

Yates' Method for Computing Sums of Squares for Nonreplicated Experiments

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Abstract

Yates Method is used to compute the effects of k factors each at two levels on a response variable. This calculation includes the sums of squares and coefficients in a factorial model for nonreplicated experiments. An overview of the Yates algorithm is presented along with different types and uses of Yates inputs and expected outputs. The estimated effect estimates are ranked and used to identify potential models which are validated by analyzing the residuals. Graphical techniques are illustrated and the statistical performances of different approaches are compared. Various formulas for variances are developed using concepts from mathematical statistics.

KeyWords: Yates Method, ANOVA, DOE, nonreplicated experiments, effects

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Introduction

Full and fractional factorial designs are common in designed experiments for engineering and scientific applications (Wiki). In these designs, each factor is assigned two levels typically called the low and high levels. A full factorial design contains all possible combinations of low/high levels for all the factors. A fractional factorial design contains a subset of these combinations chosen by using a defining contrast. A similar algorithm can be used for problems where factors have three levels (low, medium, high).

This paper provides an overview of the Yates Method in analyzing unreplicated experiments for balanced full factorial experiments and balanced fractional experiments with two and three level factors. The Yates method generates least squares estimates of effects for all factors and all interactions of interest. The Yates analysis can be used to determine:

1. a ranked list of important factors
2. the goodness-of-fit measured by the residual standard deviation for the various models

Yates order

Before performing a Yates analysis, the data should be arranged in "Yates order". That is, given k factors, the k^{th} column consists of $2^{(k-1)}$ minus signs (i.e., the low level of the factor) followed by $2^{(k-1)}$ plus signs (i.e., the high level of the factor). Table 1 describes the structure of a design matrix for an experiment with three two level factors.

Table 1. Design Matrix Structure

Column k	Column 2^{k-1}	Column Description
1	1	one negative (-) followed by one positive (+) sign
2	2	two negative (-) followed by two positive (+) signs
3	4	four negative (-) followed by four positive (+) signs

Table 2 presents four different notations used for a design matrix of a full factorial in three factors. Determining the Yates order for fractional factorial designs requires knowledge of the confounding structure of the fractional factorial design.

Table 2. Four Design Matrix Notations

1	2			3			4		
tc	A	B	C	A	B	C	A	B	C
(1)	-	-	-	-1	-1	-1	0	0	0
a	+	-	-	1	-1	-1	1	0	0
b	-	+	-	-1	1	-1	0	1	0
ab	+	+	-	1	1	-1	1	1	0
c	-	-	+	-1	-1	1	0	0	1
ac	+	-	+	1	-1	1	1	0	1
bc	-	+	+	-1	1	1	0	1	1
abc	+	+	+	1	1	1	1	1	1

Yates Algorithms. Two algorithms are presented. The first is for a 2^k design and the second is for a 3^k design. Both algorithms have the same purpose -perform the sums of squares calculations for an ANOVA table and compute the effects.

Yates algorithm for computing the contrasts in a 2^k design

1. Arrange the k factors in standard order.
2. Successively pair consecutive responses (averages or totals)
3. Generate a new column whose first $k/2$ entries are the sums of the successive pairs of entries and whose next $k/2$ entries are the differences (second entry – first entry) for the two response entries in each pair.
4. Generate a second column from the new column in step 3 by performing the same additions and subtractions on successive pairs of entries in the new column.
5. Working each newly generated column, continue the patterned addition and subtraction of successive pairs of entries until a total of k new columns have been generated.
6. Effects can be normalized (standardized) three different ways.

- a. After k cycles, divide the k^{th} column by the square root (2^k). Milliken and Johnson use this approach in *Analysis of Messy Data, Volume 2 Nonreplicated Experiments (AMD2)*.
- b. Divide the first entry in the k^{th} column by 2^k ; divide the remaining entries in this column by 2^{k-1} . This column of values now contains the constant effect (mean), the main effects, and interaction effects of the model factors. The following authors have used this approach: Montgomery, Box, Hunter and Hunter, and Mason, Gunst, and Hess.
- c. After k cycles, the k^{th} column contains the total of the treatment means and the effect contrasts with \pm coefficients applied to the treatment means. These results are in standard order. To get the grand mean and effects, divide column k by $2^k = n$. Oehlert uses this approach.

