# Multi-Factor Discretely Observed Vasicek Term Structure Models with non-Gaussian Innovations and Its Applications to the Japanese Government Bond Markets

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**Abstract**

In this paper, we propose a multi-factor model in which the discretely observed short-term interest rates follow a non-Gaussian and dependent process. The state space formulation has the advantages of taking into account both the cross-sectional and time-series restrictions on the data and measurement errors in the observed yield curve. Clarifying the non-Gaussianity and dependency of the dynamics of short-term interest rates, we show that these features are important to capture the dynamics of the observed yield curve. Applications to the estimation of the Japanese government bond yield are illustrated.

**Key Words:** Asymptotic expansion, Kalman filter state space models, term structure models, Vasicek model

## 1. Introduction

Term structure of interest rates describes the relationship between the yield on a zero-coupon bond and its maturity. Learning about the nature of bond yield dynamics plays a critical role in monetary policy, derivative pricing and forecasting, and risk-management analysis. It is necessary to capture accurately the term structure of interest rates in order to evaluate the price of interest rate derivatives. A number of theoretical term structure models have been proposed in the literature. The early models which are still widely used include these by Vasicek[13] and Cox et al.[4].

Although single-factor Vasicek model has been widely used in the theoretical literature, empirical research reports that it fails to appropriately capture the behavior of short rates. The aim of this paper is to develop a closed-form valuation for pricing zero-coupon bonds for the multi-factor Vasicek term structure models where the innovations of underlying short rate processes have non-Gaussian and dependent processes. Honda et al.[8] and Shiohama and Tamaki[12] consider the higher-order asymptotic valuation for zero-coupon bonds and the European call options on zero-coupon bonds using single-factor discretely observed Vasicek models with non-Gaussian and dependent error structure. Miura et al.[10] develop a closed-form valuation for pricing defaultable bonds incorporating a stochastic risk-free interest rate and defaultable intensity processes have non-Gaussian and dependent processes.

The estimation of out proposed non-Gaussian term structure modeling is formulated via state-space representation which involves the specification of measurement system and transition sytem. This state-space modeling is estimated using Kalman filtering methods for one-, two-, three-factor models using the Japanese government bond yields data. Examples for the use of Kalman filtering methods for the estimation of the term structure of interest rates include Duan and Simonato[6], O'Sullivan[11], and Date and Wang[5]. Nowman[9] estimates one-, two-, three-factor models using Japanese monthly data and finds the evidence that the two-, and three-factor models provides a good description of the Japanese yield curves. We extend Nowman[9] results where the underlying short rates

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dynamics have non-Gaussian innovation, and find that the non-Gaussian modeling is have better performance compared with usual Gaissian models for the one-, two-, three-factor modeling.

This paper is organized as follows: Section 2 explains the multi-factor term structure models with discretely observed Vasicek models with non-Gaussian innovations. The analytic expression for the approximate zero-coupon bond prices is obtained. Section 3 discusses the state-space formulation of the model and the estimation procedures. Section 4 presents the data used and the empirical results for the proposed models are illustrated. Finally, some conclusions are offered in Section 5.

## 2. The Multi-Factor Models

The model for the analysis is the discretely sampled short rates with interval $\Delta$. The spot interest rate is assumed to be the sum of $K$ state variables $X_{j,t}$

$$r_t = \sum_{j=1}^{K} X_{j,t},$$

and the state variable are driven by the non-Gaussain and dependent innovations. These models are considered in Honda et al.[8], Shiohama and Tamaki[12], and Miura et al.[10]. The factors $X_{j,t}$ are of the form

$$X_{j,t} - X_{j,t-1} = \kappa_j(\mu_j - X_{j,t-1})\Delta + \Delta^{1/2} Z_{j,t}, \quad j = 1, \ldots, K, \tag{1}$$

where $Z_{j,t}$ are independent such that $E[Z_{i,t}Z_{j,t}] = 0$ for $i \neq j$, $\mu_j$ are the long-term mean of $X_{j,t}$, $\kappa_j$ are their mean reversion parameters. The innovations $\{Z_{j,t}\}$ are forth order stationary in the following sense.

