# A New Transformed Test for Analysis of Variance for Skewed Distributions

Khairul Islam[1] and Tanweer Shapla
Eastern Michigan University, Ypsilanti, Michigan 48197 USA

**Abstract**

Analysis of variance (ANOVA) is one of the most popular statistical techniques for comparing different groups or treatments with respect to their means. One of the important assumptions for the validity of ANOVA F test is the assumption of normality of the groups being compared. However, many real-life data do not follow normal distributions. In the violation of normality, the non-parametric Kruskal-Wallis test is often preferable. In this paper, we propose a new transformed test for one way ANOVA for skewed distributions. The performance of the new test is compared with the standard F and the non-parametric analogue of ANOVA by examples and simulations. Our results suggest that the new transformed test is appropriate for estimating the level of significance and is more powerful than standard F test and the non-parametric test for skewed distributions.

**Key Words**: ANOVA F test, transformed test, Kruskal-Wallis test, level of significance, power of the test, simulation.

## 1. Introduction

Let us consider the test of equality of $k$ population means given samples $\{\boldsymbol{X_{ij}}: i = 1, 2, \ldots, n_j; j = 1, 2, \ldots, k\}$ from $j$th populations with mean $\mu_j$, variance $\sigma^2$ and distribution $F\{\sigma^{-1}(x - \mu_j)\}$. We want to test

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k$$

versus

$$H_1: \mu_j \neq \mu_{j'} \text{ for some } j \neq j'$$

Under the assumption that the samples come from normal distributions with common variance, the test statistic to test $H_0$ is given by

$$F = \frac{SS_{Treat}/(k-1)}{SS_{Error}/(n-k)} \sim F(k-1, n-k)$$

where,

$$n = \sum_{j=1}^{k} n_j$$

$$SS_{Treat} = \sum_{j=1}^{k} n_j \left(\bar{x}_j - \bar{x}\right)^2$$

$$SS_{Error} = \sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(x_{ij} - \bar{x}_j\right)^2$$

[1] Correspondence: kislam@emich.edu

For a level $\alpha$ test, reject $H_0$ if $F > F_{k-1,n-k;1-\alpha}$ or $P(F_{k-1,n-k} > F) < \alpha$, where $F_{k-1,n-k;1-\alpha}$ is the $(1-\alpha)th$ percentile of F distribution with numerator degrees of freedom $(k-1)$ and the denominator degrees of freedom $(n-k)$.

In real-life data, however, the assumption of normality is often invalid. As such, the usual ANOVA F test fails and we proceed with the widely used non-parametric Kruskal–Wallis test (Kruskal and Wallis, 1952). This test compares $k$ sample means by using ranks of the combined dataset from $k$ samples.

When there are no ties, the test statistic is given by

$$K = K^* = \frac{12}{n(n+1)} \times \sum_{j=1}^{k} \frac{R_j^2}{n_j} - 3(n+1) \sim \chi^2_{k-1}$$

where,
$R_j$ is the sum of ranks of jth sample in the combined set of $n$ observations from $k$ samples.

If there are ties, the test statistic is given by

$$K = \frac{K^*}{1 - \frac{\sum_{i=1}^{g}(t_i^3 - t_i)}{n^3 - n}} \sim \chi^2_{k-1}$$

where
$g$ is the number of groups with tied values
$t_i$ is the number of observations with tie in $i$th group, $i = 1,2,\dots,g$.

For a level $\alpha$ test, reject $H_0$ if $K > \chi^2_{k-1,1-\alpha}$ or $P(\chi^2_{k-1} > K) < \alpha$.

