A Generalization to the Counting Process and its Consequences

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Abstract

We have inaugurated the generalized form of the Counting process along with its various characteristics. The creditability and the consequences for the generalization of the counting process have also been explained in building generalized Poisson process, etc.

Key Words: Exponential Distribution, Memory Lapse Property, Shape parameter.

1. Introduction

How rare is "Rare?" The knowledge of the occurrence of rare events is an important issue for our survival. The occurrences of these rare events have their own patterns which may not be entirely known. If the behaviors of the occurrences of each type of rare events have been studied more than a single or a couple of centennial or decades or years or even hours depending on the life-span of its; the patterns of the occurrences of the typical rare events can be extended. The occurrence of the rare events is known as Poisson distribution which one is an independently and identically distributed Deterministic model and whose behaviors can be studied in several ways. On the other hand the Non-Deterministic or the Stochastic Model of rare events having Poisson behavior is known as Poisson process.

In Poisson process, the non-overlapping waiting time for the occurrences of the successive rare events follow Exponential Distribution. So, if the inter-arrival times can be treated as following the generalized exponential distributions as Adnan et el [1] as the Poisson process can be convoluted to the Generalized Poisson process. Similarly the probability of an event happening in per unit time is independent of any other unit time assuming that the inter-arrival time is identically distributed as exponential variant [31].
The generalized inter-arrival exponential time belonging to the Generalized Poisson process can be used to construct generalized Renewal Theory and Queuing Theory etc.

Literature Review

The Poisson distribution was first brought to light by Siméon Denis Poisson, in 1837.
Later many authors like Stein (1972), Barbour (1988), Aratia (1989), Barbour (1992, 2005), Brown (1994, 1995, 2000, 2001), Zeger (1994), Conway (1999), Salvador (2003), Chen (2004), Schuhmacher (2005, 2008), Xia (1994, 2008), Zhang (2005) worked on Poisson process. Authors such as Ibe (2005), Karlin (1975), Knill (2009), Ross (2000), Suhov (2008) explained Poisson process in their texts. The generalized Poisson probability model has many applications in areas such as engineering, manufacturing, survival analysis, genetic, shunting accidents, queuing, and branching processes. Various generalizations of the family of counting processes have been considered by a couple of authors [14, 16-19, 24, 30, 32]. Similarly, properties of family of counting processes can also be obtained from various generalized Poisson processes. Generalization of Poisson distribution had widely been studied by numerous authors like Satterthwaite (1942), Lakshmi (2012), Hubert and Lauretto (2009). Various generalized form of Poisson

process [18, 29] and its family such as Renewal process [16, 20, 25], Continuous time Markov chain and Branching process [23, 27, 44] with representations of the new form are presented in terms of definitions. Related theorems, properties and parameter estimations are also presented here with derivation. And also we have presented whether there is any difference between the usual process and generalized process. For inference procedure, we have tested the result of generalized form for different values of b of the extra parameter h^b. And our findings are at the end of this paper.

Several texts books are out there about the family of counting processes such as Veerarajan (2008), Ross (2004), Medhi (2002), Bening and Korolev (2002) and Ross (2013). Haight (1959)Fel'dman (1983)& Bening and Korolev (2002) discussed about the Models of generalized Poisson processes with applications. To study the estimation of generalized Poisson distribution, Fel'dman (1992) used the weighted discrepancies method between observed and expected frequencies and found this better than the chi-square method which links very well with the method of maximum likelihood. Famoye and Consul (1995) define univariate generalized Poisson distribution which is correlated bivariate version. Estimation of its parameters and some properties are also discussed. To allow the assignment of varying weights to events, Satterthwaite (1942) generalized the Poisson distribution when the number of events follows the Poisson law. Hubert, Lauretto and Stern (2009) studied the empirical properties of the Full Bayesian Significance Test for testing the nullity of extra parameter of the generalized Poisson distribution, which is capable to offer an evidence degree on sharp hypotheses. Pacheco (2003) proposed a procedure for fitting Markov Modulated Poisson processes (MMPPs) to traffic traces that matches both the auto covariance and marginal distribution of the counting process. The number of states is not fixed a priori is the major feature of the procedure. It is an output of the fitting process, thus allowing the number of states to be adapted to the particular trace being modeled. Lekshmi and Thomas (2012) made an attempt to review count data models developed so far as generalizations of Poisson process and considered Weibull and Winkleman's gamma count model of Mc Shane et al. A Mittag-Leffler count model is developed and studied in detail with simulation. Zhang (2008) presented three nonparametric methods respectively for making inferences of doubly stochastic Poisson processes. They analyzed sequences of arrival data.

Wang and Yang (2012) proposed a nonlinear programming approach for the Kijima type GRP model I for estimating restoration factor, for repairable systems for the model II based on the conditional Weibull distribution, using negative log-likelihood as an objective function and adding inequality constraints to model parameters. The results shows that the GRP model is greater to the ordinary renewal process (ORP) and the power law non-homogeneous Poisson process model. Kijima (2002) considered a generalized renewal process (g-renewal process for short) and its applications to reliability theoryalso general repair model with full generality constructed using a general point process. Pyke (1961)studied the Markov Renewal processes having a finite number of states as well as its limiting behavior. Jacopino, Groen and Mosleh (2004)provided general insights into the behavior of the GRP model with applications and concluded that at a low number of renewals there is little difference between the two models, i.e., GRP and ORP.Hurtado, Joglar, and Modarres (2005) describes an alternative for calculating the parameters of GRP models using a Genetic Algorithm (GA) approach to solve complex MLE equations. Feller (1941) considered the behavior of the solutions of integral renewal process.

The main objective of the current study is to develop and find the consequences of the generalized Counting processes with an extra parameter to be applied affluently in multiple disciplines.

Chapter 2 ensembles the constructions in counting process, Poisson processes. The new forms of generalized Poisson processes have briefly been discussed with related theorems and derivations. Generalized Renewal process has been discussed in Chapter 3. Conclusion appears in the last chapter. The summery of the differences among the traditional generalized process have been displayed in the Appendix. The ultimate part is the reference section.

2. Generalized Poisson process

A stochastic process $[N(t), t \geq 0]$ is said to be a counting process if $N(t)$ represents the total number of "events" that have occurred up to time t. The Poisson process is a collection $\{N(t) : t \geq 0\}$ of random variables, where $N(t)$ is the number of events that have occurred up to time t (starting from time 0) [37]. λt is the occurrence rate for the events being counted within he tome $(0, t]$. Poisson processes mathematically as

$$P[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Where λt is mean and variance. Since in the traditional Poisson distribution the pdf of the Poisson distribution $P[t = n] = \frac{e^{-\lambda}\lambda^n}{n!}$ represents the probability of the n number of events to be occurred per unit time or within the time interval $(0,1]$ and the pdf of the Poisson process

$$P[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Representing the number of events to be occurred per unit time interval$(0, t]$. When the time unit is unique, the probabilistic model of the rare events is Poisson distribution and for the unit time interval the model claims Poisson process. As a result the average rate of occurrences within the unit time in the Poisson distribution is $\lambda(1 - 0) = \lambda.1 = \lambda$ and the average rate of happening within the unit time interval is equivalent to $\lambda(t - 0) = \lambda t$. An extra parameter b named shape parameter has been introduced in the pdf of the generalized Poisson process to present the power of the subset h of the unit time (unit time interval$(0,1]$) where $(0, h] \subset (0,1]$. So, depending on the value of $b(b > 0), h^b$will be either greater than unit time 1 or less than 1. So the average rate of occurrence λ will be replaced by λh^b to present the average rate of events to be occurred per unit time interval h^b. So, interval $(0, h^b]$ will be less than or greater than 1 for $b > 1$,$b < 1$ & $h < 1$. The probability of occurrence of n number of events per unit time (time interval) h^b is

$$P[N = n] = \frac{e^{-\lambda h^b}(\lambda h^b)^n}{n!};$$

where λh^b is the rate of event to be occurred per unit time (time interval) h^b $\forall\ b \geq 0$ to be known as Poisson probability such that

$$e^{\lambda h^b} = \frac{(\lambda h^b)^0}{0!} + \frac{(\lambda h^b)^1}{1!} + \frac{(\lambda h^b)^2}{2!} + \cdots => 1 = \sum_{n=0}^{\infty} P(N = n) = \sum_{n=0}^{\infty} \frac{(\lambda h^b)^n e^{-\lambda h^b}}{n!}$$