**Assumption 1** *For $j \in \{1, 2, \ldots, K\}$, the process $\{\boldsymbol{Z} = (Z_{1,t}, \ldots, Z_{K,t})'\}$ is fourth-order stationary in the sense that*

1. $E[Z_{j,t}] = 0$,
2. *cum*$(Z_{j,t}, Z_{j,t+u}) = c_{Z_j}(u)$,
3. *cum*$(Z_{j,t}, Z_{j,t+u_1}, Z_{j,t+u_2}) = c_{Z_j}(u_1, u_2)$,
4. *cum*$(Z_{j,t}, Z_{j,t+u_1}, Z_{j,t+u_2}, Z_{j,t+u_3}) = c_{Z_j}(u_1, u_2, u_3)$.

**Assumption 2** *The $k$-th order cumulants $c_{Z_j}(u_1, \ldots, u_{k-1})$ of $Z_{j,t}$, $j = 1, \ldots, K$, for $k = 2, 3, 4$ satisfy*

$$\sum_{u_1, \cdots, u_{k-1} = -\infty}^{\infty} |c_{Z_j}(u_1, \ldots, u_{k-1})| < \infty.$$

Assumptions 1 and 2 are satisfied by a wide class of time series models containing the usual bivariate ARMA and GARCH processes.

Hereafter we assume that the current time is set at $t = 0$, and that the initial factors $X_{j,0}$ are observable and fixed. Then $r_t$ is discretely sampled at times $0, \Delta, 2\Delta, \ldots, n\Delta (\equiv T)$ over $[0, T]$. For the notational convenience, we use following notation. Let

$$A_{j,u} = \mu_j(u\Delta - B_{j,u}), \qquad B_{j,u} = \frac{1}{2\kappa_j}(1 + \upsilon_j)(1 - \upsilon_j^u),$$

$$a_{j,u} = \frac{2}{\kappa_j \Delta}\left\{1 - \frac{1}{2}\upsilon_j^{u-1}(1 + \upsilon_j)\right\},$$

where $\upsilon_j = 1 - \kappa_j\Delta$ for $j = 1, \dots, K$ and $u = 1, \dots, n$. Then it follows from Honda et al.[8] that

$$\begin{aligned}
P(0,T) &= E_0^Q\left[\exp\left(-\int_0^T r_t dt\right)\right] = E_0^Q\left[\exp\left(-\sum_{j=1}^K \int_0^T X_{j,t}dt\right)\right] \\
&= \prod_{j=1}^K E_t^Q\left[\exp\left(\int_0^T X_{j,t}dt\right)\right] \approx \prod_{j=1}^K E_0^{\tilde{Q}}\left[\exp\left\{-\Delta\left(\frac{1}{2}r_0 + \sum_{u=1}^{n-1} r_u + \frac{1}{2}r_n\right)\right\}\right] \\
&= \prod_{j=1}^K \exp(-A_{j,n} - B_{j,n}r_0)AF_{j,n}
\end{aligned}$$

where $E_0^{\tilde{Q}}$ is the expectation under the asymptotic risk-neutral measure, which is discussed in Miura et al.[10], and

$$AF_{j,n} = E_0^{\tilde{Q}}\left[\exp\left(-\frac{\Delta^{3/2}}{2}\sum_{u=1}^n a_{j,u}Z_{j,n-u+1}\right)\right]. \tag{2}$$

Let

$$Y_{j,n} = \Delta^{1/2}\sum_{u=1}^n b_{j,u}Z_{j,n-u+1} \quad \text{and} \quad b_{j,u} = \frac{\Delta}{2}a_{i,j} = \frac{1}{\kappa_j}\left\{1 - \frac{1}{2}\upsilon_j^{u-1}(1+\upsilon_j)\right\}. \tag{3}$$