Yates algorithm for computing the contrasts in a 3^n design.

1. Arrange the factors in standard order.
2. Successively group response averages into groups of three observations each.
3. Fill in k columns using Yates algorithm on groups of three.
 - Fill in upper 1/3 of each column with the sum of three entries in a group of three
 - Fill in middle 1/3 of each column with the differences of the 3rd entry in a group – 1st entry in a group of three. (linear component)
 - Fill in lower 1/3 of each column by 1st entry in a group – 2 times 2nd entry in a group + 3rd entry in a group of three. (quadratic component)
4. There are k steps starting with an initial column of responses (raw data). The column is updated k times. The last $(k+1)^{\text{th}}$ step involves a standardization of each effect in the model.

Parameter estimates as terms are added to a linear model

In most cases of least squares fitting, the model coefficients for previously added terms change depending on what was successively added. For example, the X_1 coefficient might change depending on whether or not an X_2 term was included in the model. This is not the case when the design is orthogonal, as is a 2^3 full factorial design.

For orthogonal designs, the estimates for the previously included terms do not change as additional terms are added. This means the ranked list of effect estimates simultaneously serve as the least squares coefficient estimates for progressively more complicated models.

Model selection and validation using Yates Method (Wiki)

An important part of a Yates analysis is selecting the best model from the available list of potential models. Determining the most appropriate model requires balancing the following two goals.

1. We want the model to include all important factors and interactions and to omit the unimportant factors and interactions.
2. We want the model to be parsimonious (as simple as possible).

Note that the residual standard deviation alone is insufficient for determining the most appropriate model as it will always be decreased by adding additional factors. Instead, seven criteria are utilized to define important factors. These seven criteria are not all equally important, nor will they necessarily yield identical subsets, in which case a consensus subset or a weighted consensus subset must be extracted. In practice, some of these criteria may not apply in all situations, and some analysts may have additional criteria. These criteria are given as useful guidelines. Most analysts will focus on those criteria that they find most useful.

- Practical significance of effects as determined by a subject matter specialist
- Statistical significance of effects (1/2 normal plot)
- Order of magnitude of effects (ranks of important effects by size)
- Probability plots of effects (normal plot)
- Practical significance of residual standard deviation
- Statistical significance of residual standard deviation (various models can be compared)

The first four criteria focus on effect sizes with two numeric criteria and two graphical criteria. The last two criteria focus on the residual standard deviation of the model. Once a tentative model has been selected, the error term should follow the assumptions for a univariate measurement process. That is, the model should be validated by analyzing the residuals from various models.

Graphical presentations

A Yates analysis is often accompanied by a number of graphical techniques. In particular, the following plots and charts may be useful: DOE mean plot, QQ plot, normal plot, half-normal plot and Pareto charts.

Example 1. An analysis procedure for data obtained in a nuclear pilot plant study as seen in Box, Hunter and Hunter's Statistics for Experimenters.

A pilot plant run consists of (1) cleaning the reactor, (2) inserting the appropriate catalyst charge, (3) running the apparatus at a given temperature, pressure and feed concentration for 3 hours to allow the process to settle down at the chosen experimental conditions, (4) sampling the output every 15 minutes during the final hour of running, and (5) combining chemical analyses made in these samples into a single response variable conversion (%).

The data is a full fraction in four variables each having two levels (2^4 design). Only unbiased parameter estimates of model coefficients are used in computing effects; an effect = 2 * model coefficient. The variables are seen in Table 3.

A DOE mean plot showing response means at low and high values of the predicting variables (Catalyst charge (lb.), Temperature ($^{\circ}$ C), Pressure (psi), and Concentration (%)) is given in figure 1.

In preparation for a Yates Analysis the data is put into standard Yates order. The actual order of the 16 experiments was randomized. See table 4 below.