Now, $P[N = 1] = \frac{e^{-\lambda h^b}(\lambda h^b)^1}{1!} = \lambda h^b e^{-\lambda h^b} = \lambda h^b - \frac{(\lambda h^b)^2}{1!} + \frac{(\lambda h^b)^3}{2!} - \cdots$

$$\therefore P[N = 1] = \lambda h^b + higher\ order\ term\ of\ h^b = \lambda h^b + O(h^b) \qquad (1)$$

Then, $P[N=2]=\frac{e^{-\lambda h^b}(\lambda h^b)^2}{2!}=\frac{(\lambda h^b)^2}{2!}e^{-\lambda h^b}=\frac{(\lambda h^b)^2}{2!}\left[1-\frac{(\lambda h^b)^1}{1!}+\frac{(\lambda h^b)^2}{2!}-\cdots\right]$(2)

$$= sum\ of\ the\ higher\ order\ term\ of\ h^b = O(h^b) \therefore P[N=2]=O(h^b)$$

Now if λh^b is the rate of event the occurred per unit time(time interval) h^b, $b \geq 0$ with the Poisson probability $P[N=n]=\frac{e^{-\lambda h^b}(\lambda h^b)^n}{n!}$; then $N(t)$ will be the counted number of events to be occurred in the time interval t and $\{N(t), t \geq 0\}$ is said to be a counting process with λh^b if and only if

I. $N(0)=0$.
II. The process has stationary and independent increments
III. $P\{N(h)=1\}=\lambda h^b + o(h^b)$. [from (1)]
IV. $P\{N(h)\geq 2\}=o(h^b)$. [from (2)]

The Generalized Poisson process and its various properties have been developed through various theorems.

Theorem 1: *TheProbability of* n *number of events occurring in the time interval* t *will be* $P_n(t)=P[N(t)=n]=\frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1}t)^n}{n!}$; $\forall\ n=0,1,2,3,\ldots,\infty$ *and* $b \geq 1$ *which can also be referred as the probability distribution function of the generalized Poisson process.*

Theorem 2: *Counting process starts from Bernoulli, binomial then Poisson distribution.* $P\{X=k\}=\binom{n}{k}P^k(1-P)^{n-k}$. *Binomial distribution converges to Poisson distribution.*

Theorem 3: *If* $N(t)$ *is a counting process with the pdf,* $P_n(t)=\frac{e^{-\lambda h^{b-1}x}(\lambda h^{b-1}x)^n}{n!}$ *with occurrence rate* λh^{b-1}, *then the inter-arrival times* $X_1, X_2, \ldots$ *are an independent and identically distributed random sequence of* X_i *where each* X_i *has exponential probability density .*

Theorem 4: *If* $N(t)$ *is a Poisson process along with the pdf* $P_n(t)=\frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1}t)^n}{n!}$ *having the inter-arrival times* $t_1, t_2, \ldots$ *having exponential pdf with the form* $f(t)=\lambda h^{b-1}e^{-\lambda t h^{b-1}}$; $t>0$. *Then* $S_n=\sum_{i=1}^{n} t_i$ *is the waiting time until the* n^{th} *event occurs* $\forall n=1,2,\ldots$ *Then each of the sequence of the waiting time* $S_1, S_2, \ldots, S_n$ *follows gamma distribution with parameters* (n,λ).

Theorem 5: *The counting process* $\{N(t), t\geq 0\}$ *having the pdf* $P_n(t)=\frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1}t)^n}{n!}$ *along with the rate* λh^{b-1} *iff*

I. *The number of arrivals in any interval* $[t_0, t_1]$, $N(t_1)-N(t_0)$ *is a Poisson random variable with expected value* $\lambda h^{b-1}[t_1-t_0]$.
II. *For any pair of non-overlapping interval* $[t_0, t_1], [t_0', t_1']$. *The no of arrivals in each interval,* $N(t_1)-N(t_0)$ *and* $N(t_1')-N(t_0')$ *respectively are independent random variables.*

Theorem 6: *For a Poisson process* $\{N(t), t \geq 0\}$ *having the pdf* $P_n(t)=\frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1}t)^n}{n!}$ *along with the rate of occurrences* λh^{b-1}, *the joint probability mass*

function of $N(t_1), N(t_2), \ldots, N(t_{k-1}), N(t_k)$; $t_1 < t_2 < \cdots < t_{k-1} < t_k$ *is* $P_{N(t_1),N(t_2),\ldots,N(t_{k-1}),N(t_k)}(n_1, n_2, \ldots, n_{k-1}, n_k) =$
$\frac{e^{-\lambda h^{b-1}t_1}\left(\lambda h^{b-1}t_1\right)^{n_1}}{n_1!} \cdot \frac{e^{-\lambda h^{b-1}[t_2-t_1]}\left(\lambda h^{b-1}[t_2-t_1]\right)^{n_2-n_1}}{(n_2-n_1)!} \ldots \frac{e^{-\lambda h^{b-1}[t_k-t_{k-1}]}\left(\lambda h^{b-1}[t_k-t_{k-1}]\right)^{n_k-n_{k-1}}}{(n_k-n_{k-1})!}$

Theorem 7:*A counting process* $N(t)$ *having pdf* $P_n(t) = \frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1}t)^n}{n!}$*& independent exponential arrivals* $X_1, X_2, \ldots$ *with mean* $E(X_i) = \frac{1}{\lambda h^{b-1}}$ *is a Poisson process of rate* λh^{b-1} *which is memoryless since* $P[X_n > t' + t | X_n > t'] = P(X_n > t)$ *for all* $t, t' > 0$.

Theorem 8: *If* $N(t)$ *is a Poisson process along with the pdf* $P[N(t) = n] = \frac{e^{-\lambda h^{b-1}t}\left(\lambda h^{b-1}t\right)^n}{n!}$, *the mean* $E[N(t)]$ *and Variance* $V[N(t)]$ *will be* $\lambda h^{b-1}t$ *; the correlation coefficient,* $corr[N(t), N(t+r)] = \sqrt{\frac{t}{t+r}}$, *Covariance,* $Cov[N(t), N(t+r)]$=$\lambda h^{b-1}t$ *and moment generating function,* $M_{N(t)}(t) = e^{\lambda h^{b-1}t(e^{t'}-1)}$.

Theorem 9: *A Poisson process* $N(t)$ *having pdf* $P[N(t) = n] = \frac{e^{-\lambda h^{b-1}t}\left(\lambda h^{b-1}t\right)^n}{n!}$ *is a Markov process.*

Theorem10: *If* $N_1(t)$ *and* $N_2(t)$ *are Poisson processes along with their pdf* $P[N_1(t) = r] = \frac{e^{-\lambda h^{b-1}t}\left(\lambda h^{b-1}t\right)^r}{r!}$ *and* $P[N_2(t) = n - r] = \frac{e^{-\lambda h^{b-1}t}\left(\lambda h^{b-1}t\right)^{n-r}}{(n-r)!}$ *where* $P[N_1(t) + N_2(t) = n]$*then* $N_1(t) + N_2(t)$ *is also a Poisson process.*

Theorem 11: *If* $N_1(t)$ *with parameters* (λ_1, b) *and* $N_2(t)$ *with parameters* (λ_2, b) *are Poisson process along with their pdf* $P[N_1(t) = r] = \frac{e^{-\lambda h^{b-1}t}\left(\lambda h^{b-1}t\right)^r}{r!}$ *and* $P[N_2(t) = n - r] = \frac{e^{-\lambda h^{b-1}t}\left(\lambda h^{b-1}t\right)^{n-r}}{(n-r)!}$ *where* $P[N_1(t) + N_2(t) = n]$ *with mgf* $e^{\lambda_1 h^{b-1}t(e^{t'}-1)}$ *and* $e^{\lambda_2 h^{b-1}t(e^{-t'}-1)}$ *respectively then* $N_1(t) - N_2(t)$ *is not a Poisson process.*

Theorem 12: $\frac{N_1(t)}{N_1(t)+N_2(t)}$ *follows Binomial variant along with the pdf* $P[N_1(t) = r | N_1(t) + N_2(t) = n] = \binom{n}{r}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r\left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-r}$.