Using the process $\{Y_{j,n}\}$, we express the product of the $AF_{j,n}$ terms as

$$\prod_{j=1}^K AF_{j,n} = E_0^{\tilde{Q}}\left[\exp\left(-\sum_{j=1}^K Y_{j,n}\right)\right].$$

We give an analytic approximation of the zero coupon bond prices for the multi-factor discretely observed Vasicek term structure models with non-Gaussian and dependent innovations by the Edgeworth expansion of the joint density function of $\boldsymbol{Y}_n = (Y_{1,n}, \dots, Y_{K,n})'$. It is easy to observe that the processes $\{Y_{j,n}\}$, $j = 1, \dots, K$ are fourth-order stationary with $\mathrm{Var}(Y_{j,n}) = \sigma_{j,n}^2$, and the third and fourth order cumulant is denoted by

$$\mathrm{cum}(Y_{j,n}, Y_{j,n}, Y_{j,n}) = n^{-1/2}C_{Y_j}^{(3)} \ \text{ and } \ \mathrm{cum}(Y_{j,n}, Y_{j,n}, Y_{j,n}, Y_{j,n}) = n^{-1}C_{Y_j}^{(4)}.$$

We need following assumption.

**Assumption 3** *The $J$-th order ($J \geq 5$) cumulants of $\{Y_{j,n}\}$, $j = 1, \dots, K$ are of order $O(n^{-J/2+1})$.*

Since we calibrate this model to the market interest rates, we need to include the risk premium before we pricing the zero-coupon bonds. We assume that the $j$th factor's market price of risk $\lambda_j$ is constant and define $\bar{\mu}_j = \mu_j - \lambda_i\sigma_{X_j}/\kappa_j$.

By using the asymptotic expansion for the defaultable bond price of Miura et al.[10], we can derive the following formula for the nominal price of a pure discount bond with face value 1 maturing at time $T$.

**Theorem 1** *Under Assumptions 1–3, the current bond price of the $K$-factor discretely observed Vasicek term structure model is expressed as*

$$P(0,T) = \exp\left(A(T) - \sum_{j=1}^K B_{j,n}X_{j,0}\right)D(T) \tag{4}$$

*where*

$$A(T) = \sum_{j=1}^{K} A_{j,n} = \sum_{j=1}^{K} \left[ \bar{\mu}_j (n\Delta - B_{j,n}) + \frac{1}{2}\sigma_{j,n}^2 \right],$$
$$D(T) = \prod_{j=1}^{K} \exp\left( -\frac{1}{6\sqrt{n}} C_{Y_j}^{(3)} + \frac{1}{24n} C_{Y_j}^{(4)} \right),$$
$$B_{j,n} = \frac{1}{2\kappa_j}(2 - \kappa_j\Delta)(1 - (1 - \kappa_j\Delta)^n).$$

The proof of Theorem 1 is omitted, since it is analogous to the results obtained from Honda et al.[8] and Miura et al.[10].

The analytic expressions for the bond price and yield given in Theorem 1 are based on the discrete time models with non-Gaussian and dependent innovations. According to this expression, the linkage between continuous and discrete scheme for short rate models are apparent. If $Z_{j,t}$s are standard normal distribution, then as $\Delta \to 0$, bond price tends to the standard multi-factor Vasicek term structures.

## 3. State Space Representation and Estimation

The application of Kalman filtering methods in the estimation of term structure models using cross-sectional and time series data has been investigated by Duan and Simonate[6], Chen and Schott[3], and Babbs and Nowman[1, 2].

To estimate the model, we use the state-space representation of the term structure models with non-Gaussian innovations. Our proposed models is discrete scheme with non-Gaussian driven innovations, hence the corresponding state-space model is also non-Gaussian, however the Kalman filter can still be applied to obtain approximate moments of the model and the resulting filter is quasi-optimal.