Table 3. Variable list –Pilot Plant Study

#	Nickname	Variable	Low (-)	High (+)
1	A	Catalyst charge(lb.)	10	20
2	B	Temperature (°C)	220	240
3	C	Pressure (psi)	50	80
4	D	Concentration (%)	10	12
	Y Chemical = response	Conversion (%)	Continuous random variable	

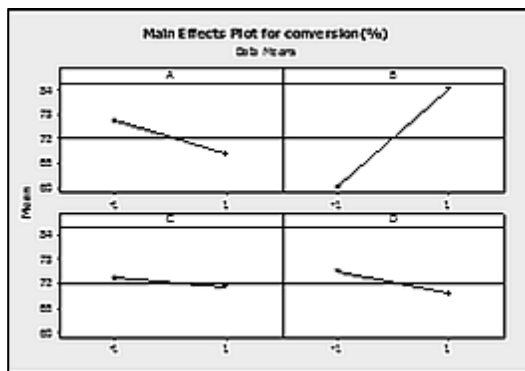


Figure 1 DOE Main effects Plot

Table 4. Design matrix for Pilot Plant Data in standard (Yates) order

Std Order	Intercept and Factors					% Conv	Order of Runs
	I	A	B	C	D		
1	1	-1	-1	-1	-1	71	(8)
2	1	1	-1	-1	-1	61	(2)
3	1	-1	1	-1	-1	90	(10)
4	1	1	1	-1	-1	82	(4)
5	1	-1	-1	1	-1	68	(15)
6	1	1	-1	1	-1	61	(9)
7	1	-1	1	1	-1	87	(1)
8	1	1	1	1	-1	80	(13)
9	1	-1	-1	-1	1	61	(16)
10	1	1	-1	-1	1	50	(5)
11	1	-1	1	-1	1	89	(11)
12	1	1	1	-1	1	83	(14)
13	1	-1	-1	1	1	59	(3)
14	1	1	-1	1	1	51	(12)
15	1	-1	1	1	1	85	(6)
16	1	1	1	1	1	78	(7)
Effect Contrast	1156	-64	192	-18	-44		

Procedure to find sigma using the half- normal plot (Pilot Plant Data)

- 1 . Do a Yates ANOVA on an unrestricted model. A good estimate of error variance will be obtained by running the half-normal plot two times.
- 2 . Construct a half-normal plot of the effects using the method described in Analysis of Messy Data, Vol. 2 – Nonreplicated Experiments (AMD2). The method is nicknamed DEJ in this paper.

The significance of an effect is determined graphically using the half normal plot. This involves computation of a variable named V. V is the quantile from a half-normal distribution associated with a probability P. V is constructed using the ranks and absolute value of the effects (|X_i|) as follows:

$$\begin{aligned} \text{Let } R_i &= \text{rank of } |X_i| \\ \text{Then } R_i^* &= (R_i - .5) / n \\ P_i &= .5(R_i^* + 1) \\ V &= p^{\text{th}} \text{ quantile from a } n(0,1) \text{ pdf} \end{aligned}$$

Each effect has a (P, V) pair. The absolute value of an effect vs its V value is plotted in the half-normal plot. Significance of an effect is based upon whether the (|X_i|, V) pair has a value of V greater than one. This is discussed in AMD2. Some pertinent relationships are:

$$\frac{SS(effect)}{\sigma^2} \sim ChiSquare(1) \qquad \sqrt{\frac{SS(effect)}{\sigma^2}} \sim half - normal(0,1)$$

Significant effects noted in the half normal plot below are B, A, D, BD, and C. These are removed and a new half normal plot is run without the significant effects. See steps three and four.