Theorem 13: *If* $N(t)$ *is a Poisson process along with the pdf* $P_n(t) = \frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1}t)^n}{n!}$ *having the inter-arrival times* $t_1, t_2, \ldots$ *having exponential pdf with the form* $f(t) = \lambda h^{b-1}e^{-\lambda t h^{b-1}}$; $t > 0$. *Follows independent and stationary increments then time*$[0, t]$, *that is for* $s < t$ *and the conditional distribution of inter* $-$ *arrival* $f_{S_n}(t) = \frac{(\lambda h^{b-1})^n}{(n-1)!}e^{-\lambda h^{b-1}t}t^{n-1}$; $t \geq 0$ *then* $P\{T_1 < S | N(t) = 1\} = \frac{s}{t} =$ *constant.*

3. Generalized Renewal Process

From the generalized process we obtained a new form of exponential distribution for inter-arrival time along with the pdf,

$$f(t) = \lambda h^{b-1} e^{-\lambda t h^{b-1}};\ \ t > 0$$

Based on this generalized exponential variant we have derived the generalized forms for various properties of Renewal process. A counting process $\{N(t), t \geq 0\}$ for which the times between successive events are independently and identically distributed arbitrary random variables is known as renewal process.A Poisson process is a renewal process for which the times between successive events are independently and identically distributed exponential random variables. So renewal process is a special kind of Poisson process

$$P\{\mathrm{N(t)} \geq \mathrm{n}\} = P\{S_n \leq t\}$$

Where waiting time until n^{th}event is, $S_n = \sum_{i=1}^{n} T_i$, $N(t)$ is number of renewals; $N(t) = Sup\{n\text{: } S_n \leq t\}$. That is$N(t)$ is a Renewal process.Then we can say, if $N(t)$is a counting process, for a Poisson process of rate λh^b, the inter-arrival times $T_1, T_2, \ldots$ are an iid random sequence with exponential distribution

$$f_T(t) = \begin{cases} \lambda h^{b-1} e^{-\lambda t h^{b-1}}, & t > 0 \\ 0 & otherwise \end{cases}$$

Theorem 14: *The distribution of the renewal process* $N(t)$ *will be* $Pr\{N(t) = n\} = F_n(t) - F_{n+1}(t)$

Theorem 15: *The Renewal function* $P\{S_n \leq t\}$ *Where waiting time until* n^{th} *event is,* $S_n = \sum_{i=1}^{n} T_i$*will be* $m(t) = E\{N(t)\}$*, where* $m(t)$ *is the mean value of the function and* $m(t) = E[N(t)] = \lambda t h^{b-1}$.

Theorem 16: *If* $N(t)$ *is a Poisson process along with the pdf* $P_n(t) = \frac{e^{-\lambda h^{b-1} t} (\lambda h^{b-1} t)^n}{n!}$ *having the inter-arrival times* $t_1, t_2, \ldots$ *having exponential pdf with the form* $f(t) = \lambda h^{b-1} e^{-\lambda t h^{b-1}};\ \ t > 0$ *and inter-arrival distribution.If* $\{N(t), t \geq 0\}$ *is a Renewal Process of a Stochastic Process thenMean,*$E[N(t)] = Variance, V[N(t)] = \lambda t h^{b-1}$.

Theorem 17: *The average renewal rate by time t converges with probability* 1 *to* $\lambda h^{b-1} t$*as* $t \to \infty$*. Such that* $\lim_{t\to\infty} \left\{\frac{N(t)}{t} \to \lambda h^{b-1} t\right\} \xrightarrow{W.P} 1$ *where,* $E(T_n) = \frac{1}{\lambda h^{b-1} t} \leq \infty$.

Conclusion

The twenty first century makes the lives fast and furious. People are concerned about reducing the consequences of the effect of the occurrences used to happen less frequently or rarely. Ensuring the best emergency services may demand the prior knowledge of forecasting the local and global intensities of the random behavior of the occurrences of rare hazards. The knowledge of rare events may ensure a good public health and healthy economy.The family of counting process plays an important role to model the rare events with the probabilistic approach. This is one of the most important random processes in probability theory.The goal of this work is to develop and study the properties of the family of Generalized counting processes and its consequences in building the generalized Renewal process, etc. Attempts have been made to find the differences between the properties of usual processes and the generalized processes.

The generalized forms of the Poisson process, Renewal process etc can be unfolded through the proper specification of the parameters of the Poisson processes. The explicit form of the various generalized process can be applied to several capricious situations in multiple disciplines. The way of generalization of the Poisson process and Renewal process etc. can be vastly extended due to the inequality of the shape parameters or of the small time interval of two Poisson processes $(b_1 \neq b_2)$ and/or $(h_1 \neq h_2)$.

Appendix A

Table A1: Explicit forms of the Poisson process and its properties

$\boldsymbol{P[N(h)=0]}$	$\boldsymbol{1-\lambda h^b+o(h^b)}$	$\boldsymbol{1-\lambda h^b+o(h^b)}$	
		$\boldsymbol{b=1}$	$\boldsymbol{b=2}$
No. of events occurred by time t , $P_n(t)$	$\frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1}t)^n}{n!}$	$\frac{e^{-\lambda t}(\lambda t)^n}{n!}$	$\frac{e^{-\lambda ht}(\lambda ht)^n}{n!}$
Inter-arrival Density function $f(t)=F'(t)$	$\lambda h^{b-1}e^{-\lambda h^{b-1}t}$; $t>0$	$\lambda e^{-\lambda t}$	$\lambda h e^{-\lambda ht}$
Probability of occurring event for non-overlapping time	$\frac{e^{-\lambda h^{b-1}[t_1-t_0]}(\lambda h^{b-1}[t_1-t_0])^n}{n!}$	$\frac{e^{-\lambda[t_1-t_0]}(\lambda[t_1-t_0])^n}{n!}$	$\frac{e^{-\lambda h[t_1-t_0]}(\lambda h[t_1-t_0])}{n!}$
Mean, $E[N(t)]$	$\lambda h^{b-1}t$	λt	λht
Variance, $V[N(t)]$	$\lambda h^{b-1}t$	λt	λht
Covariance, $[N(t),N(t+r)]$	$\lambda h^{b-1}t$	λt	λht
Correlation, $[N(t),N(t+r)]$	$\frac{\sqrt{t}}{\sqrt{t+r}}$	$\frac{\sqrt{t}}{\sqrt{t+r}}$	$\frac{\sqrt{t}}{\sqrt{t+r}}$
Joint Conditional distribution, $P[X_1(t)=r/X_1(t)+X_2(t)=n]$	$\binom{n}{r}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-r}$	$\binom{n}{r}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-r}$	$\binom{n}{r}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-r}$
Conditional distribution of waiting time	$\frac{s}{t}$	$\frac{s}{t}$	$\frac{s}{t}$
Waiting time, $f_{S_n}(t)$	$\frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1})^n t^{n-1}}{(n-1)!}$	$\frac{e^{-\lambda t}\lambda^n t^{n-1}}{(n-1)!}$	$\frac{e^{-\lambda ht}(\lambda h)^n t^{n-1}}{(n-1)!}$

