}(D) \neq \emptyset}} \left[\sum_{\emptyset \neq I \subseteq \mathcal{J}(C)} \sum_{\emptyset \neq J \subseteq \mathcal{K}(D)} (-1)^{|I|+|J|}\right] h(C, D) \\
&= \sum_{\emptyset \neq I \subseteq \{1, \cdots, k\}} \sum_{\emptyset \neq J \subseteq \{1, \cdots \ell\}} (-1)^{|I|+|J|} \left[\sum_{\substack{C \subseteq E_1 \\ \mathcal{J}(C) \supseteq I}} \sum_{\substack{D \subseteq E_2 \\ \mathcal{K}(D) \supseteq J}} h(C, D)\right] \\
&= \sum_{\emptyset \neq I \subseteq \{1, \cdots, k\}} \sum_{\emptyset \neq J \subseteq \{1, \cdots \ell\}} (-1)^{|I|+|J|} H\left(\bigcap_{i \in I} A_i, \bigcap_{j \in J} B_j\right).
\end{aligned}
$$

Similarly, we obtain

$$
\text{(II)} + \text{(I)} = \sum_{\emptyset \neq J \subseteq \{1, \cdots \ell\}} (-1)^{|J|+1} H\left(\bigcup_{i=1}^{k} A_i, \bigcap_{j \in J} B_j\right),
$$

and

$$
\text{(III)} + \text{(I)} = \sum_{\emptyset \neq I \subseteq \{1, 2, \cdots, k\}} (-1)^{|I|+1} H\left(\bigcap_{i \in I} A_i, \bigcup_{j=1}^{\ell} B_j\right).
$$

Therefore the Property (iii) holds. $\square$

It turns out that the properties (i), (ii) and (iii) of H in the Theorem 2.1 characterize the joint distribution function of a (nonempty) random set vector.

Definition 2.2 *A set function $H : 2^{E_1} \times 2^{E_2} \to [0,1]$ satisfying the properties (i), (ii) and (iii) in the Theorem 2.1 is said to be the* **joint belief function** *of random set vector $(\mathcal{S}_1, \mathcal{S}_2)$.*

The following result shows that for any given joint belief function H of $(\mathcal{S}_1, \mathcal{S}_2)$, there exists a probability density function $h : 2^{E_1} \times 2^{E_2} \to [0,1]$ corresponding to H.

Theorem 2.2 *If $H : 2^{E_1} \times 2^{E_2} \to [0,1]$ is such that*

(i) $H(\emptyset,\emptyset) = H(\emptyset, B) = H(A, \emptyset) = 0$*, and* $H(E_1, E_2) = 1$*,*

(ii) H *is monotone of infinite order on each component, and*

(iii) H *is joint monotone of infinite order. then for any $(A,B) \in 2^{E_1} \times 2^{E_2}$, there exists a nonnegative set function $h : 2^{E_1} \times 2^{E_2} \to [0,1]$, called the* **Möbius inverse** *of H, such that*

$$H(A,B) = \sum_{C \subseteq A} \sum_{D \subseteq B} h(C,D) \tag{5}$$

and

$$\sum_{C \subseteq E_1} \sum_{D \subseteq E_2} h(C,D) = 1. \tag{6}$$

Proof. Let $h : 2^{E_1} \times 2^{E_2} \to [0,1]$ be defined by

$$h(A,B) = \sum_{C \subseteq A} \sum_{D \subseteq B} (-1)^{|A \setminus C| + |B \setminus D|} H(C,D), \tag{7}$$

where $A \setminus C = A \cap C^c$ and C^c is the complement of C. First we need to show h is nonnegative.

From (i), it is easy to see $h(\emptyset,\emptyset) = h(A,\emptyset) = h(\emptyset, B) = 0$, where $A \subseteq E_1, B \subseteq E_2$. Also, it is obvious $h(\{x\},\{y\}) = H(\{x\},\{y\}) \geq 0$, for any $x \in E_1, y \in E_2$.

For any $A \subseteq E_1$, $y \in E_2$ with $|A| \geq 2$, we assume that $A = \{x_1, x_2, \cdots, x_k\}$. Let $A_i = A \setminus \{x_i\}$, $i = 1, 2, \cdots, k$. Then from (7), we have

$$\begin{aligned}
h(A,\{y\}) &= H(A,\{y\}) - \sum_{i=1}^{k} H(A_i,\{y\}) + \sum_{i_1 < i_2} H(A_{i_1} \cap A_{i_2}, \{y\}) + \cdots \\
&\quad + (-1)^{k-1} \sum_{i_1 < \cdots < i_{k-1}} H\left(\bigcap_{j=1}^{k-1} A_{i_j}, \{y\}\right) \\
&= H(A,\{y\}) - \sum_{\emptyset \neq I \subseteq \{1,2,\cdots,k\}} (-1)^{|I|+1} H\left(\bigcap_{i \in I} A_i, \{y\}\right),
\end{aligned}$$

by Property (ii), $h(A,\{y\}) \geq 0$. Similarly, we obtain $h(\{x\}, B) \geq 0$ for any $x \in E_1$, $B \subseteq E_2$ with $|B| \geq 2$. Finally, for any $A \subseteq E_1$ and $B \subseteq E_2$ with $|A| \geq 2$ and $|B| \geq 2$, say $A = \{x_1, x_2, \cdots, x_k\}$ and $B = \{y_1, y_2, \cdots, y_\ell\}$. Let $A_i = A \setminus \{x_i\}$, $i = 1, 2, \cdots, k$ and $B_j = B \setminus \{y_j\}$, $j = 1, 2, \cdots, \ell$. Then,