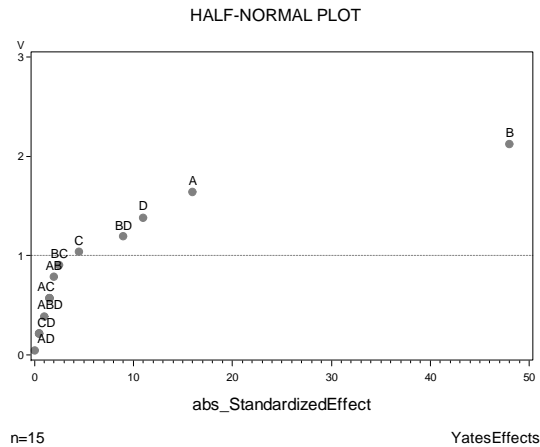


Figure 2 Half –Normal Plot of the effects

- 3 . Remove significant effects from the dataset. (- B, A, D, BD, C)

4. Construct a second half-normal plot (- significant effects). As expected the remaining points are close about the fitted regression line. This is a check on whether the correct significant factors are found.

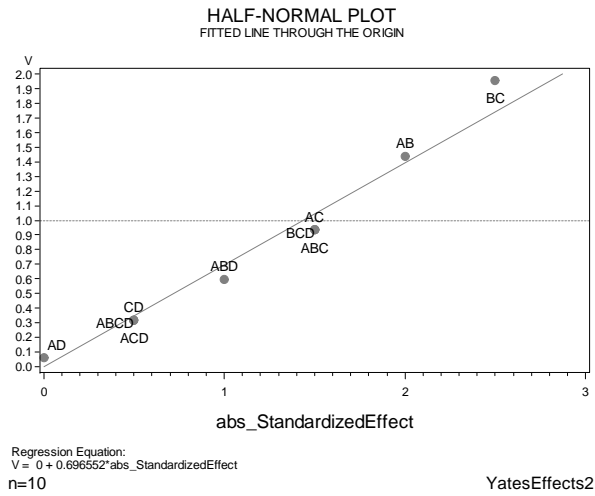


Figure 3 Half-Normal plot to estimate sigma.

5. Find $\hat{\sigma}$. Estimated sigma is 1.42. It is found by dropping a vertical line from the intersection of the regression and horizontal lines.
6. Compute $\hat{\sigma}^2$ ($= 2.02$). This is an approximation based upon the effects.
7. Compare $\hat{\sigma}^2 = 2.02$ to the mean square of the four way interaction term $A*B*C*D = .25$ from a conventional ANOVA.
8. Check results with a half- normal plot using the parameter estimates of the beta vector.

Figure 4 indicates that B, B*D, A and B*C are statistically significant if we use the cutoff line of $V=1$.

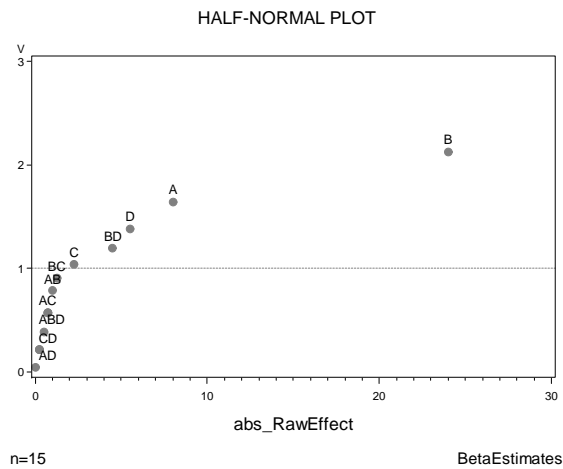


Figure 4 Half- normal plot based upon Parameter Estimates

The overall ANOVA and type III SS's are the same for models having categorical predictors and models having continuous predictors. However the parameter estimates differ. The unbiased estimates of model coefficients obtained using continuous predictors are presented in table 5.

Standardizing Effects. In the literature there are different methods of standardizing the effects in a Yates analysis. The methods differ in how they standardize the data in the final step. I give the methods the nicknames DEJ, BOX, MONT, OEH, and HT. MGH refers to the method of using the parameter estimates from the ANOVA as opposed to using Yates Algorithm. Table 5 is the ANOVA with continuous predictors for the nuclear pilot plant data.