Table A2: Mathematical expressions of the Renewal process and its various forms

Characteristics	**For new exponent variant,** $\boldsymbol{f(t)=\lambda h^{b-1}e^{-t\lambda h^{b-1}}}$	$\boldsymbol{f(t)=\lambda h^{b-1}e^{-t\lambda h^{b-1}}}$	
		$\boldsymbol{b=1}$	$\boldsymbol{b=2}$
Mean, $E[N(t)]$	$\lambda t h^{b-1}$	λt	λht
Variance, $Var[N(t)]$	$\lambda t h^{b-1}$	λt	λht
$\lim_{t\to\infty}\frac{N(t)}{t}$	$\frac{1}{\mu}$	$\frac{1}{\mu}$	$\frac{1}{\mu}$
Number of renewals, $N(t)$	$\frac{(t\lambda h^{b-1})^n e^{-t\lambda h^{b-1}}}{n!}$	$\frac{(\lambda t)^n e^{-\lambda t}}{n!}$	$\frac{(\lambda ht)^n e^{-\lambda ht}}{n!}$
Density function, $f(t)$	$\lambda h^{b-1}e^{-\lambda t h^{b-1}}$	$\lambda e^{-\lambda t}$	$\lambda h e^{-\lambda ht}$
Waiting time, S_n	$\frac{t^{n-1}e^{-t\lambda h^{b-1}}(\lambda h^{b-1})^n}{n!}$	$\frac{t^{n-1}e^{-\lambda t}\lambda^n}{n!}$	$\frac{t^{n-1}e^{-\lambda ht}(\lambda h)^n}{n!}$

Appendix B

B1 ***Theorem 1 Proof:*** $P\{N(h)=0\}+P\{N(h)=1\}+P\{N(h)=2\}+\cdots=1\Rightarrow$ $P\{N(h)=0\}=1-P\{N(h)=1\}-P\{N(h)\geq 2\}$

$$\Rightarrow P\{N(h)=0\}=1-\left[\lambda h^b+o(h^b)\right]-o(h^b)$$

$\therefore P\{N(h)=0\}=1-\lambda h^b+o(h^b)$ [Since $o(h^b)$ is very small

Now, probability of$n$ events to be occurred in $(t+h)$ interval of time is as follows where h is very small.

$$\therefore\ P[N(t+h)=n]=P[N(t)=n\ and\ N(h)=0\ or,N(t)=n-1\ and\ N(h)=1\ or,N(t)=n-2\ and\ N(h)=2\ \ldots]$$

$$\Rightarrow P[N(t+h)=n]=\ P\{N(t)=n\ and\ N(h)=0\}+P\{N(t)=n-1\ and\ N(h)=1\}+P\{N(t)=n-2\ and\ N(h)=2\}+\cdots$$

$$=P_n(t)\left[1-\ \lambda h^b+o\left(h^b\right)\right]+P_{n-1}(t)\left[\ \lambda h^b+o\left(h^b\right)\right]+o(h^b)$$

$$\Rightarrow \lim_{h\to 0}\frac{P_n(t+h)-P_n(t)}{h}\ =-\ \lambda h^{b-1}P_n(t)+\lambda h^{b-1}P_{n-1}(t)+\lim_{h\to 0}\frac{o\left(h^b\right)}{h}$$

$\therefore \frac{d}{dt}[P_n(t)]=-\lambda h^{b-1}P_n(t)+\lambda h^{b-1}P_{n-1}(t)$..(3) $\because \lim_{h\to 0}\frac{o(h^b)}{h}=0\ with\ a\ greater\ rate$

If we put $n=1$ in the above equation (3), we get

$$\frac{d}{dt}[P_1(t)]=\ -\lambda h^{b-1}P_1(t)+\lambda h^{b-1}P_0(t) \qquad (4)$$

$$P_0(t)=P[N(t)=0]\therefore P_0(t+h)=P[N(t+h)=0]=P_0(t)\left[1-\lambda h^b+o\left(h^b\right)\right]$$

$$\Rightarrow \lim_{h\to 0}\frac{P_0(t+h)-P_0(t)}{h}\ =-\lambda h^{b-1}P_0(t)+\lim_{h\to 0}\frac{o\left(h^b\right)}{h}$$

$$\therefore\ \frac{d}{dt}[P_0(t)]=P_0'(t)=-\lambda h^{b-1}P_0(t)\ \therefore \frac{P_0'(t)}{P_0(t)}=-\lambda h^{b-1}$$

By integrating the above equation,

$$\Rightarrow \int \frac{P_0'(t)}{P_0(t)}\ dt=-\ \lambda h^{b-1}\int dt \Rightarrow \ln P_0(t)=\ -\ \lambda t h^{b-1}+C\ \therefore P_0(t)=e^{-\ \lambda t h^{b-1}+C}$$

If we put $t=0,\ P_0(0)=e^{-\ \lambda h^{b-1}0+C}\Rightarrow e^0=e^C\ \therefore C=0\ \therefore\ P_0(t)=e^{-\ \lambda h^{b-1}t}$

So from equation (4) we get,

$$\frac{d}{dt}[P_1(t)]=P_1'(t)=-\lambda h^{b-1}P_1(t)+\lambda h^{b-1}P_0(t)=-\lambda h^{b-1}P_1(t)+\lambda h^{b-1}e^{-\ \lambda h^{b-1}t}$$

We can rewrite equation (3)as, $\frac{d}{dt}[P_n(t)]=\ P_n'(t)=-\lambda h^{b-1}P_n(t)+\lambda h^{b-1}P_{n-1}(t)$

$$\Rightarrow\ P_n'(t)+\lambda h^{b-1}P_n(t)\ =\lambda h^{b-1}P_{n-1}(t)$$

By multiplying $e^{\ \lambda h^{b-1}t}$ in both sides we get-

$$e^{\ \lambda h^{b-1}t}[P_n'(t)+\lambda h^{b-1}P_n(t)]\Rightarrow \frac{d}{dt}[e^{\ \lambda h^{b-1}t}\ P_n(t)]=\lambda h^{b-1}e^{\ \lambda h^{b-1}t}\ P_{n-1}(t)$$

If we put $n=1$ the value of $P_0(t)$

$$\frac{d}{dt}[e^{\ \lambda h^{b-1}t}\ P_1(t)]=\lambda h^{b-1}e^{\ \lambda h^{b-1}t}\ P_0(t)=\ \lambda h^{b-1}e^{\ \lambda h^{b-1}t}e^{-\ \lambda h^{b-1}t}\ \ =\lambda h^{b-1}$$

By integrating the above equation,

$$\int \frac{d}{dt}[e^{\ \lambda h^{b-1}t}\ P_1(t)]\,dt=\ \lambda h^{b-1}\int dt \Rightarrow e^{\ \lambda h^{b-1}t}\ P_1(t)=\lambda h^{b-1}t+C=\lambda h^{b-1}t$$

[We know, C=0] $\therefore P_1(t)=\lambda h^{b-1}t\ \ e^{-\lambda h^{b-1}t}$

If we put $n=2$ and use the value of $P_1(t)=\lambda h^{b-1}t\ \ e^{-\lambda h^{b-1}t}$ we get-

$$\frac{d}{dt}[e^{\ \lambda h^{b-1}t}\ P_2(t)]=\lambda h^{b-1}e^{\ \lambda h^{b-1}t}\ P_1(t)=\ (\ \lambda h^{b-1})^2 t$$

Since we know, C=0; By integrating the above equation we get,

$$\int \frac{d}{dt}\left[e^{\ \lambda h^{b-1}t}\ P_2(t)\right]dt=\ (\ \lambda h^{b-1})^2\int t dt \Rightarrow e^{\ \lambda h^{b-1}t}\ P_2(t)=(\ \lambda h^{b-1})^2\left[\frac{t^2}{2}+C\right]$$

$$\therefore\ P_2(t)=\ e^{-\lambda h^{b-1}t}\left[\frac{(\ \lambda h^{b-1}t)^2}{2!}+0\right]=\ \frac{e^{-\lambda h^{b-1}t}(\ \lambda h^{b-1}t)^2}{2!}$$

If we put $n=n+1$ we get-

$$\frac{d}{dt}[e^{\lambda h^{b-1}t}\, P_{n+1}(t)] = \lambda h^{b-1} e^{\lambda h^{b-1}t} \frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1}t)^n}{n!} = (\lambda h^{b-1})^{n+1}\frac{t^n}{n!}$$

By integrating the above equation, $\int \frac{d}{dt}\left[e^{\lambda h^{b-1}t}\, P_{n+1}(t)\right] dt = \frac{(\lambda h^{b-1})^{n+1}}{n!}\int t^n dt$

$$\Rightarrow e^{\lambda h^{b-1}t}\, P_{n+1}(t) = \frac{(\lambda h^{b-1})^{n+1}}{n!\,(n+1)}[t^{n+1} + C] \therefore\; P_{n+1}(t) = e^{-\lambda h^{b-1}t}\frac{(\lambda h^{b-1}t)^{n+1}}{(n+1)!}$$

Since the following equation is true for $n = 1, 2, 3, \dots, n+1$ so, according to the iterative method we can claim that the following equation, $P_n(t) = \frac{e^{-\lambda h^{b-1}t}(\lambda h^{b-1}t)^n}{n!}$; is true for $n = 1, 2, 3, \dots, \infty$ and $b \geq 1$.