$$\begin{aligned}
h(A,B) &= H(A,B) - \sum_{\emptyset \neq J \subseteq \{1,2,\cdots,\ell\}} (-1)^{|J|+1} H\left(A, \bigcap_{j \in J} B_j\right) \\
&\quad - \sum_{\emptyset \neq I \subseteq \{1,2,\cdots,k\}} (-1)^{|I|+1} H\left(\bigcap_{i \in I} A_i, B\right) \\
&\quad + \sum_{\emptyset \neq I \subseteq \{1,2,\cdots,k\}} \sum_{\emptyset \neq J \subseteq \{1,2,\cdots,\ell\}} (-1)^{|I|+|J|} H\left(\bigcap_{i \in I} A_i, \bigcap_{J \in J} B_j\right)
\end{aligned}$$

therefore, by Property (iii), $h(A,B) \geq 0$.

Next, we need to show $\sum_{C\subseteq E_1}\sum_{D\subseteq E_2} h(C,D)=1$. Note that

$$\begin{aligned}\sum_{C\subseteq A}\sum_{D\subseteq B} h(C,D) &= \sum_{C\subseteq A}\sum_{D\subseteq B}[\sum_{E\subseteq C}\sum_{F\subseteq D}(-1)^{|C\setminus E|+|D\setminus F|}H(E,F)] \\ &= \sum_{E\subseteq C\subseteq A}\sum_{F\subseteq D\subseteq B}(-1)^{|C\setminus E|+|D\setminus F|}H(E,F).\end{aligned}$$

If $E=A$ and $F=B$, the last expression is $H(A,B)$. If $E\neq A$ or $F\neq B$, then $A\setminus E$ has $2^{|A\setminus E|}$ subsets and $B\setminus F$ has $2^{|B\setminus F|}$ subsets, so there are even number pair of subsets (C,D) such that $E\subseteq C\subseteq A$ and $F\subseteq D\subseteq B$, exactly half of which will make $(-1)^{|C\setminus E|+|D\setminus F|}$ to be 1 and half are -1. Thus

$$\sum_{E\subseteq C\subseteq A}\sum_{F\subseteq D\subseteq B}(-1)^{|C\setminus E|+|D\setminus F|}H(E,F)=0, \quad \text{for each } E\neq A \quad \text{or } F\neq B.$$

Therefore, $\sum_{C\subseteq A}\sum_{D\subseteq B} h(C,D)=H(A,B)$.

In particular, $1=H(E_1,E_2)=\sum_{C\subseteq E_1}\sum_{D\subseteq E_2} h(C,D)$, so that h is a joint probability density on $2^{E_1}\times 2^{E_2}$. $\square$

The explanations of Theorem 2.1 and Theorem 2.2 are given below.

Remark 2.1 (a). Consider the set function $F_1(A)=H(A,E_2), A\in 2^{E_1}$. It is easy to show that $F_1(A)$ is a belief function of random set $\mathcal{S}_1$ over E_1, which is called the **marginal belief function** of random set $\mathcal{S}_1$. Similarly, $F_2(B)=H(E_1,B), B\in 2^{E_2}$ is the marginal belief function of random set $\mathcal{S}_2$ over E_2. More details on belief functions of univariate random sets are given in Nguyen[5].

(b). For any given $B\subseteq E_2$, let $f_2(B)$ be the Möbius inverse of $F_2(B)$. Then

$$P(\mathcal{S}_1\subseteq A, \mathcal{S}_2=B)=\sum_{C\subseteq A} h(C|B)f_2(B)=H_{\mathcal{S}_1|\mathcal{S}_2}(A|B)f_2(B),$$

where $h(C|B)=P(\mathcal{S}_1=C|\mathcal{S}_2=B)$ is the conditional probability of $\mathcal{S}_1=C$ given $\mathcal{S}_2=B$. We call $H_{\mathcal{S}_1|\mathcal{S}_2}(A|B)$ be the **conditional belief function** of $\mathcal{S}_1$ given $\mathcal{S}_2=B$. Similarly, we can obtain $H_{\mathcal{S}_2|\mathcal{S}_1}(B|A)$ the **conditional belief function** of $\mathcal{S}_2$ given $\mathcal{S}_1=A$. For a given joint belief function $H(A,B)$ of random set vector $(\mathcal{S}_1,\mathcal{S}_2)$, we say $\mathcal{S}_1$ and $\mathcal{S}_2$ are **independent** if and only if $H(A,B)=F_1(A)F_2(B)$, for all $A\in 2^{E_1}$ and $B\in 2^{E_2}$.

(c) In Theorem 2.1, if $B_1=B_2=\cdots=B_\ell=B$, the Property (iii) is reduced to an equality, so Property (ii) is needed for characterizing the marginal belief functions of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively.

(d) In view of the direct product $(2^{E_1}\times 2^{E_2},\leq)$ of two locally finite posets $(2^{E_1},\subseteq)$ and $(2^{E_1},\subseteq)$, where $(C,D)\leq(A,B)$ means $C\subseteq A$ and $D\subseteq B$, with its Möbius function

$$\mu:(2^{E_1}\times 2^{E_2})\times(2^{E_1}\times 2^{E_2})\to\mathbb{Z} \quad \text{with } \mu((C,D),(A,B))=(-1)^{|A\setminus C|+|B\setminus D|},$$

we have

$$H(A,B)=\sum_{(C,D)\leq(A,B)} h(C,D),$$

where $h(A,B)$ is the **Möbius inverse** of H,

$$h(A,B)=\sum_{(C,D)\leq(A,B)}(-1)^{|A\setminus C|+|B\setminus D|}H(C,D). \tag{8}$$

$h(.,.)$ is also called the probability assignment of random set vector $(\mathcal{S}_1, \mathcal{S}_2)$. Note that there is a bijection between the joint belief function H and its the corresponding joint density h (See e.g. Nguyen [7]).