Let $R_t(\tau)$ denote the continuously compounded yield on a zero-coupon bond of maturity $\tau$ with corresponding discrete sample size $\tau/\Delta = n$. The state-space formulation of the model consists of the measurement and transition equations. To construct measurement equation, we need $N$ zero-coupon rates and use the following relationship between the zero-coupon yield and the price of zero-coupon bonds.

$$R_t(\tau) = -\frac{\ln P(0,\tau)}{\tau} = -\frac{1}{\tau}\left( (-A(\tau) + \ln D(\tau)) + \sum_{i=1}^{K} B_{j,n} X_{j,n} \right).$$

Then the measurement equation has the following form with $K = 3$

$$\begin{bmatrix} R_t(\tau_1) \\ R_t(\tau_2) \\ \vdots \\ R_t(\tau_N) \end{bmatrix} = \begin{bmatrix} \frac{A(\tau_1)+\ln D(\tau_1)}{\tau_1} \\ \frac{A(\tau_2)+\ln D(\tau_2)}{\tau_2} \\ \vdots \\ \frac{A(\tau_N)+\ln D(\tau_N)}{\tau_N} \end{bmatrix} + \begin{bmatrix} \frac{B_{1,n_1}}{\tau_1} & \frac{B_{2,n_1}}{\tau_1} & \frac{B_{3,n_1}}{\tau_1} \\ \frac{B_{1,n_2}}{\tau_2} & \frac{B_{2,n_2}}{\tau_2} & \frac{B_{3,n_2}}{\tau_2} \\ \vdots & \vdots & \vdots \\ \frac{B_{1,n_N}}{\tau_N} & \frac{B_{2,n_N}}{\tau_N} & \frac{B_{3,n_N}}{\tau_N} \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t,1} \\ \varepsilon_{t,2} \\ \vdots \\ \varepsilon_{t,N} \end{bmatrix},$$

or

$$\boldsymbol{R}_t = \boldsymbol{A}(\boldsymbol{\Psi}) + \boldsymbol{H}(\boldsymbol{\Psi})\boldsymbol{X}_t + \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\Psi}$ denotes the unknown parameter vectors to be estimated and $\boldsymbol{\varepsilon}_t \sim N(\boldsymbol{0}, \boldsymbol{V}_\varepsilon)$ with $\boldsymbol{V}_\varepsilon = \mathrm{diag}(h_1^2, \ldots, h_N^2)$.

To obtain the transition equation for the state-space model, we need conditional mean and variance of the state variable process. Using recursive substitution in (1) and remind that $v_j = 1 - \kappa_j \Delta$, $X_{j,n}$ can be represented as

$$X_{j,n} = (1 - v_j^n)\bar{\mu}_j + v_j^n X_{j,0} + \Delta^{1/2} \sum_{u=1}^{n} v_j^{u-1} Z_{j,n-u+1}.$$

For simplicity, we assume that sequence $\{Z_{j,n}\}$ is i.i.d. with zero mean and finite variance $\sigma^2_{Z_j}$. Then the variance of $X_{j,n}$ becomes

$$\sigma^2_{X_j} = \sigma^2_{Z_j} \left[ \frac{1 - v_j^{2(n-1)}}{2\kappa_j - \kappa_j^2 \Delta} \right]. \tag{5}$$

The exact discrete-time models is a VAR(1), and the transition system as follows

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{bmatrix} = \begin{bmatrix} \bar{\mu}_1 \kappa_1 \Delta \\ \bar{\mu}_2 \kappa_2 \Delta \\ \bar{\mu}_3 \kappa_3 \Delta \end{bmatrix} + \begin{bmatrix} 1 - \kappa_1 \Delta & 0 & 0 \\ 0 & 1 - \kappa_2 \Delta & 0 \\ 0 & 0 & 1 - \kappa_3 \Delta \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{3,t-1} \end{bmatrix} + \begin{bmatrix} \eta_{t,1} \\ \eta_{t,2} \\ \eta_{t,3} \end{bmatrix},$$

or

$$\boldsymbol{X}_t = \boldsymbol{C}(\boldsymbol{\Psi}) + \boldsymbol{F}(\boldsymbol{\Psi}) X_{t-1} + \boldsymbol{\eta}_t(\boldsymbol{\Psi})$$

where $\eta_t \sim N(\mathbf{0}, \boldsymbol{V}_\eta)$ with $\boldsymbol{V}_\varepsilon = \mathrm{diag}(\sigma^2_{X_1}, \ldots, \sigma^2_{X_K})$.