## 2. The New Transformed ANOVA Test

If $k$ groups deviate from normality, an alternative to Kruskal-Wallis test, we propose a new transformed test with the Box-Cox type power transformation (Box and Cox, 1964). The idea is to estimate the transformation parameter by a univariate normal goodness-of-fit. We provide an algorithm to perform the new test, and compare the performance of the new test with the traditional ANOVA F test and the non-parametric Kruskal-Wallis test with examples and a simulation study.
Given $k$ samples $\{\boldsymbol{X}_j : j = 1, 2, \dots, k\}$ and a scalar $\lambda$, the Box-Cox power transformation (Box and Cox, 1964) to $\boldsymbol{X}_j$ is defined by

$$\boldsymbol{X}_j(\lambda) = \begin{cases} (\boldsymbol{X}_j^{\lambda} - 1)/\lambda, & \text{if } \lambda \neq 0 \\ \log(\boldsymbol{X}_j), & \text{if } \lambda = 0 \end{cases} ; j = 1, 2, \dots, k \qquad (1)$$

As expected with Box-Cox transformation (Box and Cox, 1964), the transformed $\boldsymbol{X}_j(\lambda), j = 1, 2, \dots, k$, follows a normal distribution, say, $\boldsymbol{X}_j(\lambda) \sim N(\mu_j(\lambda), \sigma_j^2(\lambda))$. Then $\boldsymbol{Z}_j(\lambda) = \frac{\boldsymbol{X}_j(\lambda) - \mu_j(\lambda)}{\sigma_j(\lambda)}$ is independent N(0,1), where, the term $\boldsymbol{X}_j(\lambda) - \mu_j(\lambda)$ allows element-wise subtraction of mean $\mu_j(\lambda)$ from the vector $\boldsymbol{X}_j(\lambda)$. Then, $\boldsymbol{Z}(\lambda) = (\boldsymbol{Z}_1(\lambda), \boldsymbol{Z}_2(\lambda), \dots, \boldsymbol{Z}_k(\lambda))$ represents a sample of size $n = \sum_{j=1}^{k} n_j$ from N(0,1).

In order to estimate $\lambda$, we use the fact that the $\boldsymbol{Z}(\lambda)$ is as close as possible to a N(0,1) distribution by a goodness-of-fit criteria. Viewing this problem as a goodness-of-fit to a normal distribution, we test the hypothesis:
$H_0$: $Z_1(\lambda), Z_2(\lambda), \dots, Z_n(\lambda)$ is coming from a N(0,1) distribution, against
$H_1$: $Z_1(\lambda), Z_2(\lambda), \dots, Z_n(\lambda)$ is not a N(0,1) distribution.
Following Shapiro and Wilk (1965), we use the test statistic $W_Z(\lambda)$ to test $H_0$, which is given by
$W_{\boldsymbol{Z}}(\lambda) = \frac{\left[\sum_{i=1}^{n} a_i Z_{(i)}(\lambda)\right]^2}{\sum_{i=1}^{n}(Z_i(\lambda)-\bar{Z}(\lambda))^2}$, where
$Z_{(i)}(\lambda), i = 1, \dots, n$ represents the ith order statistic of the sample $\boldsymbol{Z}(\lambda)$,
$\bar{Z}(\lambda) = (\sum_{i=1}^{n} Z_i(\lambda))/n$,
$(a_1, \dots, a_n) = \frac{m^T V^{-1}}{(m^T V^{-1} V^{-1} m)^{1/2}}$,
$m = (m_1, \dots, m_n)^T$,
$m_i = E\left(Z_{(i)}(\lambda)\right), i = 1, \dots, n,$ is the expected value of the ith order statistic $Z_{(i)}(\lambda)$,
$V = \left(v_{i,i'}\right)$ is the variance-covariance matrix of order $n \times n$, and
$v_{i,i'} = Cov\left(Z_{(i)}(\lambda), Z_{(i')}(\lambda)\right), i, i' = 1, \dots n,$ is the covariance between $i$th and $i'$th order statistics.

While the value of $W_Z(\lambda)$ lies between zero and one, the small value of $W_Z(\lambda)$ leads to the rejection of normality, whereas a value close to one indicates normality. In other words, given a level of significance $\alpha$ one may reject the null hypothesis if $p$-value $p(\lambda) = P(W \le w_Z(\lambda)) \le \alpha$ and accept otherwise. We propose to estimate $\lambda$ by observing the maximum $p$-value associated with $W_Z(\lambda)$ over all possible values of $\lambda$ to achieve the desired normality of the transformed data. In other words, the new estimate $\hat{\lambda}_n$ using the goodness-of-fit to N(0,1) distribution satisfies the equation

$$p(\hat{\lambda}_n) = \max_{\lambda \in [a,b]} P(W \le w_Z(\lambda))$$

Once $\hat{\lambda}_n$ is obtained, we re-express the original samples and apply an ANOVA F-test to the transformed data to compare the group means.