Remark 2.2 Similar to property of Theorem 2.2, there is a property called completely monotone in each component, given by Schmelzer [8, 9] and Nguyen [7] as follows.

A set function $H_1 : 2^{E_1} \times 2^{E_2} \to [0,1]$ is said to be **completely monotone in each component**, if for any $k \geq 2$ and $(A_i, B_i) \in 2^{E_1} \times 2^{E_2}$, $i = 1, 2 \cdots, k$,

$$H_1\left(\bigcup_{i=1}^{k} A_i, \bigcup_{i=1}^{k} B_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,2,\cdots,k\}} (-1)^{|I|+1} H_1\left(\bigcap_{i \in I} A_i, \bigcap_{i \in I} B_i\right). \tag{9}$$

The difference between (2)-(4) and (9) is that (A_i, B_j)'s in (2)-(4) are distinct sets while (A_i, B_i)'s in (9) are not necessary distinct sets and can be duplicated many times if needed. In the following, we will show that (9) is equivalent to (2)-(4).

Proposition 2.1 *If $H : 2^{E_1} \times 2^{E_2} \to [0,1]$ is such that $H(\emptyset, \emptyset) = H(\emptyset, B) = H(A, \emptyset) = 0$, and $H(E_1, E_2) = 1$, then H is completely monotone in each component given (9) if and only if H is monotone of infinite order on each component given in (2), and (3) and H is joint monotone of infinite order given in (4).*

Proof. For "only if" part, assume that (9) holds. Let $A_1, A_2 \cdots, A_k \in 2^{E_1}$ be distinct, $B \in 2^{E_2}$, if we set $B_1 = B_2 = \cdots = B_k = B$, then (9) is reduced to

$$\begin{aligned} H\left(\bigcup_{i=1}^{k} A_i, B\right) &= H\left(\bigcup_{i=1}^{k} A_i, \bigcup_{i=1}^{k} B_i\right) \\ &\geq \sum_{\emptyset \neq I \subseteq \{1,2,\cdots,k\}} (-1)^{|I|+1} H_1\left(\bigcap_{i \in I} A_i, \bigcap_{i \in I} B_i\right) \\ &= \sum_{\emptyset \neq I \subseteq \{1,2,\cdots,k\}} (-1)^{|I|+1} H_1\left(\bigcap_{i \in I} A_i, B\right), \end{aligned}$$

so that (2) holds. Simiarly, if $A \in 2^{E_1}$, we set $A_1 = A_2 = \cdots = A_k = A$, and $B_1, B_2 \cdots, B_\ell \in 2^{E_2}$ be distinct, then (9) implies (3).

Now let $A_1, \cdots, A_k \in 2^{E_1}$ and $B_1, \cdots, B_\ell \in 2^{E_2}$ be distinct. Define sets C_t and D_t, $1 \leq t \leq k + \ell$ by

$$C_t = \begin{cases} A_t & \text{if } 1 \leq t \leq k \\ \bigcup\limits_{i=1}^{k} A_i & \text{if } k+1 \leq t \leq k+\ell, \end{cases} \qquad D_t = \begin{cases} \bigcup\limits_{j=1}^{\ell} B_j & \text{if } 1 \leq t \leq k \\ B_{t-k} & \text{if } k+1 \leq t \leq k+\ell. \end{cases}$$

Then, (9) can be written to:

$$
\begin{aligned}
& H\left(\bigcup_{i=1}^{k} A_i, \bigcup_{j=1}^{\ell} B_j\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\cdots,k+\ell\}} (-1)^{|I|+1} H\left(\bigcap_{t\in I} C_t, \bigcap_{t\in I} D_t\right) \\
= & \sum_{\emptyset \neq I \subseteq \{1,\cdots,k\}} (-1)^{|I|+1} H\left(\bigcap_{t\in I} C_t, \bigcap_{t\in I} D_t\right) \\
& + \sum_{\emptyset \neq I \subseteq \{k+1,\cdots,k+\ell\}} (-1)^{|I|+1} H\left(\bigcap_{t\in I} C_t, \bigcap_{t\in I} D_t\right) \\
& + \sum_{\substack{I\cap\{1,\cdots,k\}\neq\emptyset \\ I\cap\{k+1,\cdots,k+\ell\}\neq\emptyset}} (-1)^{|I|+1} H\left(\bigcap_{t\in I} C_t, \bigcap_{t\in I} D_t\right) \\
= & \sum_{\emptyset \neq I \subseteq \{1,\cdots,k\}} (-1)^{|I|+1} H\left(\bigcap_{i\in I} A_i, \bigcup_{j=1}^{\ell} B_j\right) \\
& + \sum_{\emptyset \neq J \subseteq \{1,\cdots,\ell\}} (-1)^{|J|+1} H\left(\bigcup_{i=1}^{k} A_i, \bigcap_{j\in J} D_j\right) \\
& - \sum_{\emptyset \neq I \subseteq \{1,\cdots,k\}} \sum_{\emptyset \neq J \subseteq \{1,\cdots,\ell\}} (-1)^{|I|+|J|} H\left(\bigcap_{i\in I} A_i, \bigcap_{j\in J} B_j\right),
\end{aligned}
$$

So that (4) holds.