Now that we have placed our models in state-space form, we can construct the Kalman filter for the three-factor model in which we want to minimize the mean squared error between $R_t(\tau_i)$ and $\widehat{R_t}(\tau_i)$.

**Example** Let $\{Z_{j,t}\}$ follows a GARCH(1,1) process

$$Z_{j,t} = h_j^{1/2} \varepsilon_{j,t}, \quad h_{j,t} = \omega_j + \alpha_j Z^2_{j,t-1} + \beta_j h_{j,t-1},$$

where $\{\varepsilon_{t,j}\}$ is a sequence of i.i.d. standard Normal random variables. The parameter values must satisfy $\omega_j > 0$, $\alpha_j, \beta_j \geq 0$, $\alpha_j + \beta_j < 1$, and $1 - 2\alpha_j^2 - (\alpha_j + \beta_j)^2 > 0$. Accordingly, $\sigma_{X_j^2}$ in (5) should be

$$\sigma^2_{X_j} = \frac{\omega_j}{1 - \alpha_j - \beta_j} \left[ \frac{1 - v_j^{2(n-1)}}{2\kappa_j - \kappa_j^2 \Delta} \right].$$

$C^{(3)}_{Y_j}$ and $C^{(4)}_{Y_j}$ in the definition of $D(T)$ in Theorem 1 should become

$$\begin{aligned} C^{(3)}_{Y_j} &= 0, \\ C^{(4)}_{Y_j} &= \frac{3}{n} \int_{-\pi}^{\pi} |B_{j,2}(\lambda)^2| f_{Z_j^2}(\lambda) \mathrm{d}\lambda - 2 \frac{3\{(1 - (\alpha_j + \beta_j)^2)\}}{1 - (\alpha_j + \beta_j)^2 - 2\alpha_j^2} \frac{1}{n} \sum_{u=1}^{n} b^4_{j,u}, \end{aligned}$$

where $B_2(\lambda) = \sum_{u=1}^{n} b^2_{j,u} e^{\boldsymbol{i} j \lambda}$ and

$$f_{Z_j{}^2,2}(\lambda) = \frac{\sigma^2_{\nu_j}}{2\pi} \frac{1 + \beta_j^2 - 2\beta_j \cos\lambda}{1 + (\alpha_j + \beta_j)^2 - 2(\alpha_j + \beta_j)\cos\lambda}$$

with

$$\sigma^2_{\nu_j} = \frac{2\omega_j^2 (1 + \alpha_j + \beta_j)}{\{1 - (\alpha_j + \beta_j)\}\{1 - 2\alpha_j^2 - (\alpha_j + \beta_j)^2\}}.$$

Using this parametrization in the state space representation, we can estimate the GARCH(1,1) driven multi-factor term structure models explicitly.