B2 ***Theorem 2 Proof:*** If $n\ and\ k$ are the number of trials in any random experiment and success within a time interval h^b then binomial distribution with parameter (n, p) is
$n \rightarrow$No. of trials, $k \rightarrow$no. of success within time h^b;
This forms a Binomial distribution from Counting process k; i.e., $P(n) = \binom{n}{k} p^k q^{n-k}$.
$If\, \frac{k}{n} = p \rightarrow 0 => n \rightarrow \infty\ \therefore\ n.\frac{k}{n} \rightarrow \infty. 0 = constant = \lambda h^{b-1}$, So it becomes a Poisson random variable with PMF, $P(n) = \frac{e^{-n.\frac{k}{n}}\left(n.\frac{k}{n}\right)^k}{k!} = \frac{e^{-\lambda h^{b-1}}(\lambda h^{b-1})^k}{k!}$

$$As\ np \rightarrow \lambda h^{b-1}; \frac{\llcorner n}{\llcorner (n-k) \llcorner x}\left(\frac{np}{n}\right)^k \left(1 - \frac{np}{n}\right)^{n-k} = \frac{e^{-np}(np)^k}{k!} = \frac{e^{-\lambda h^{b-1}}(\lambda h^{b-1})^k}{k!}$$

$$k \rightarrow Bernoulli; \sum k_i = S_1 \rightarrow Binomial; k_1 + k_2 = S_2 \rightarrow Poissontime, k\ \rightarrow infinite$$

$$P(K = k) = \frac{e^{-np}(np)^k}{k!} = \frac{e^{-\lambda h^{b-1}}(\lambda h^{b-1})^k}{k!}; \qquad k = 1, 2, 3, \dots, \infty$$

B3 ***Theorem 3 Proof:*** Given $X_1 = x_1, X_2 = x_2, \dots, X_{n-1} = x_{n-1}$ arrival $n-1$ occurs at time, $t_{n-1} = x_1 + x_2 + \cdots + x_{n-1}$. For $x > 0$, $X_n > x$ if and only if there are no arrivals in the interval $(t_{n-1}, t_{n-1} + x)$. The number of arrivals in $(t_{n-1}, t_{n-1} + x)$ is independent of the past history described by $X_1, X_2, \dots, X_{n-1}$. This implies

$$P(X_n > x / X_1 = x_1, X_2 = x_2, \dots, X_{n-1} = x_{n-1}) = P[N(t_{n-1} + x) - N(t_{n-1}) = 0]$$

$$\therefore P(X_n > x) = \frac{e^{-\lambda h^{b-1}x}(\lambda h^{b-1}x)^0}{0!} = e^{-\lambda h^{b-1}x} \qquad (5)$$

So the distribution function, $F(x) = 1 - P(X_n > x) = 1 - e^{-\lambda h^{b-1}x}$

$$\frac{d}{dx}F(x) = 0 - (-\lambda)e^{-\lambda h^{b-1}x} \qquad \therefore f(x) = \lambda h^{b-1}e^{-\lambda h^{b-1}x}$$

Thus X_n is independent of $X_1, X_2, \dots, X_{n-1}$. So the waiting time distribution $f(x)$ follows exponential distribution which is one kind of Poisson distribution.
We will derive theorems of generalized Poisson process for this new exponential variant.

B4 ***Theorem 4 Proof:*** Let t_i be the random variable representing the interval between two successive occurrences of Poisson process $[\mathrm{N(t)}, \mathrm{t} \geq 0]$ and let $F(t)$ be its distribution function. We know from the previous theorem (*); $t \sim \exp\ (\lambda, b)$ with probability density function, $f(t) = F'(t) = \lambda h^{b-1}e^{-\lambda t h^{b-1}}$; $\qquad t > 0$
Then the distribution of S_n, waiting time until the n^{th} event occurs is denoted by

$$f\left[S_n(t) = \sum_{i=1}^{n} t_i\,;\ \text{where}\ f(t) = \lambda h^{b-1}e^{-\lambda t h^{b-1}}\right] = \frac{(\lambda h^{b-1}t)^{n-1}\lambda h^{b-1}e^{-\lambda h^{b-1}t}}{(\mathrm{n}-1)!}$$

This follows Gamma distribution, that is, $f_{S_n}(t) = \frac{t^{n-1}(\lambda h^{b-1})^n e^{-\lambda h^{b-1}t}}{\Gamma n} \quad ; t \geq 0$

B5 ***Theorem 6 Proof:*** Let the number of arrival in any time is $N(t_i)$ for $i = 2, \dots, k$. Then in $t_1, t_2, \dots, t_{k-1}, t_k$ time events are $N(t_1), N(t_2), \dots, N(t_{k-1}), N(t_k)$ where $t_1 < \cdots < t_{k-1} < t_k$. If the arrival of any two consecutive times interval is $N(t_i) - N(t_{i-1})$. By the definition of Poisson process, the joint probability mass function of total events is

$$P_{N(t_1),N(t_2),\dots,N(t_{k-1}),N(t_k)}(n_1, \dots, n_k) = P\{N(t_1)\}P\{N(t_2)\} \dots P\{N(t_{k-1})\}P\{N(t_k)\}$$
$$= P[N(t_1) = n_1]P[N(t_2) = n_2|N(t_1) = n_1] \dots P[N(t_{k-1}) = n_{k-1}|N(t_{k-2}) = n_{k-2}]P[N(t_k) = n_k|N(t_{k-1}) = n_{k-1}] \quad (6)$$
$$[Since, P(A|B) = P(A) when\ P(A) and\ P(B) are\ independent$$
$$= P[N(t_1) = n_1]P[N(t_2 - t_1) = n_2 - n_1] \dots P[N(t_{k-1} - t_{k-2}) = n_{k-1} - n_{k-2}]P[N(t_k - t_{k-1}) = n_k - n_{k-1}]$$
$$= \frac{e^{-\lambda h^{b-1}t_1}\left(\lambda h^{b-1}t_1\right)^{n_1}}{n_1!} \cdot \frac{e^{-\lambda h^{b-1}[t_2-t_1]}\left(\lambda h^{b-1}[t_2 - t_1]\right)^{n_2-n_1}}{(n_2 - n_1)!} \dots \frac{e^{-\lambda h^{b-1}[t_k-t_{k-1}]}\left(\lambda h^{b-1}[t_k-t_{k-1}]\right)^{n_k-n_{k-1}}}{(n_k-n_{k-1})!}$$

Where $n_1 \leq n_2 \leq \cdots \leq n_k$

The probability of an arrival during any instant is independent of the past history of the process.

B6 ***Theorem 7 Proof:*** If X_n denotes the time between the $(n-1)^{st}$ and the n^{th} event then the probability of n^{th} inter-arrival is $P(X_n > t) = e^{-\lambda h^{b-1}t}$; if and only if no more than $(n-1)^{st}$ event occur before t as it follows exponential distribution. If t' is the additional time and given that, number of occurrence by time t', the additional time until the arrival $X_n - t'$, has the same exponential distribution as X_n .The conditional probability that $X_n - t' > t$ given $X_n > t'$, is

$$P[X_n - t' > t|X_n > t'] = \frac{P[X_n > t + t',\ X_n > t']}{P[X_n > t']} = \frac{P[X_n > t + t']}{P[X_n > t']} = \frac{e^{-\lambda h^{b-1}(t+t')}}{e^{-\lambda h^{b-1}t'}} = e^{-\lambda h^{b-1}t} = P(X_n > t) \text{for all } t, t' > 0$$

That is, no matter how long we have waited for the arrival, the remaining time until the arrival always has an exponential distribution with mean $\frac{1}{\lambda h^{b-1}}$. That is, if the lifetime (or the holding time) has exceeded the value t' then, conditionally, the residual lifetime $X_n - t'$ is still $EXP(\lambda)$. So the memory less property of the Poisson process can also be seen in the exponential inter-arrival Times.