For "if" part, assume (2)-(4) holds. By Theorem 2.2, there exists a nonnegative set function $h : 2^{E_1} \times 2^{E_2} \to [0,1]$, such that

$$
H(A,B) = \sum_{C\subseteq A} \sum_{D\subseteq B} h(C,D), \quad \text{and} \quad \sum_{C\subseteq E_1} \sum_{D\subseteq E_2} h(C,D) = 1.
$$

Now, for any $k \geq 2$ and $(A_i, B_i) \in 2^{E_1} \times 2^{E_2}$, $i = 1, 2, \cdots, k$. For any $C \subseteq E_1$ and $D \subseteq E_2$, let $\mathcal{J}(C,D) = \{i = 1, 2, \cdots, k \text{ such that } C \subseteq A_i, \text{and } D \subseteq B_i\}$. Then $C \subseteq \bigcap_{i\in\mathcal{J}(C,D)} A_i$ and $D \subseteq \bigcap_{i\in\mathcal{J}(C,D)} B_i$ if $\mathcal{J}(C,D) \neq \emptyset$. Clearly,

$$
\begin{aligned}
& H\left(\bigcup_{i=1}^{k} A_i, \bigcup_{i=1}^{k} B_i\right) = \sum_{C\subseteq \bigcup_{i=1}^{k} A_i} \sum_{D\subseteq \bigcup_{i=1}^{k} B_i} h(C,D) \geq \sum_{\substack{C,D \\ \mathcal{J}(C,D)\neq\emptyset}} h(C,D) \\
= & \sum_{\substack{C,D \\ \mathcal{J}(C,D)\neq\emptyset}} \left[\sum_{\emptyset\neq I\subseteq\mathcal{J}(C,D)} (-1)^{|I|+1}\right] h(C,D) \\
= & \sum_{\emptyset \neq I \subseteq \{1,\cdots,k\}} (-1)^{|I|+1} \left[\sum_{\substack{C,D \\ \mathcal{J}(C,D)\supseteq I}} h(C,D)\right] \\
= & \sum_{\emptyset \neq I \subseteq \{1,\cdots,k\}} (-1)^{|I|+1} H\left(\bigcap_{i\in I} A_i, \bigcap_{i\in I} B_i\right).
\end{aligned}
$$

And hence (9) holds. □

Given a set function $H : 2^{E_1} \times 2^{E_2} \to [0,1]$, it is nature to ask whether if it is a well-defined joint belief function. By the conditions shown in Theorem 2.1, we only need to check all distinct sets $A_1, \cdots, A_k$ and $B_1, \cdots, B_\ell$.

3. Connections between joint belief functions and subcopulas

From the joint belief function of random set vector$(\mathcal{S}_1, \mathcal{S}_2)$, it is not easy to tell the dependence relationship between $\mathcal{S}_1$ and $\mathcal{S}_2$. Copula is a useful tool for modeling dependence of random variables as they account for the dependence structure and provide a flexible representation, see Nelson[4], Sklar[11], Hung[7], and Wei et al.[12]. Copulas connect marginals to obtain possible joint distributions. Therefore, it is necessary to built a bridge for connecting the joint belief functions of random set vector and copulas.

Definition 3.1 (Sklar 1959)[11] *A* **copula** $\mathcal{C}$ *is a function* $\mathcal{C}(.,.) : [0,1]^2 \to [0,1]$ *satisfying:*
(i) $\mathcal{C}(u,0) = \mathcal{C}(0,v) = 0$*, for* $u, v \in [0,1]$,
(ii) $\mathcal{C}(u,1) = u$, $\mathcal{C}(1,v) = v$*, for* $u, v \in [0,1]$, *and*
(iii) for any $(u_1, v_1) \le (u_2, v_2)$, $\mathcal{C}(u_2, v_2) - \mathcal{C}(u_1, v_2) - \mathcal{C}(u_2, v_1) + \mathcal{C}(u_1, v_1) \ge 0.$

Let H be the joint distribution function of a random vector (X, Y) with marginals F and G, then there exists a copula $\mathcal{C}$ such that $H(x,y) = \mathcal{C}(F(x), G(y))$. Furthermore, if the marginals are continuous, then the copula $\mathcal{C}$ is unique. In order to investigate both discrete and continuous distributions, we consider a slightly more general concept, namely subcopula. A bivariate **subcopula** is a function $\mathcal{C}'(.,.) : I_1 \times I_2 \to [0,1]$ where $I_1, I_2 \subseteq [0,1]$ containing 0 and 1, such that (i), (ii) and (iii) in Definition 3.1 are satisfied on its domain. An initial approach of using copulas for random sets was discussed by Alvarez [1].

In this section, we are going to investigate some connections between joint belief functions of discrete random set vector and copulas(subcopulas).

3.1 An algorithm for constructing joint belief functions through copulas

Given two univariate belief functions $F_1(A) = P(S_1 \subseteq A), A \in 2^{E_1}$ and $F_2(B) = P(S_2 \subseteq B), B \in 2^{E_2}$ on finite domains E_1 and E_2 , respectively, what are all possible joint belief functions, $H(A,B)$, with these given marginals? In the following, we will introduce a method for constructing joint belief functions from given marginal belief functions and copula.