| | Vasicek model (a) | | | non-Gaussian model (b) | | | Difference(%) (b)/(a)-1 | | |
|---|---|---|---|---|---|---|---|---|---|
| | $K=1$ | $K=2$ | $K=3$ | $K=1$ | $K=2$ | $K=3$ | $K=1$ | $K=2$ | $K=3$ |
| 3 Month | 31.84 | 2.94 | 1.88 | 28.83 | 4.11 | 1.67 | -9.43 | 39.81 | -11.03 |
| 6 Month | 27.61 | 0.93 | 0.63 | 28.20 | 1.78 | 0.57 | 2.12 | 90.58 | -9.76 |
| 1 Year | 22.57 | 1.29 | 1.23 | 23.16 | 1.20 | 1.23 | 2.64 | -6.45 | -0.31 |
| 2 Year | 15.35 | 5.25 | 3.48 | 13.67 | 3.37 | 3.10 | -10.97 | -35.77 | -10.94 |
| 3 Year | 12.13 | 7.48 | 3.59 | 9.34 | 4.71 | 3.40 | -23.05 | -37.10 | -5.38 |
| 4 Year | 10.49 | 7.87 | 3.33 | 8.35 | 5.72 | 3.21 | -20.41 | -27.34 | -3.73 |
| 5 Year | 11.03 | 7.64 | 3.15 | 9.74 | 6.25 | 3.10 | -11.66 | -18.28 | -1.44 |
| 6 Year | 13.34 | 7.87 | 3.77 | 12.73 | 6.12 | 3.65 | -4.58 | -22.21 | -3.26 |
| 7 Year | 15.17 | 7.56 | 4.79 | 15.22 | 6.06 | 4.66 | 0.29 | -19.80 | -2.82 |
| 8 Year | 19.59 | 9.09 | 6.71 | 19.76 | 7.14 | 6.57 | 0.89 | -21.39 | -2.17 |
| 9 Year | 19.47 | 7.56 | 5.94 | 19.37 | 6.39 | 5.89 | -0.52 | -15.39 | -0.77 |
| 10 Year | 18.29 | 6.78 | 5.36 | 17.84 | 6.34 | 5.39 | -2.43 | -6.43 | 0.52 |
| 15 Year | 13.02 | 10.73 | 10.73 | 14.84 | 13.44 | 11.04 | 14.03 | 25.22 | 2.95 |
| 20 Year | 20.34 | 12.61 | 6.03 | 20.88 | 14.54 | 6.12 | 2.70 | 15.33 | 1.57 |
| 30 Year | 33.38 | 27.95 | 6.88 | 33.51 | 22.86 | 7.15 | 0.41 | -18.22 | 4.05 |
| Total | 283.61 | 123.54 | 67.50 | 275.45 | 110.03 | 66.75 | -2.88 | -10.93 | -1.11 |

**Table 1**: Sum of the squared errors with different maturities and models

## 4. Data Analysis

The data used consist of Japanese Government Bond (JGB) yields which are zero-coupon adjusted obtained from Bloomberg. We use weekly sampled data and set $\Delta = 1/52$. Data cover the period October 1, 1999 to December 27, 2013, a total of $T = 744$ observations. The maturities included are 1/4, 1/2, 1,2,3,4,5,6,7,8,9,10,15,20, and 30 years, a total of $N = 15$ different maturities. Application of the Kalman filter to the one-, two- and three-factor models with discretely observed non-Gaussian models are discussed. For fair comparison, we also estimate corresponding multi-factor Vasicek term structure models.

Table 1 gives the sum of the squared errors for estimated models with various maturities. For the bond yield with $\tau_i$ maturity, the entry in the cell is given by

$$SSE(\tau_i) = \sum_{t=1}^{T} (R_t(\tau_i)^{(obs)} - \widehat{R}_t(\tau_i)^{(model)})^2,$$

and the total mean squared error is calculated as

$$Total\ SSE = \sum_{i=1}^{N} \sum_{t=1}^{T} (R_t(\tau_i)^{(obs)} - \widehat{R}_t(\tau_i)^{(model)})^2.$$

We see from Table 1 that the total sum of the squares errors are small for the non-Gaussian models compared with those corresponding one-, two- three-factors of Vasicek model. As the number of factors increase, the calibration errors get smaller. We also observe that the non-Gaussian models perform better for the maturities no longer than 10 years, whereas for the long maturities Vasicek term structure models perform better. This is because, the distribution of $\{Y_{j,n}\}$ tends be normal as the sample size $n$ increases by the Central Limit Theorem. Hence the non-Gaussian modelling is much better to fit the short maturities of bond yield, where the underlying short rates exhibit highly non-Gaussian behavior.