Table 5. Nuclear Pilot Plant Study ANOVA with continuous predictors

Source	DF	Type III SS	SS / MS	Unbiased Parameter Estimate	Standardized Effect (DEJ)	Standardized Effect (Box, Montgomery, Mason, Gunst and Hess)
Model	15	2801.000	186.733			
Intercept				72.250		72.25
A	1	256.000	256.000	-4.000	-16.0	-8
B	1	2304.000	2304.000	12.000	48.0	24
A*B	1	4.000	4.000	0.500	2.0	1
C	1	20.250	20.250	-1.125	-4.5	-2.25
A*C	1	2.250	2.250	0.375	1.5	0.75
B*C	1	6.250	6.250	-0.625	-2.5	-1.25
A*B*C	1	2.250	2.250	-0.375	-1.5	-0.75
D	1	121.000	121.000	-2.750	-11.0	-5.5
A*D	1	0.000	0.000	0.000	0.0	0
B*D	1	81.000	81.000	2.250	9.0	4.5
A*B*D	1	1.000	1.000	0.250	1.0	0.5
C*D	1	0.250	0.250	-0.125	-0.5	-0.25
A*C*D	1	0.250	0.250	-0.125	-0.5	-0.25
B*C*D	1	2.250	2.250	-0.375	-1.5	-0.75
A*B*C*D	1	0.250	0.250	-0.125	-0.5	-0.25
Error	0	0.000				
Corrected Total	15	2801.000				

The equivalent Yates analysis is given in Table 7. The raw effects computing using Yates Method are divided by a constant to compute the equivalent standardized or normalized effects. Divisors seen in six different textbooks that are used with Yates analyses are given in table 6 below. Table 7 is an application of these methods for the pilot plant study.

Table 6. Normalization (Standardization) Methods

Author	Method Nickname	Divisor*	Stdz effect	MS=SS=Effect ²
Dallas E. Johnson	DEJ	\sqrt{n}	Contrast / \sqrt{n}	Yes
Box, Montgomery and Mason Gunst and Hess	BOX, MONTE, MGH	n , n/2	Contrast/(n,n/2)	No
Hicks & Turner	HT	$r2^{f-1}$	Contrast/($r2^{f-1}$)	No
Hicks & Turner	HT (1/2 fraction)	$.5r2^{f-1}$	Contrast/($.5r2^{f-1}$)	No
Oehlert	OEH	n	Contrast/ 2^f	No

* f=number of factors, r=number of observations per treatment combination (tc),
Unbiased parameter estimate=coefficient=effect/2

Table 7. Pilot Plant Study

%conc				Raw Effect	Box Mont	DEJ stdz	OEH Stdz	HT Stdz		
step0	step1	step2	step3	step4	Step5	Step6	Step7	Step8	SS / Mean Square	Effect
71	132	304	600	1156	72.25	289	72.25	144.5	$2801=162+482+22+\dots(-.5)^2$	Total Mean
61	172	296	556	-64	-8	-16	-4	-8	$256=(-16)^2$	A
90	129	283	-32	192	24	48	12	24	$2304=(48)^2$	B
82	167	273	-32	8	1	2	0.5	1	$4=(2)^2$	AB
68	111	-18	78	-18	-2.25	-4.5	-1.125	-2.25	$20.25=(-4.5)^2$	C
61	172	-14	114	6	0.75	1.5	0.375	0.75	$2.25=(1.5)^2$	AC
87	110	-17	2	-10	-1.25	-2.5	-0.625	-1.25	$6.25=(-2.5)^2$	BC
80	163	-15	6	-6	-0.75	-1.5	-0.375	-0.75	$2.25=(-1.5)^2$	ABC
61	-10	40	-8	-44	-5.5	-11	-2.75	-5.5	$121=(-11)^2$	D
50	-8	38	-10	0	0	0	0	0	0	AD
89	-7	61	4	36	4.5	9	2.25	4.5	$81=(9)^2$	BD
83	-7	53	2	4	0.5	1	0.25	0.5	1	ABD
59	-11	2	-2	-2	-0.25	-0.5	-0.125	-0.25	$.25=(-0.5)^2$	CD
51	-6	0	-8	-2	-0.25	-0.5	-0.125	-0.25	$.25=(-0.5)^2$	ACD
85	-8	5	-2	-6	-0.75	-1.5	-0.375	-0.75	$2.25=(-1.5)^2$	BCD
78	-7	1	-4	-2	-0.25	-0.5	-0.125	-0.25	$.25=(-0.5)^2$	ABCD