Let $f_1(A) = P(\mathcal{S}_1 = A)$ and $f_2(B) = P(\mathcal{S}_2 = B)$ be the densities of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Since E_i's are finite sets, we can order the elements of $\mathcal{F}(f_1) = \{A \subseteq E_1 : f_1(A) > 0\}$, by the Lexicographical order(also known as lexical order, dictionary order) as

$$\mathcal{F}(f_1) = \{A_1, A_2, \cdots, A_m\}.$$

Similarly, we can obtain

$$\mathcal{F}(f_2) = \{B \subseteq E_2 : f_2(B) > 0\} = \{B_1, B_2, \cdots, B_n\}.$$

Let $\mathcal{B}_1$ is the collection of all Borel subsets of $[0,1]$ and $\lambda(dx)$ is the Lebesgue measure on $\mathcal{B}_1$. Consider the probability space $([0,1], \mathcal{B}_1, \lambda(dx))$. For $F_1(A)$, partition $[0,1]$ into m intervals $I_1, I_2, \cdots, I_m$ with length $f_1(A_i)$, $A_i \in \mathcal{F}(f_1)$. Similarly, for $F_2(B) = P(\mathcal{S}_2 \subseteq B)$, partition $[0,1]$ into n intervals $J_1, J_2, \cdots, J_n$ with length $f_2(B_j)$, $B_j \in \mathcal{F}(f_2)$. Define

$$\mathcal{S}_1' : [0,1] \to \mathcal{F}(f_1) \qquad \mathcal{S}_1'(x) = A_i \quad \text{for } x \in I_i, i = 1, \cdots, m, \tag{10}$$

and

$$\mathcal{S}_2' : [0,1] \to \mathcal{F}(f_2) \qquad \mathcal{S}_2'(y) = B_j \quad \text{for } y \in J_j, j = 1, \cdots, n. \tag{11}$$

Observe that the Lebesgue measure $\lambda(dx)$ on $[0,1]$ corresponds, by Lebesgue-Stieltjes theorem, to the distribution function $x \to x$, on $[0,1]$, of the uniform random variable on it. As such, a joint distribution on $[0,1]^2$ with uniform marginals is preciesly some copula $\mathcal{C}$.

Note that each copula $\mathcal{C}$, as a bivariate distribution, generates a probability measure on $\mathcal{B}_1 \times \mathcal{B}_1$ of $[0,1]^2$, denoted as $d\mathcal{C}$, by

$$d\mathcal{C}\left\{[u_1,u_2] \times [v_1,v_2]\right\} = \mathcal{C}(u_1,v_1) - \mathcal{C}(u_1,v_2) - \mathcal{C}(u_2,v_1) + \mathcal{C}(u_2,v_2). \tag{12}$$

Now we can consider the random set vector $(\mathcal{S}_1', \mathcal{S}_2') : [0,1]^2 \to 2^{E_1} \times 2^{E_2}$ which has marginal densities f_1, f_2. Let $\mathcal{C}$ be a copula. If we equip the measurable space $([0,1]^2, \mathcal{B}_1 \times \mathcal{B}_1)$ with the probability measure $d\mathcal{C}$, then the function $H_{\mathcal{C}} : 2^{E_1} \times 2^{E_2} \to [0,1]$, defined by

$$\begin{aligned} H_{\mathcal{C}}(A,B) &= d\mathcal{C}\{(x,y) \in [0,1]^2 : \mathcal{S}_1'(x) \subseteq A, \mathcal{S}_2'(y) \subseteq B\} \\ &= d\mathcal{C}\{(x,y) \in [0,1]^2 : \mathcal{S}_1'(x) \times \mathcal{S}_2'(y) \subseteq A \times B\}. \end{aligned} \tag{13}$$

In summary, we have the following result,

Proposition 3.1 *For any given univariate belief functions F_1 and F_2 of random sets $\mathcal{S}_1$ and $\mathcal{S}_2$, and a copula $\mathcal{C}$, if we equip an order on their focal sets $\mathcal{F}(f_1)$, $\mathcal{F}(f_2)$, then the joint belief function of random set vector $(\mathcal{S}_1, \mathcal{S}_2)$ can be constructed by (13). Furthermore, the joint density of random set vector $(\mathcal{S}_1, \mathcal{S}_2)$ can be obtained by its Möbius inverse of $H_{\mathcal{C}}$ given in (8).*

Note that it is easy to verify that $H_{\mathcal{C}}(.,.)$ has marginal belief functions F_1 and F_2. The following example is an illustration of our construction method.

Example 3.1 *Let $E_1 = \{1,2\}$ and $E_2 = \{3,4,5\}$. Suppose the densities of random sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are given by*

$$f_1(\{1\}) = f_1(\{2\}) = 0.25, f_1(\{1,2\}) = 0.5,$$

and

$$f_2(\{3\}) = 0.2, f_2(\{4\}) = f_2(\{3,5\}) = 0.3, f_2(\{3,4,5\}) = 0.2,$$

respectively. Now if we equip $\mathcal{F}(f_i)$ with Lexicographical order, we will obtain a unique joint belief function of $\mathcal{S}_1$ and $\mathcal{S}_2$ for any given copula $\mathcal{C}$.