B7 ***Theorem 8 Proof:*** Mean, $E[N(t)] = \sum_{n=0}^{\infty} n \frac{e^{-\lambda h^{b-1}t}\left(\lambda h^{b-1}t\right)^n}{n!} = \lambda h^{b-1}t$

Variance, $Var[N(t)] = E[N(t)]^2 - [E\{N(t)\}]^2 = E[\{N(t)\}\{N(t-1)\}] + \mathrm{E}[N(t)] - [E\{N(t)\}]^2 = \sum_{n=0}^{\infty} n(n-1) \frac{e^{-\lambda h^{b-1}t}\left(\lambda h^{b-1}t\right)^n}{n!} + \lambda h^{b-1}t - \left(\lambda h^{b-1}t\right)^2 = \lambda h^{b-1}t$

We know, $Corr[N(t), N(t+r)] = \frac{Cov[N(t),N(t+r)]}{\sqrt{Var[N(t)]Var[N(t+r)]}}$

Now, Covariance, $Cov[N(t), N(t+r)] = E[N(t), N(t+r)] - E[N(t)]E[N(t+r)]$

$$\therefore E[N(t), N(t+r)] = E[N(t)]E[N(r)] + \{E[N(t)]\}^2 + V[N(t)]$$
$$= \lambda h^{b-1}t\lambda h^{b-1}r + (\lambda h^{b-1}t)^2 + \lambda h^{b-1}t$$

$[the\ occurance\ for\ the\ number\ of\ events\ for\ the\ time (0, t] is\ independent\ that\ for\ the\ time\ (t, r)$

$$\therefore Cov[N(t), N(t+r)]$$
$$= \lambda h^{b-1}t\lambda h^{b-1}r + \lambda h^{b-1}t + \left(\lambda h^{b-1}t\right)^2 - \lambda h^{b-1}t\lambda h^{b-1}(t+r)$$
$$= \lambda h^{b-1}t \qquad \therefore Cov[N(t), N(t+r)] = \lambda h^{b-1}t$$

$$\therefore Corr[N(t), N(t+r)] = \frac{\lambda h^{b-1}t}{\sqrt{\lambda h^{b-1}t\lambda h^{b-1}(t+r)}} = \frac{t}{\sqrt{t(t+r)}} = \frac{\sqrt{t}}{\sqrt{t+r}} = \sqrt{\frac{t}{t+r}}$$

Now, moment generating function, $M_{N(t)}(t) = E\left(e^{t'N(t)}\right) = \sum_{n=0}^{\infty} \frac{e^{nt'}e^{-\lambda h^{b-1}t}\left(\lambda h^{b-1}t\right)^n}{n!}$

$= e^{\lambda h^{b-1}t(e^{t'}-1)}$ ...(7) $\quad [Since, \sum_{n=0}^{\infty} \frac{\left(e^{t'}\lambda h^{b-1}t\right)^n}{n!} is\ the\ power\ series\ for\ e^{e^{t'}\lambda h^{b-1}t}$

B8 ***Theorem 9 Proof:*** Let the number of events occurred in t_1, t_2, t_3 ... time interval is $n_1, n_2, n_3, \ldots$ respectively. So the number of events occurred by $t_{1,}t_2$ is $N(t_1) = n_1$, $N(t_2) = n_2$. Then the inter-arrival time $(t_2 - t_1)$ indeed produces$(n_2 - n_1)$ number of events occurred.Now the probability of n_3 event in the time interval t_3; given that $n_1\ and\ n_2$ events occur at $t_1\ and\ t_2$ time interval respectively; will be

$$\therefore P[N(t_3) = n_3|N(t_2) = n_2, N(t_1) = n_1];\ t_1 < t_2 < \cdots < t_n$$

$$= \frac{\frac{e^{-\lambda h^{b-1}t_1}(\lambda h^{b-1}t_1)^{n_1}}{n_1!} \cdot \frac{e^{-\lambda h^{b-1}(t_2-t_1)}[\lambda h^{b-1}(t_2-t_1)]^{n_2-n_1}}{(n_2-n_1)!} \cdot \frac{e^{-\lambda h^{b-1}(t_3-t_2)}[\lambda h^{b-1}(t_3-t_2)]^{n_3-n_2}}{(n_3-n_2)!}}{\frac{e^{-\lambda h^{b-1}t_1}(\lambda h^{b-1}t_1)^{n_1}}{n_1!} \cdot \frac{e^{-\lambda h^{b-1}(t_2-t_1)}[\lambda h^{b-1}(t_2-t_1)]^{n_2-n_1}}{(n_2-n_1)!}}$$

$$= \frac{e^{-\lambda h^{b-1}(t_3-t_2)}[\lambda h^{b-1}(t_3-t_2)]^{n_3-n_2}}{(n_3-n_2)!}\ [from\ theorem\ equation\ (6)$$

The process $N(t_3)$ only depends on the immediate past $N(t_2)$. So, the process $N(t_3)$ is a Markov process. Thus we can say that $P[N(t_k) = n_k|N(t_{k-1}) = n_{k-1}, \ldots, N(t_1) = n_1] = P[N(t_k) = n_k|N(t_{k-1}) = n_{k-1}]$. So, $N(t)$ is a Markov Process.

B9 ***Theorem 10 Proof:*** $N_1(t)\ and\ N_2(t)$ follows Poisson process. Let total number of events occur in total time t is n. That is if $N_1(t) = r, N_2(t) = n - r$.

$$\therefore P[N(t) = n] = P[N_1(t) + N_2(t) = n] = \sum_{r=0}^{n} P[N_1(t) = r, N_2(t) = n - r]$$

$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 h^{b-1}t}\left(\lambda_1 h^{b-1}t\right)^r}{r!} \cdot \frac{e^{-\lambda_2 h^{b-1}t}\left(\lambda_2 h^{b-1}t\right)^{n-r}}{(n-r)!} = \frac{e^{-(\lambda_1+\lambda_2)h^{b-1}t}[(\lambda_1+\lambda_2)h^{b-1}t]^n}{n!}$ So, $N_1(t) + N_2(t)$ is also a Poisson process.

B10 ***Theorem 11 Proof:*** Let $N_1(t)\ and\ N_2(t)$follows Poisson process. Number of events occur within time t is n. So the difference between these two processes will be

$$M_{N_1-N_2}(t) = E\left[e^{t'(N_1-N_2)}\right] = E\left[e^{t'N_1}\right] - E\left[e^{t'N_2}\right] = e^{\lambda_1 h^{b-1}t\left(e^{t'}-1\right)} \cdot e^{\lambda_2 h^{b-1}t\left(e^{-t'}-1\right)}$$

$$= e^{\lambda_1 h^{b-1}t(e^{t'}-1)} \cdot e^{-\lambda_2 h^{b-1}t(1-\frac{1}{e^{t'}})} = e^{h^{b-1}t(\lambda_1 e^{t'}+\lambda_2 e^{-t'})} \cdot e^{-(\lambda_1+\lambda_2)h^{b-1}t}[From\ theorem\ (8)$$

This is not a Poisson process.

B11 ***Theorem 12 Proof*:** Let $N_1(t)\ and\ N_2(t)$follows Poisson process and total number of events in t is n; if the number of events in $N_1(t)$ is r so the number of events in $N_2(t)$ is $n - r$ in then$\frac{N_1(t)}{N_1(t)+N_2(t)}$will be $P[N_1(t) = r|N_1(t) + N_2(t) = n]$

$$= \frac{\frac{e^{-\lambda_1 h^{b-1}t}\left(\lambda_1 h^{b-1}t\right)^r}{r!} \frac{e^{-\lambda_2 h^{b-1}t}\left(\lambda_2 h^{b-1}t\right)^{n-r}}{(n-r)!}}{\frac{e^{-(\lambda_1+\lambda_2)h^{b-1}t}\left[(\lambda_1+\lambda_2)h^{b-1}t\right]^n}{n!}} = \binom{n}{r}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r \left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-r}$$

$[from\ theorem\ \ (6)]$. Which is the probability mass function of Binomial distribution.