| parameter | Vasicek model | | | non-Gaussian model | | |
|---|---|---|---|---|---|---|
| | 1-factor | 2-factor | 3-factor | 1-factor | 2-factor | 3-factor |
| $\mu_1$ | -0.290 | 0.140 | 7.330 | 0.189 | -0.873 | -0.802 |
| $\mu_2$ | | 6.030 | -4.370 | | -1.210 | 2.200 |
| $\mu_3$ | | | 3.350 | | | 1.280 |
| $\lambda_1$ | 2.380 | -2.040 | -0.039 | 2.430 | -12.000 | 8.120 |
| $\lambda_2$ | | 0.909 | 3.010 | | 3.720 | -10.000 |
| $\lambda_3$ | | | 1.800 | | | -0.507 |
| $\kappa_1$ | 0.009 | 0.560 | 0.251 | 0.003 | 0.325 | 0.212 |
| $\kappa_2$ | | 0.094 | 0.396 | | 0.093 | 0.415 |
| $\kappa_3$ | | | 0.009 | | | 0.023 |
| $\sigma_1$ | 0.012 | 0.114 | 0.001 | 0.015 | 0.003 | 0.019 |
| $\sigma_2$ | | 0.085 | 0.003 | | 0.183 | 0.019 |
| $\sigma_3$ | | | 4.27E-04 | | | 0.002 |
| $C_1^{(3)}$ | | | | 0.709 | 2.020 | -0.622 |
| $C_1^{(4)}$ | | | | -0.636 | 7.320 | -0.103 |
| $C_2^{(3)}$ | | | | | -1.480 | -0.451 |
| $C_2^{(4)}$ | | | | | 0.487 | 3.110 |
| $C_3^{(3)}$ | | | | | | 0.999 |
| $C_3^{(4)}$ | | | | | | 2.460 |

**Table 2**: Estimates of one-, two- , and three- factor Vasicek models and non-Gaussian models

Table 2 shows the parameter estimation results. As for the sum of the long-run mean levels, the Vasicek models tends to have quite high levels with 6.1% for two-factor and 6.3% for three-factor models, whereas those with non-Gaussian models have -2% and 2.6% for two- and three- factor models, respectively. We see that the three-factor models with non-Gaussian models can appropriately capture the long-run interest rate level. Most of the estimates for the sum of the risk premiams are negative. This is because, in general, the risk in a bond associate with the spot rate is proportional to the sensitivity of the bond price, that is $\partial P(0,T)/\partial X_{j,0} < 0$.

For the skewness effects on the zero-coupon yield can be seen as the parameter values of $C_j^{(3)}$, and these values vary from -1.5 to 2.0 among ont-, two-, and three-factors. According to these values, we see that the effect of the skewness of underlying innovation process is small. On the other hand, the kurtosis effect on the zero-coupon yields is apparent for some factors in two- and three- factor models.

The observed term structure of Japanese Government Bond (JGB) yield with fitted yield curve with various models estimated are displayed in Figure 1. We choose JGB yield of December 20, 2013 as an example. This figure shows a typical shape for the JGB yield under the Quantitative and Qualitative easing policy with low interest rate level for short maturities. According to these two figures, the fitting performances for the non-Gaussian modelling is superior to those with Vasicek term structure modelling.

## 5. Summary and Conclusions

In this paper we have introduced the multi-factor discretely observed Vasicek term structure models, and presented a method to estimate these models by the Kalman filter. The