Consider the orders given by

$$\mathcal{F}(f_1) = \{\{1\},\{2\},\{1,2\}\} \quad \textit{and} \quad \mathcal{F}(f_2) = \{\{3\},\{4\},\{3,5\},\{3,4,5\}\}.$$

Define $\mathcal{S}_1' : [0,1] \to \mathcal{F}(f_1)$ and $\mathcal{S}_2' : [0,1] \to \mathcal{F}(f_2)$ respectively by by

$$\mathcal{S}_1'(x) = \begin{cases} \{1\}, & \textit{if } x \in [0,0.25], \\ \{2\}, & \textit{if } x \in (0.25,0.5], \\ \{1,2\}, & \textit{if } x \in (0.5,1] \end{cases} \qquad \mathcal{S}_2'(y) = \begin{cases} \{3\}, & \textit{if } y \in [0,0.2], \\ \{4\}, & \textit{if } y \in (0.2,0.5], \\ \{3,5\}, & \textit{if } y \in (0.5,0.8], \\ \{3,4,5\}, & \textit{if } y \in (0.8,1]. \end{cases}$$

If we apply Farlie-Gumbel-Morgenstern copula

$$\mathcal{C}(u, v) = uv(1 + \rho(1 - u)(1 - v))$$

with $\rho = \frac{1}{2}$, then from (13), we obtain the joint distribution given bellow,

$H_{\mathcal{C}}$	{3}	{4}	{5}	{3,4}	{3,5}	{4,5}	{3,4,5}
{1}	13/200	267/3200	0	19/128	421/3200	367/3200	1/4
{2}	11/200	249/3200	0	17/128	407/3200	249/3200	1/4
{1,2}	1/5	3/10	0	1/2	1/2	3/10	1

Table 1: Joint distribution of $(\mathcal{S}_1, \mathcal{S}_2)$.

Then, from (8), we can calculate the joint density, $h_{\mathcal{C}}$, of $(\mathcal{S}_1, \mathcal{S}_2)$ given below.

$h_{\mathcal{C}}$	{3}	{4}	{5}	{3,4}	{3,5}	{4,5}	{3,4,5}
{1}	13/200	267/3200	0	0	213/3200	0	7/200
{2}	11/200	249/3200	0	0	231/3200	0	9/200
{1,2}	2/25	111/800	0	0	129/800	0	3/25

Table 2: Joint density of $(\mathcal{S}_1, \mathcal{S}_2)$.

Remark 3.1 Note that the construction method given above shows that the joint distribution H_C depends not only on copula C but also on the order of $\mathcal{F}(f_1)$ and $\mathcal{F}(f_2)$. Nguyen [7] suggested us to use the principle of maximum entropy for selecting the orders of A_i's and B_j's so that the definition of joint distribution is determined. However, this selection of A_i's and B_j's is not unique so that same maximum entropy can result different joint distribution of $(\mathcal{S}_1, \mathcal{S}_2)$.

Example 3.2 *For example, let $E_1 = \{1, 2\}$ and $E_2 = \{3, 4\}$, $f_1(\{1\}) = 1/3$, $f_1(\{1, 2\}) = 2/3$ and $f_2(\{4\}) = 3/4$, $f_2(\{3, 4\}) = 1/4$, all four different orders on $\mathcal{F}(f_1) = \{A_1 = \{1\}, A_2 = \{1, 2\}\}$ and $\mathcal{F}(f_2) = \{B_1 = \{4\}, B_2 = \{3, 4\}\}$ give different distributions. However, all four different orders gives the same entropy,*

$$\mathcal{E}_{nt}(h_C) = - \sum_{A \in 2^{E_1}, B \in 2^{E_2}} h_C(A, B) \log_2 h_C(A, B) = 0.5183131.$$

As an application of the construction method given in this section, the joint belief function can be applied in game theory. Given two correlated univariate games, one can further calculate the joint Shapley's value based on the copula based joint belief function. More detials about the connections between the joint belief function and its applications on the joint game can be found in Wei et al [13].

Example 3.3 (See Fernandez [2] with some values changed) *Consider three cell-phone operators (namely $O1$, $O2$, and $O3$) that want to enter a new market. There are two criteria that must be considered in the process. On the one hand, there is the profit that has been estimated from the market analysis. On the other hand, there is the coverage, which is regulated by law. Thus, the percentage of population covered by each operator or by merging is fixed by the government. Cover- age is very important because it is known to improve the return in the medium and long run. Let us assume that profit is measured in millions of dollars and coverage in percent. We represent by vectors with two entries the values obtained by each operator: the first entry is the profit and the second one is the* *coverage. Let us consider the following data that represent the values obtained in different cooperation situations:*

Coalition	{O1}	{O2}	{O3}	{O1,O2}	{O1,O3}	{O2,O3}	{O1,O2,O3}
ν_1	2	3	3	6	6	8	12
ν_2	20	40	10	70	30	50	100

Table 3: Two correlated games ν_1 and ν_2.

Let set functions ν_1 and ν_2 be the profit and the coverage of each coalition, respectively. Note that ν_1 and ν_2 are converted to standardized games ν_1' and ν_2' which are belief functions, given in Table 4.

Coalition	{O1}	{O2}	{O3}	{O1,O2}	{O1,O3}	{O2,O3}	{O1,O2,O3}
ν_1'	1/6	1/4	1/3	1/2	1/2	2/3	1
ν_2'	0.2	0.4	0.1	0.7	0.3	0.5	1

Table 4: ν_1' and ν_2' .

If we use the correlation coefficient of the profit and the coverage $\rho = 0.83$, and adopt Farlie-Gumbel-Morgenstern copula $C_\rho(u, v)$ to construct the joint game (or the joint belief function) ν, then we have the following Table 5.

ν	{O1}	{O2}	{O3}	{O1,O2}	{O1,O3}	{O2,O3}	{O1,O2,O3}
{O1}	.052	.076	.013	.139	.065	.089	1/6
{O2}	.064	.107	.022	.191	.086	.129	1/4
{O3}	.059	.13	.035	.225	.094	.164	1/3
{O1,O2}	.126	.213	.045	.38	.171	.258	1/2
{O1,O3}	.111	.206	.048	.363	.159	.253	1/2
{O2,O3}	.131	.266	.067	.464	.199	.333	2/3
{O1,O2,O3}	0.2	0.4	0.1	0.7	0.3	0.5	1

Table 5: The joint game ν of ν_1 and ν_2 .