B12 ***Theorem 13 Proof:*** Let T_1denotes the time of the first event. To find the distribution of it, we know the event $\{T_1 < s\}$takes place if and only if, the first event of the Poisson process occur in the interval $[0, s]$and each interval in$[0, t]$ of equal length should have the same probability of containing the event. Thus if the number of event is 1 in t time interval, $P\{T_1 < s|N(t) = 1\} = \frac{P\{T_1<s \,\&\, N(t)=1\}}{P\{N(t)=1\}}$

$$= \frac{P\{1\ event\ occurs\ at\ (0,s) and\ no\ events\ during\ (s,t)\}}{P\{N(t) = 1\}}$$

$$= \frac{P\{N(s) = 1\}\, P\{N(t-s) = 0\}}{P\{N(t) = 1\}} = \frac{\lambda h^{b-1} s e^{-\lambda s h^{b-1}} e^{-\lambda h^{b-1}(t-s)}}{\lambda h^{b-1} t e^{-\lambda h^{b-1} t}} = \frac{s}{t} = constant$$

B13 ***Theorem 14 Proof:*** Suppose that $T_1, T_2, \ldots, T_n$ be a sequence of inter-arrival time. Then, $S_1 = T_1$ =Time until the first renewal occurs

$S_2 = T_1 + T_2$ = Time of the second renewal

$\vdots$

$S_n = T_1 + T_2 + T_3 + \cdots + T_n$ = Time of the n^{th} renewal

If $N(t)$ is the number of independent and identically distributed random variables with a common distribution F, then, the number of renewals, $N(t)$ by time t depends on n^{th} renewal S_n. That is, $N(t) \geq n \leftrightarrow S_n \leq t$. Thus we obtain $Pr\{N(t) = n\} = Pr\{N(t) \geq n\} - Pr\{N(t) \geq n+1\} = Pr\{S_n \leq t\} - Pr\{S_{n+1} \leq t\} = F_n(t) - F_{n+1}(t)$.

B14 ***Theorem 15 Proof:*** If $N(t)$ is the number of independent and identically distributed random variables with an arbitrary common distribution F then the renewal function will be, $m(t) = E[N(t)] = \sum_{n=0}^{\infty} nP_n(t) = \sum_{n=1}^{\infty} n[F_n(t) - F_{n+1}(t)] = \sum_{n=1}^{N(t)} F_n(t)$

Where, $F_n(.)$ is the n^{th} convolution of $F(.)$.

If the inter-arrival time T_n have gamma distribution with density $f(t) = \lambda h^{b-1} e^{-\lambda t h^{b-1}}$.

Then the density function of $S_n = \sum_{i=1}^{n} T_i$ is, $F'(t) = \frac{(\lambda h^{b-1})^n}{\Gamma n} e^{-\lambda h^{b-1} t} t^{n-1}; t \geq 0$.

$F_n(t) = \int_0^t \frac{(\lambda h^{b-1})^n}{\Gamma n} e^{-\lambda h^{b-1} t} t^{n-1} dt = 1 - e^{-\lambda h^{b-1} t} \sum_{r=0}^{n-1} \frac{(\lambda h^{b-1} t)^r}{r!}$

$$\therefore P_n(t) = e^{-\lambda h^{b-1} t} \frac{(\lambda h^{b-1} t)^n}{n!} \therefore m(t) = \sum_{n=0}^{\infty} n \frac{e^{-\lambda h^{b-1} t} (\lambda h^{b-1} t)^n}{n!} = \lambda h^{b-1} t$$

B15 ***Theorem 16 Proof:*** If $\{N(t), t \geq 0\}$ is a Renewal Process of a Stochastic Process then $V[N(t)] = E[N(t)]^2 - \{E[N(t)]\}^2$ ... (1) where $E[N(t)] = \lambda h^{b-1} t$ [Theorem 15

We know the inter-arrival time T_n have gamma distribution with density $f(t) = \lambda h^{b-1} e^{-\lambda t h^{b-1}}$.

Then the density function of $S_n = \sum_{i=1}^{n} T_i$ is $F'(t) = \frac{(\lambda h^{b-1})^n}{\Gamma n} e^{-\lambda h^{b-1} t} t^{n-1}; \quad t \geq 0$

$F_n(t) = 1 - e^{-\lambda h^{b-1} t} \sum_{r=0}^{n-1} \frac{(\lambda h^{b-1} t)^r}{r!}$ And $P_n(t) = e^{-\lambda h^{b-1} t} \frac{(\lambda h^{b-1} t)^n}{n!}$ [From theorem 15

$V[N(t)] = (\lambda h^{b-1} t)^2 + \lambda h^{b-1} t - \left(\lambda h^{b-1} t\right)^2 = \lambda t h^{b-1}$ [From theorem 8

B16 ***Theorem 17 Proof:*** Here the distribution function $F(t) = 1 - P[T > t] = 1 - e^{-\lambda t h^{b-1}}$. $\therefore we\ know, \mu = \int_0^{\infty} t f(t) dt = \int_0^{\infty} t dF(t) = \frac{1}{\lambda h^{b-1} t}$

Let $S_{N(t)+1}$ be the time of the first renewal after time t where time of the last renewal prior to t or at time t is $S_{N(t)}$. Then as $N(t) \leq n < N(t) + 1$; we have, $S_{N(t)} \leq t < S_{N(t)+1}$ Now, $\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} \therefore \frac{S_{N(t)}}{N(t)} = \frac{T_1 + T_2 + \cdots + T_{N(t)}}{N(t)} \Longrightarrow \frac{S_{N(t)}}{N(t)} = \frac{T_1}{N(t)} + \frac{T_2 + \cdots + T_{N(t)}}{N(t)}$.

If we take limit $, \lim_{N(t)\to\infty} \frac{S_{N(t)}}{N(t)} = \lim_{N(t)\to\infty} \frac{T_1}{N(t)} + \lim_{N(t)\to\infty} \frac{T_2 + \cdots + T_{N(t)}}{N(t)} = 0 + \frac{1}{\lambda h^{b-1} t} = \frac{\sum_{n=1}^{N(t)} T_n}{N(t)}$

$[by\ the\ strong\ law\ of\ large\ number$

So we can write, $\lim_{t\to\infty}\frac{S_{N(t)+1}}{N(t)} = \lim_{t\to\infty}\frac{S_{N(t)+1}}{N(t)+1}.\lim_{t\to\infty}\frac{N(t)+1}{N(t)} = \lim_{t\to\infty}\frac{S_{N(t)+1}}{N(t)+1}\frac{N(t)}{N(t)} =$
$\lim_{t\to\infty}\frac{\sum_{n=1}^{N(t)+1}T_n}{N(t)+1}.1$ [$Since,\ \frac{N(t)+1}{N(t)} \to 1\ \ as\ t\to\infty$]. $So, \lim_{N(t)\to\infty}\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} = \frac{1}{\lambda h^{b-1}t}$
Now from (i) we get, $\lim_{t\to\infty}\frac{S_{N(t)}}{N(t)} \le \lim_{t\to\infty}\frac{t}{N(t)} < \lim_{t\to\infty}\frac{S_{N(t)+1}}{N(t)} \Longrightarrow \frac{1}{\lambda h^{b-1}t} \le \lim_{t\to\infty}\frac{t}{N(t)} < \frac{1}{\lambda h^{b-1}t}$
That is, $\lim_{t\to\infty}\frac{t}{N(t)} \to \frac{1}{\lambda h^{b-1}t} \Longrightarrow \lim_{t\to\infty}\frac{N(t)}{t} \to \lambda h^{b-1}t \therefore \lim_{t\to\infty}\left\{\frac{N(t)}{t} = \lambda h^{b-1}t\right\} \xrightarrow{W.P} 1$

References

[1] Adnan, M.A.S. and Kiser, H. (2011). A Generalization to the Family of Exponential and Beta Distributions.Proceedings.*Joint Statistical Meetings*, American Statistical Association, pp. 2336-2347.