(Step)

(0) = raw data

(4) = the grand total in 1st row followed by contrasts in std order

(5) = estimated effects using Box and Montgomery's approach

(6) = DEJ Standardized Effects

 $(6)^2 = SS(\text{DEJ})$ DEJ standardized effect squared $(4)^2/2^4 = SS(\text{Box})$ Raw effect squared divided by 2^f

It is to be noted that we can compute the SS by squaring the unstandardized effects in step4 and dividing by 2^4 or by squaring the standardized effect computed using the DEJ method.

Estimation of the mean square error (MSE).

Estimation of the mean square error (MSE) is done in one of three ways. Pooling methods used described by Box and Montgomery give different results than those of Natrella.

1. Graphical method using the half-normal plot described above was developed by Milliken and Johnson (AMD2).
2. Pooling to find a good estimate of σ^2 as described by both Box and Montgomery.

If we assume all three and four factor interactions are negligible we can compute an estimate for the mean square error by first summing the squares of these effects and then dividing by the number of assumed negligible terms. The estimated standard error of an effect is computed as the square root. The pilot data set is pooled in table 8.

3. Computational method seen in Experimental Statistics by Mary Gibbons Natrella

This approach involves finding the sum of the squares of the effects corresponding to interactions of three or more factors. Divide the sum of squares by $v \cdot 2^n$ where v is the number of interactions included to find s^2 . Taking the square root gives an estimate of the standard error (RMSE). See table 8 below.

Complete factorial experiments having fewer than six factors should be analyzed by the half-normal plot technique while those with six or more factors should be analyzed by pooling the higher-order interactions to estimate the experimental error variance.

The pooling method of Natrella is illustrated below with only four factors to illustrate its use and to compare it to results obtained using Box's technique.

Table 8. Estimation of the mean square error (MSE) for the pilot plant data

Effect Name	Raw Effect	Box Stdz Effect	Raw Effect ²	Box and Montgomery MSE	Natrella MSE
ABC	-6	-.75	.5625		
ABD	4	.50	.2500		
ACD	-2	-.25	.0625		
BCD	-6	-.75	.5625		
ABCD	-2	-.25	.0625		
SUM			1.500		
Divisor				5	$5 \times 2^4 = 80$
MSE				$1.5 / 5 = .30$	$1.5 / 80 = 0.18745$
RMSE				0.5500	$\text{sqrt}(0.18745) = 0.1369$

Significance testing of main effects and interactions.

Significance testing of main effects and interactions can be done by computing $w = (2^n)^{\frac{1}{2}} t_{v,1-\alpha/2} s$ where v = error degrees of freedom. If the absolute value of any main effect or interaction estimate obtained using Yates $> w$, it is concluded that the effect is different from zero.

Three level experiments.

Example 2. Yates Algorithm for the 3^2 battery design study as seen in Montgomery, Design and Analysis of Experiments -8th edition, Example 5.1 table 5.4.

Table 9. Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature(°F)								y_i		
	15		70		125						
1	130	155	539	34	40	229	20	70	230	998	
	74	180		80	75		82	58			
2	150	188	623	136	122	479	25	70	198	1300	
	159	126		106	115		58	45			
3	138	110	576	174	120	583	96	104	342	1501	
	168	160		150	139		82	60			
Totals			1738			1291				3799	= $y_{...}$
			$y_{.i}$								

This 3^2 design investigates the effect of material type (A) and temperature (B) on the life of a battery in hours. There are $r = 4$ replicates. The Yates algorithm for three level factors described earlier is used. The sums of squares (SS) seen in the conventional ANOVA analysis are subdivided using orthogonal polynomials. See table 10 below.