Note that the last row and the last column of ν can be treated as the standardized the vector-valued game μ in Table 3.

Remark 3.2 From Example 3.3, we can see that the vector valued game can be treated as special case of of the joint belief function whose marginals are games of the vector valued game respectively. For more details of the joint belief function and the joint game see Wei et al[13].

3.2 Constructing a subcopula from the joint belief function

The previous subsection shows that given marginal belief functions and a copula, we can construct a joint belief function. Conversely, given joint belief function, we can obtain a subcopula. Note that any subcopula can be extended to a copula, but its extension is not generally unique.

Now given a joint belief function $H : 2^{E_1} \times 2^{E_2} \to [0,1]$ of discrete random set vector $(\mathcal{S}_1, \mathcal{S}_2)$, we can find its marginal belief functions $F_1(A) = H(A, E_2)$, $F_2(B) = H(E_1, B)$ and the corresponding marginal densities $f_1 : 2^{E_1} \to [0,1]$, $f_2 : 2^{E_2} \to [0,1]$. By the construction method given in (10) to (13), we can solve for a subcopula $\mathcal{C}'$.

Proposition 3.2 *Given a joint belief function $H : 2^{E_1} \times 2^{E_2} \to [0,1]$ of discrete random set vector $(\mathcal{S}_1, \mathcal{S}_2)$, if we equip an order on their marginal focal sets $\mathcal{F}(f_1)$, $\mathcal{F}(f_2)$, then there is a unique subcopula $\mathcal{C}'$, such that $H(A,B) = \mathcal{C}'(F_1(A), F_2(B))$, for any $A \subseteq E_1, B \subseteq E_2$, where F_1 and F_2 are marginal belief functionals.*

Proof: Given $H : 2^{E_1} \times 2^{E_2} \to [0,1]$, we can find two marginal belief functions $F_1 : 2^{E_1} \to [0,1]$ and $F_2 : 2^{E_2} \to [0,1]$ by $F_1(A) = H(A, E_2)$ and $F_2(B) = H(E_1, B)$ for any $A \subseteq E_1$ and $B \subseteq E_2$. Furthermore, we can find two marginal densities f_1 and f_2. Since E_i's are finite sets, we can order the elements of $\mathcal{F}(f_1) = \{A \subseteq E_1 : f_1(A) > 0\}$, by the Lexicographical order as

$$\mathcal{F}(f_1) = \{A_1, A_2, \cdots, A_m\}.$$

Similarly, we can obtain

$$\mathcal{F}(f_2) = \{B \subseteq E_2 : f_2(B) > 0\} = \{B_1, B_2, \cdots, B_n\}.$$

For $F_1(A)$, partition $[0,1]$ into m intervals $I_1, I_2, \cdots, I_m$ with length $f_1(A_i)$, $A_i \in \mathcal{F}(f_1)$, assume the partition is $0 = i_0 < i_1 < \cdots < i_{m-1} < i_m = 1$. Similarly, for $F_2(B) = P(\mathcal{S}_2 \subseteq B)$, partition $[0,1]$ into n intervals $J_1, J_2, \cdots, J_n$ with length $f_2(B_j)$, $B_j \in \mathcal{F}(f_2)$, assume the partition is $0 = j_0 < j_1 < \cdots < j_{n-1} < j_n = 1$.. Define

$$\mathcal{S}'_1 : [0,1] \to \mathcal{F}(f_1) \qquad \mathcal{S}'_1(x) = A_i \quad \text{for } x \in I_i, i = 1, \cdots, m, \tag{14}$$

and

$$\mathcal{S}'_2 : [0,1] \to \mathcal{F}(f_2) \qquad \mathcal{S}'_2(y) = B_j \quad \text{for } y \in J_j, j = 1, \cdots, n. \tag{15}$$

Then, $\mathcal{S}'_i$s are two marginal random sets with densities f_i. Let $I_1 = \{i_0, \cdots, i_m\}$ and $I_2 = \{j_0, \cdots, j_n\}$, then $I_i \subseteq [0,1]$ which contains 0 and 1. Then we can define $\mathcal{C}'$: $I_1 \times I_2 \to [0,1]$ by (13): Define $\mathcal{C}'(u,0) = 0 = \mathcal{C}'(0,v)$, and $\mathcal{C}'(i_1,j_1) = \mathcal{C}'(i_1,j_1) - \mathcal{C}'(i_1,0) - \mathcal{C}'(0,j_1) + \mathcal{C}'(0,0) = H(A_1,B_1)$, following this pattern, we can define $\mathcal{C}'$ for all $i \in I_1$ and $j \in I_2$ recursively and it is easy to verify $\mathcal{C}'$ is indeed a subcopula by the definition. $\square$

The following example is an illustration of the above construction algorithm.

Example 3.4 *Given a joint belief function H in the following table,*

H	{3}	{4}	{5}	{3,4}	{3,5}	{4,5}	{3,4,5}
{1}	1/12	1/12	1/12	1/6	1/6	1/6	1/3
{2}	1/12	1/12	1/12	1/6	1/6	1/6	1/3
{1,2}	1/6	1/6	1/6	1/3	1/3	1/3	1

Table 6: The joint belief function H.

From the last row and the last column of the joint belief function, we can calculate the marginal densities, f_1 and f_2, which are given below.