[2] Barbour, A.D. (1988). Stein’s method and poisson process convergence. *J. Appl. Probab.* Vol. 25A, pp. 175-184.

[3] Barbour, A.D. and Brown, T.C. (1992). Stein’s method and point process approximation. *Stochastic Process. Appl.* Vol. 43, pp. 9-31.

[4] Barbour, A.D. and Chen, L.H.Y., eds. (2005). An introduction to Stein’s method. Lecture Notes Series. Institute for Mathematical Sciences. National University of Singapore. Singapore: Singapore University Press.

[5] Bening, V.E. and Korolev, V.Y., (2002), “*Generalized Poisson Models and Their Applications in Insurance and Finance*”, First edition. The Netherlands.

[6] Brown, T.C. and Greig, D. (1994). On Stein’s method and point process approximation.*Stochastic Process. Appl.* Vol. 54, pp. 291-296.

[7] Brown, T.C. and Xia, A. (1995). On metrics in point process approximation. *Stochastics Rep.* 52 247-263.

[8] Brown, T.C. and Xia, A. (1995). On Stein-Chen factors for Poisson approximation.*Statist.Probab.Lett.*Vol. 23, pp. 327-332.

[9] Brown, T.C., Weinberg, G.V. and Xia, A. (2000). Removing logarithm from Poisson process error bounds.*Stochastic Process. Appl.* Vol. 87, pp. 149-165.

[10] Brown, T.C. and Xia, A. (2001). Steins method and birth-death process.*Annals of Probability*.Vol.29, pp. 1373-1403.

[11] Chen, L.H.Y. and Xia, A. (2004). Stein’s method, Palm theory and Poisson process approximation. *Annals of Probability*. Vol. 32, pp. 2545-2569.

[12] Conway, J.H. and Sloane, N.J.A. (1999). Sphere Packings, Lattices and Groups 3rd ed. *Grundlehren der MathematischenWissenschaften [Fundamental Principles of Mathematical Sciences]* 290. New York: Springer.

[13] Famoye, F. and Consul, P.C., (1995). “Bivariate generalized Poisson distribution with some applications”. *Metrika*. Volume 42, Issue 1, pp 127-138.

[14] Fel'dman, G.M., (1983). “Generalized Poisson distribution on groups”.*Journal of Soviet Mathematics*. Volume 23, Issue 3, pp 2335-2351.

[15] Fel'dman, G.M., (1992). “Estimation of Generalized Poisson Distribution by the Method of Weighted Discrepancies”.*Computing Science and Statistics*. Springer New York, pp 332-335.

[16] Feller, W., (1941). “On the Integral Equation of Renewal Theory”. The Annals of Mathematical Statistics. Volume 12, Number 3, pp 243-267.

[17] Girish, M.K. and Hu, J.Q. (2000) “Higher order approximations for the single server queue with splitting, merging and feedback”. European Journal of Operational Research, Vol. 124, pp. 447-467.

[18] Haight, F.A., (1959). “The generalized Poisson distribution”. Annals of the Institute of Statistical Mathematics. Volume 11, Issue 1 Supplement, pp 101-105.

[19] Hubert, P.C., Lauretto, M.S. and Stern, J.M., (2009). “FBST for Generalized Poisson Distribution”.*Bayesian Inference and Maximum Entropy Methods in Science and Engineering*: The 29th International Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering. AIP Conference Proceedings, Volume 1193, pp. 210-217.

[20] Hurtado, J.L., Joglar, F. and Modarres, M., (2005) “Generalized Renewal Process: Models, Parameter Estimation and Applications to Maintenance Problems”. *International Journal of Performability Engineering*. Volume 1, Number 1, Paper 3, pp. 37 – 50.

[21] Ibe, O.C. (2005). *Fundamentals of Applied Probability and Random Processes*. Elsevier.

[22] Jacopino, A., Groen, F. and Mosleh, A., (2004). “Behavioural study of the general renewal process”.*Reliability and Maintainability,* pp 237 – 242.

[23] Jagers, P. and Klebaner, F.C., (2000). “Population-size-dependent and age-dependent branching processes”.*Stochastic Processes and their Applications*. Volume 87, pp 235-254.

[24] Karlin, S. and Taylor, H.M. (1975). *A First Course in Stochastic Process*.Academic Press.

[25] Kijima, M., (2002). “*Generalized Renewal Processes and General Repair Models*”, Third edition.Springer Berlin Heidelberg.

[26] Knill, O. (2009). *Probability Theory and Stochastic Processes with Applications*.Overseas Press.

[27] Kobayashi, H. and Mark, B.L., (2002) “Generalized Loss models and queuing loss networks”. International transactions in Operational Research, Vol. 9, pp. 97-112.

[28] Lekshmi, S. and Thomas, C., (2012). “Some Generalizations of Poisson Processes”.*Journal of Statistical Theory and Applications*. Volume 11, Number 3, pp. 225-235.

[29] Medhi, J., (2002), “*Stochastic Models in Queuing Theory*”, Second edition. Academic Press.

[30] Noortwijk, J.M. and Weide, J.A.M., (2006). "Computational techniques for discrete-time renewal processes".Safety and Reliability for Managing Risk, *Proceedings of European Safety and Reliability Conference 2006*, pages 571-578.

[31] Pollard, D., (1997). "*Poisson Processes*".Chapter 9. Lecture Notes in Statistics.

[32] Pyke, R., (1961). "Markov Renewal Processes with Finitely Many States". *The Annals of Mathematical Statistics*.Volume 32, Number 4, pp 951-1365.

[33] Ross, S.M., (2004), "*Introduction to Probability Models*", Eighth edition. Elsevier.

[34] Ross, S.M., (2013), "*Stochastic Processes*", Second edition. John Wiley and Sons.

[35] Salvador, P., Valadas, R. and Pacheco, A. (2003). Multiscale fitting procedure using markov modulated Poisson processes. *Telecommunications Systems*. Vol. 23, pp. 123-148.

[36] Satterthwaite, F.E., (1942). "Generalized Poisson Distribution".*The Annals of Mathematical Statistics*. Volume 13, Number 4, p 410-417.

[37] Schuhmacher, D. (2005). Estimation of distances between point process distributions. Ph.D. thesis, Switzerland: University of Zurich. Available at, http://www.dissertationen.unizh.ch/2006/schuhmacher/diss.pdf.

[38] Stein, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In, Proceedings of the Sixth Berkeley Symposium on mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability Theory pp. 583-602. California: Univ. California Press, Berkeley.

[39] Suhov, S. and Kelbert, M. (2008). Markov Chains: A primer in *Random Processes and their Applications*. Cambridge Press.

[40] Veerarajan, T., (2008), "*Probability, Statistics and Random Processes*", Third edition. Tata McGraw- Hill.

[41] Wang, Z.M. and Yang, J.G., (2012). "Numerical method for Weibull generalized renewal process and its applications in reliability analysis of NC machine tools". *Journal of Computers and Industrial Engineering*. Volume 63 (4), pp 1128-1134.

[42] Xia, A. (1994). Poisson approximations and some limit theorems for Markov processes. Australia: Ph.D. thesis, University of Melbourne.

[43] Xia, A. (2005). Stein's method and Poisson process approximation. *An Introduction to Stein's Method*. Lecture Notes Series. Institute for Mathematical Sciences. National University of Singapore. Vol. 4, pp. 115-181. Singapore: Singapore University Press.

[44] Yechiali, U., (1973) "A Queuing-Type Birth-And-Death Process Defined on a Continuous-Time Markov Chain". Operations Research, Vol. 21, No. 2, pp. 604-609.

[45] Zhang, Tingting, (2008). Nonparametric studies of doubly stochastic Poisson processes, binomial data and high dimension, low sample size data.Ph.D. Thesis, Harvard 2008, 133 pages; AAT 3334823.