Table 10. Battery Design ANOVA with Orthogonal Polynomials

Source (dependent variable is hour)	DF	SS	MS	F	Pr > F	R-Square	CV	Mean
Model	8	59416.22	7427.03	11.00	<.0001	0.765	24.624	105.528
Material Type A	2	10683.72	5341.86	7.91	0.0020			
AL	1	10542.04						
AQ	1	141.68						
Temperature B	2	39118.72	19559.36	28.97	<.0001			
BL	(1)	39042.67						
BQ	(1)	76.06						
Material Type*Temperature AB	4	9613.78	2403.44	3.56	0.0186			
ALBL	1	351.56						
ALBQ	1	6510.02						
AQBL	1	1963.52						
AQBQ	1	788.67						
Error	27	18230.75	675.21					
Corrected Total	35	77646.97						

Table 10 gives the following SS breakdown:

$$A = AL + AQ = 10542.04 + 141.68 = 10,683.72$$

$$B = BL + BQ = 39042.66 + 76.06 = 39,118.7$$

$$A*B = ALBL + ALBQ + AQBL + AQBQ = 2351.56 + 1963.52 + 6510.02 + 788.67 = 9613.77$$

Orthogonal polynomial contrasts are used to measure the linear, quadratic, cubic, etc. effects of the quantitative levels. Material type (A) is qualitative; thus, the linear and quadratic partitioning of A is not appropriate. However by coding it numerically we can compute linear and quadratic components which match the Yates analysis. Only the main effect of material type A is of interest here.

A simpler and faster way to do this is to use the Yates procedure as displayed in Table 11. The treatment combinations are written down in standard order; that is, the factors are introduced one at a time, each level being combined successively with every set of factor levels above it in the table.

The divisors are computed using the fact that four (n) replicates are in each treatment combination. Using this fact we have two methods of standardizing a three level design.

1. Dallas Johnson in AMD2 standardizes by dividing all effects by $\sqrt{4d}$, where d corresponds to the sum of products of the individual effect contrast coefficients. The square of DEJ standardized effects are equal to the SS from the conventional ANOVA as before. A half-norm plot is appropriate for this data since the standardized effects are normally distributed.

2. Montgomery, Douglas C. (2013) in Design and Analysis of Experiments Supplementary Material Chapter 9 recommends:

The entries in the divisor column are found from $2^r 3^t n$ where r is the number of factors in the effect considered, t is the number of factors in the experiment minus the number of linear terms in this effect, and n is the number of replicates. For example, B_L has the divisor $2^1 \times 3^1 \times 4 = 24$.

Table 11. Yates Algorithm for the 3² Battery Design

tc	(0) Data	(1)	(2)	(3) Effect	(4) Montgomery Divisor 2 ^r 3 ^t n	(5) Montgomery StdzEffect (2) / 2 ^r 3 ^t n	(6) DEJ Divisor d	(7) DEJ StdzEffect $\frac{ (2) }{\sqrt{n d}} = \frac{ (2) }{\sqrt{4 d}}$	(8) SS
00	539	1738	3799	Total	2 ¹ × 3 ² × 4 = 36	3799/√36 = 633.17 (Mean)	9	3799/√(4 × 9) = 633.17 (Mean)	
10	623	1291	503	A _L	2 ¹ × 3 ¹ × 4 = 24	503/√24 = 102.67	6	503/√(4 × 6) = 102.67	10542
20	576	770	-101	A _Q	2 ¹ × 3 ² × 4 = 72	-101/√72 = -11.90	18	-101/√(4 × 18) = -11.90	141.68
01	229	37	-968	B _L	2 ¹ × 3 ¹ × 4 = 24	-968/√24 = -197.59	6	-968/√(4 × 6) = -197.59	39042.70
11	479	354	75	AB _{LXL}	2 ² × 3 ⁰ × 4 = 16	75/√16 = 18.75	4	75/√(4 × 4) = 18.75	351.56
21	583	112	307	AB _{QXL}	2 ² × 3 ¹ × 4 = 48	307/√48 = 44.31	12	307/√(4 × 12) = 44.31	1963.52
02	230	-131	-74	B _Q	2 ¹ × 3 ² × 4 = 72	-74/√72 = -8.72	18	-74/√(4 × 18) = -8.72	76.06
12	198	-146	-559	AB _{LXQ}	2 ² × 3 ¹ × 4 = 48	-559/√48 = -80.68	12	-559/√(4 × 12) = -80.68	6510.02
22	342	176	337	AB _{QXQ}	2 ² × 3 ² × 4 = 144	337/√144 = 28.08	36	337/√(4 × 36) = 28.08	788.67

*r is the number of factors in the effect considered, t is the number of factors in the experiment minus the number of linear terms in the effect considered and n is the number of replicates.