A	$\{1\}$	$\{2\}$	$\{1,2\}$
f_1	1/3	1/3	1/3

B	$\{3\}$	$\{4\}$	$\{5\}$	$\{3,4\}$	$\{3,5\}$	$\{4,5\}$	$\{3,4,5\}$
f_2	1/4	1/4	1/4	0	0	0	1/4

Table 7: Marginal densities $f_1(A)$ and $f_2(B)$.

Define $\mathcal{S}_1 : [0,1] \to 2^{E_1}$ *and* $\mathcal{S}_2 : [0,1] \to 2^{E_2}$ *respectively by*

$$\mathcal{S}_1(x) = \begin{cases} \{1\}, & \textit{if } x \in [0, \frac{1}{3}], \\ \{2\}, & \textit{if } x \in (\frac{1}{3}, \frac{2}{3}], \\ \{1,2\}, & \textit{if } x \in (\frac{2}{3}, 1], \end{cases} \qquad \mathcal{S}_2(y) = \begin{cases} \{3\}, & \textit{if } y \in [0, \frac{1}{4}], \\ \{4\}, & \textit{if } y \in (\frac{1}{4}, \frac{2}{4}], \\ \{5\}, & \textit{if } y \in (\frac{2}{4}, \frac{3}{4}], \\ \{3,4,5\}, & \textit{if } y \in (\frac{3}{4}, 1]. \end{cases}$$

Solving $\mathcal{C}'(u,v)$ *in (13), we obtain a subcopula* $\mathcal{C}'$ *given in Table 8.*

$\mathcal{C}'$	1/4	2/4	3/4	1
1/3	1/12	1/6	1/4	1/3
2/3	1/6	2/6	2/4	2/3
1	1/4	2/4	3/4	1

Table 8: The subcopula $\mathcal{C}'$.

For example, for calculating $\mathcal{C}'(\frac{2}{3}, \frac{2}{4})$*, which belongs to the interval* $(\frac{1}{3}, \frac{2}{3}] \times (\frac{1}{4}, \frac{2}{4}]$*, we have*

$$\begin{aligned} d\mathcal{C}' \left\{ \left(\frac{1}{3}, \frac{2}{3} \right] \times \left(\frac{1}{4}, \frac{2}{4} \right] \right\} &= \mathcal{C}' \left(\frac{2}{3}, \frac{2}{4} \right) + \mathcal{C}' \left(\frac{1}{3}, \frac{1}{4} \right) - \mathcal{C}' \left(\frac{1}{3}, \frac{2}{4} \right) - \mathcal{C}' \left(\frac{2}{3}, \frac{1}{4} \right) \\ &= H(\{2\}, \{4\}) = \frac{1}{12}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{C}' \left(\frac{2}{3}, \frac{2}{4} \right) &= H\left(\{2\}, \{4\}\right) - \mathcal{C}' \left(\frac{1}{3}, \frac{1}{4} \right) + \mathcal{C}' \left(\frac{1}{3}, \frac{2}{4} \right) + \mathcal{C}' \left(\frac{2}{3}, \frac{1}{4} \right) \\ &= \frac{1}{12} - \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{1}{3}. \end{aligned}$$

It is easy to check that $\mathcal{C}'$ *is an independent subcopula, which in turn shows that* H *is an independent joint belief function.*

4. Acknowledgments

The authors would like to thank Professor Hung T. Nguyen for introducing this interesting topic to us.

References

[1] Alvarez, D.A.(2009), "A Monte Carlo-based method for the estimation of lower and upper probabilities of events using infinite random sets of indexable type," *Fuzzy Sets and Systems*, 160, 384–401.

[2] Fernandez, F.R., Miguel, A.H., Justo, P.: Core solutions in vector-valued games. Journal of Optimization Theory and Applications **1**12 331-360 (2002)

[3] Harry, J.(1997), *Multivariate Models and Dependence Concepts*, London, UK: Chapman & Hall.

[4] Nelsen, R.B.(2006), *An introduction to Copulas* (second edition), New York: Springer.

[5] Nguyen, H.T.(2006), *An Introduction to Random Sets,* Boca Raton, FL: CRC Press.

[6] Nguyen, H.T. and Wang, T.(1997), *Belief functions and random sets, The IMA Volumes in Mathematics and Its Applications,* New York: Springer-Verlag, pp. 243-255.

[7] Nguyen, H.T.(2013), *Lecture Notes in: Statistics with copulas for applied research,* Department of Economics, Chiang Mai University, Chiang Mai, Thailand.

[8] Schmelzer, B.(2012), "Characterizing joint distributions of random sets by multivariate capacities," *Intern. J. Approximate Reasoning* 53, 1228–1247.

[9] Schmelzer, B.(2014), "Joint distributions of random sets and their relation to copulas," in *Modeling Dependence in Econometrics*, Advances in Intelligent Systems and Computing 251, DOI: 10.1007/978-3-319-03395-2-10, ©Springer International Publishing Switzerland 2014 Copulas.

[10] Shafer, G.(1976), *A mathematical theory of evidence,* New Jersey, Princeton: Princeton University Press.

[11] Sklar, A.(1959), "Fonctions de répartition á n dimensions et leurs marges," *Publ. Inst. Statist. Univ.* 8, 229–231.

[12] Wei, Z., Wang, T. and Panichkitkosolkul, W., "Dependence and association concepts through copulas," in *Modeling Dependence in Econometrics*, Advances in Intelligent Systems and Computing 251, DOI: 10.1007/978-3-319-03395-2-7, ©Springer International Publishing Switzerland 2014 Copulas.

[13] Wei, Z., Wang, T., Li, B. and Nguyen A. P., "The joint belief function and Shapley value for the joint cooperative game," in *Modeling Dependence in Econometrics*, (to appear).