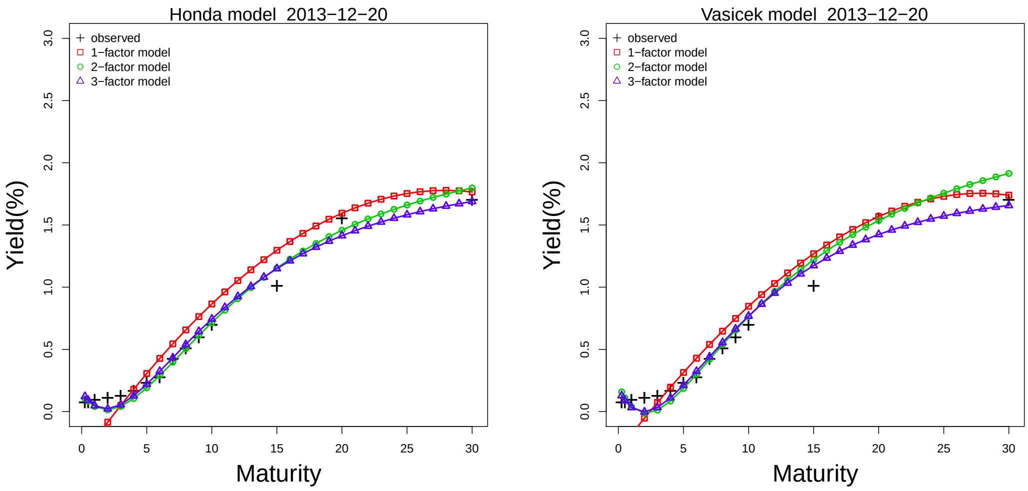


**Figure 1**: Wiener stochastic paths (left) and stochastic Logistic paths (right)

advantages of incorporating non-Gaussian effect for the short rate process are clear by investigating Japanese Government Bond yield calibration. The following is possible research topics. A particle filtering method should be used to compute estimates of the model parameters as well as the state variables. Evaluation for the various interest rate derivatives using proposed model should be investigated.

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## References

[1] Babbs, S. H., and Nowman, K.B. “An application of generalized Vasicek term structure models to the UK gilt-edged market: a Kalman filtering analysis”, *Applied Financial Economics*, 8, 637–644 (1998)

[2] Babbs, S. H., and Nowman, K. B.“Kalman filtering of generalized Vasicek term structure models”, *Journal of Financial and Quantitative Analysis*, 115–130 (1999).

[3] Chen, R.-R., and Scott, L. “Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates”, *Journal of Fixed Income*. 3, 14–31 (1993).

[4] Cox, C., Ingersoll, J., and Ross, S. A. “A theory of the term structure of interest rates”, *Econometrica*, 53, 385–407 (1985).

[5] Date, P. and Wang, C. “Linear Gaussian affine term structure models with unobservable factors: calibration and yield forecasting”, *European Journal of Operational Research*, 195, 156-66 (2009).

[6] Duan, J.-C. and Simonate, J.-G. “Estimating and testing exponential affine term structure models by Kalman filter”, *Review of Quantitative Finance and Accounting*, 13, 111–135 (1999).

[7] Duffie, J. D., Kan, R. “A yield-factor model of interest rates”, *Mathematical Finance*, **6**, 379–406 (1996).

[8] Honda, T., Tamaki, K., and Shiohama, T. “Higher order asymptotic bond price valuation for interest rates with non-Gaussian dependent innovations”, *Finance Research Letters*, 7, 60–69 (2010).

[9] Nowman, K. B. “Estimation of one-, two-, three-factor generalized Vasicek term structure models for Japanese interest rates using monthly panel data”, *Applied Financial Economics*, 21. 1069–1078 (2011).

[10] Miura, M, Tamaki, K.,and Shiohama, T. “Asymptotic expansion for term structures of defaultable bonds with non-Gaussian dependent innovations”, *Asia-Pacific Financial Markets*, **20**, 311-344 (2013).

[11] O’Sullivan, C., “Parameter uncertainty in Kalman filter estimation of the CIR term structure model”, Center for Financial Markets Working Paper Series No. WP-07-18, University College Dublin, (2009)

[12] Shiohama, T., and Tamaki, K. “ Asymptotic expansion for interest rates with non-Gaussian dependent innovations”, In Nishiyama, Y. (Ed.), *Interest rates: Term structure models, monetary policy, and prediction*. New York: Nova Science Publishers, 19–61 (2012).

[13] Vasicek, O. “An equilibrium characterization of the term structure”, *Journal of Financial Economics*, 5, 177–188 (1977).