(0) = raw data (totals)

(2) = the grand total is in first entry followed by contrasts in standard order

(5) = estimated effects using Montgomery approach

(7) = DEJ Standardized Effect

(8) = (7)² = SS (DEJ)

(8) = (2)² / (4) = SS (MONT)

Concluding Discussion

When working with Yates method it is assumed that the datasets are balanced and the contrasts are orthogonal. The sums of squares are additive in the analysis of variance computed using Yates method.

It is important to use the correct form of the variance. The equations can be classified by design type (unreplicated, replicated).

Unreplicated Designs

An expression given by Dallas Johnson for a simple effect of the difference of two means in a 2^k unreplicated experiment is

$$\begin{aligned} V(\text{effect}) &= V(\text{difference in 2 means}) & SE(\text{effect}) \\ &= \left(\frac{1}{2^k} + \frac{1}{2^k} \right) \hat{\sigma}^2 = \frac{2}{2^k} \hat{\sigma}^2 = \frac{\hat{\sigma}^2}{2^{k-1}} & = \frac{\hat{\sigma}}{\sqrt{2^{k-1}}} \end{aligned}$$

Note that when there is no replication it is appropriate to use pooling to find a good estimate of σ^2 or use the half-normal plots.

Replicated Designs.

Montgomery in the Design and Analysis of Experiments develops the theory behind the computation of an estimate of the variance of an effect for a 2^k factorial with n replicates at each run.

At each of the 2^k experimental conditions assume that there are n independent observations with variance estimate. These are pooled to obtain an overall variance estimate.

$$s^2 = \frac{1}{2^k} \sum_{i=1}^{2^k} s_i^2 = \frac{1}{2^k} \sum_{i=1}^{2^k} \sum_{j=1}^n \frac{(y_{ij} - \bar{y}_i)^2}{n-1} = \frac{1}{(n-1)2^k} \sum_{i=1}^{2^k} \sum_{j=1}^n (y_{ij} - \bar{y}_i)^2$$

This estimate is used to find the variance and estimated standard error of an effect. The standard errors of all the model coefficients are equal because the design is orthogonal.

$$V(\text{effect}) = V\left(\frac{\text{contrast}}{n2^{k-1}}\right) = \frac{1}{(n2^{k-1})^2} V(\text{contrast}) = \frac{1}{n^2 2^{2k-2}} n2^k \hat{\sigma}^2 = \frac{2^2}{n2^k} \hat{\sigma}^2 = \frac{4}{n2^k} \hat{\sigma}^2$$

Estimated s.e. (effect) = $2 \frac{s}{\sqrt{n2^k}} = \frac{s}{\sqrt{n2^{k-1}}} = 2s.e.(\hat{\beta}_{\text{effect}})$ in a regression model for the 2^k design.

A variation of this is discussed in Ledolter and Hogg in Applied Statistics for Engineers and Physical Scientists. Using the same data assumptions as given above, his approach uses

the overall variance s^2 in the computations. $n2^k$ is used as the divisor for all of the effects and the overall mean.

$$V(\text{effect})=V(\text{average})=\frac{1}{(2^k)^2} \sum_{i=1}^{2^k} \frac{s^2}{n} = \frac{1}{(2^k)^2} (2^k) \frac{s^2}{n} = \frac{s^2}{n2^k}$$

Thus the theory and data assumptions are important and must match when computing these variances. With repetition in the use of Yates procedure, there is a marked increase of speed in doing these analyses.

Acknowledgements

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Dr. William Woodall and Roanoke-Radford Chapter of ASQ
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