In this article, we employed the software R (2009) in all examples and simulation to obtain the optimum $\hat{\lambda}_n$. The search for $\hat{\lambda}_n$ is made over the interval [-1, 1] with an increment of 0.1 written hereafter as $\lambda \in \{-1: 0.1: 1\}$.

Below is an algorithm for the estimate $\hat{\lambda}_n$ and the transformed test using $\hat{\lambda}_n$.
Given $\boldsymbol{X}_j$ and a fixed $\lambda$:

1) Obtain the transformation $\boldsymbol{X}_j(\lambda)$ using equation (1).
2) Find $\boldsymbol{Z}_j(\lambda) = \frac{\boldsymbol{X}_j(\lambda) - \overline{X_j}(\lambda)}{S_{\boldsymbol{X}_j}(\lambda)}$ where $S_{\boldsymbol{X}_j}(\lambda) = \sqrt{\sum_{i=1}^{n_j}\left(X_j(\lambda) - \bar{X}_j(\lambda)\right)^2 / n_j}$. Note the term $\boldsymbol{X}_j(\lambda) - \bar{X}_j(\lambda)$ allow element-wise subtraction of sample mean $\bar{X}_j(\lambda)$ from the vector $\boldsymbol{X}_j(\lambda)$.
3) Combine $k$ samples together to form $\boldsymbol{Z}(\lambda) = \left(Z_1(\lambda), Z_2(\lambda), \dots, Z_n(\lambda)\right)$, where $n = \sum_{j=1}^{k} n_j$.
4) Compare $\boldsymbol{Z}(\lambda)$ with the N(0,1) distribution using the Shapiro and Wilk (1965) goodness-of-fit $W_{\boldsymbol{Z}}(\lambda)$ and find the $p$-value.
5) Repeat steps (1) through (4) for all $\lambda \in \{-1: 0.1: 1\}$.
6) Select the maximum $p$-value among all $p$-values from steps (1) through (5).
7) Identify the $\hat{\lambda}_n$ corresponding to the maximum $p$-value in step (6).

8) Obtain $\boldsymbol{X}_j(\hat{\lambda}_n)$.
9) Perform usual $F$-test on the basis of transformed data in step (8) and decide about the acceptance and rejection of the null hypothesis comparing observed value of $F$ with critical value of $F_{k-1,n-k;1-\alpha}$ distribution for a given level of significance $\alpha$.

The theoretical aspects of the Box-Cox transformed data analysis described above have been reported in literature. For examples, Hinkley (1975) and Hernandez and Johnson (1980) investigated the asymptotic properties of the parameter estimates; Bickel and Doksum (1981) critically examined the behavior of the asymptotic variances of the parameter estimates for regression and analysis of variance situations; Chen and Loh (1992) and Chen (1995) proved that the Box-Cox transformed $t$-test is typically more efficient asymptotically than the $t$-test without transformation. The use of transformed $t$-test is also justified by Islam and Chen (2007) by fitting a $t$ distribution to transformed data. In this paper, we empirically assess the performance of the transformed ANOVA as compared to traditional ANOVA F and non-parametric Kruskal-Wallis test.

## 3. Simulation and Result Discussion

In this section, we carry out a simulation study to compare the finite sample performance of the three ANOVA tests described in this article. All simulations are performed by using the statistical software R. For the transformed F-test (Trans F), we estimate $\hat{\lambda}_n$ from values of $\lambda \in \{-1{:}\,0.1{:}\,1\}$ as described in section 2. The samples $\{\boldsymbol{X_{ij}}: i = 1, 2, \dots, n_j; j = 1, 2, \dots, k\}$, with $k = 3$ chosen arbitrarily, are simulated from $G(\theta, \beta)$ population where $\theta$ is the shape parameter and $\beta$ is the scale parameter.

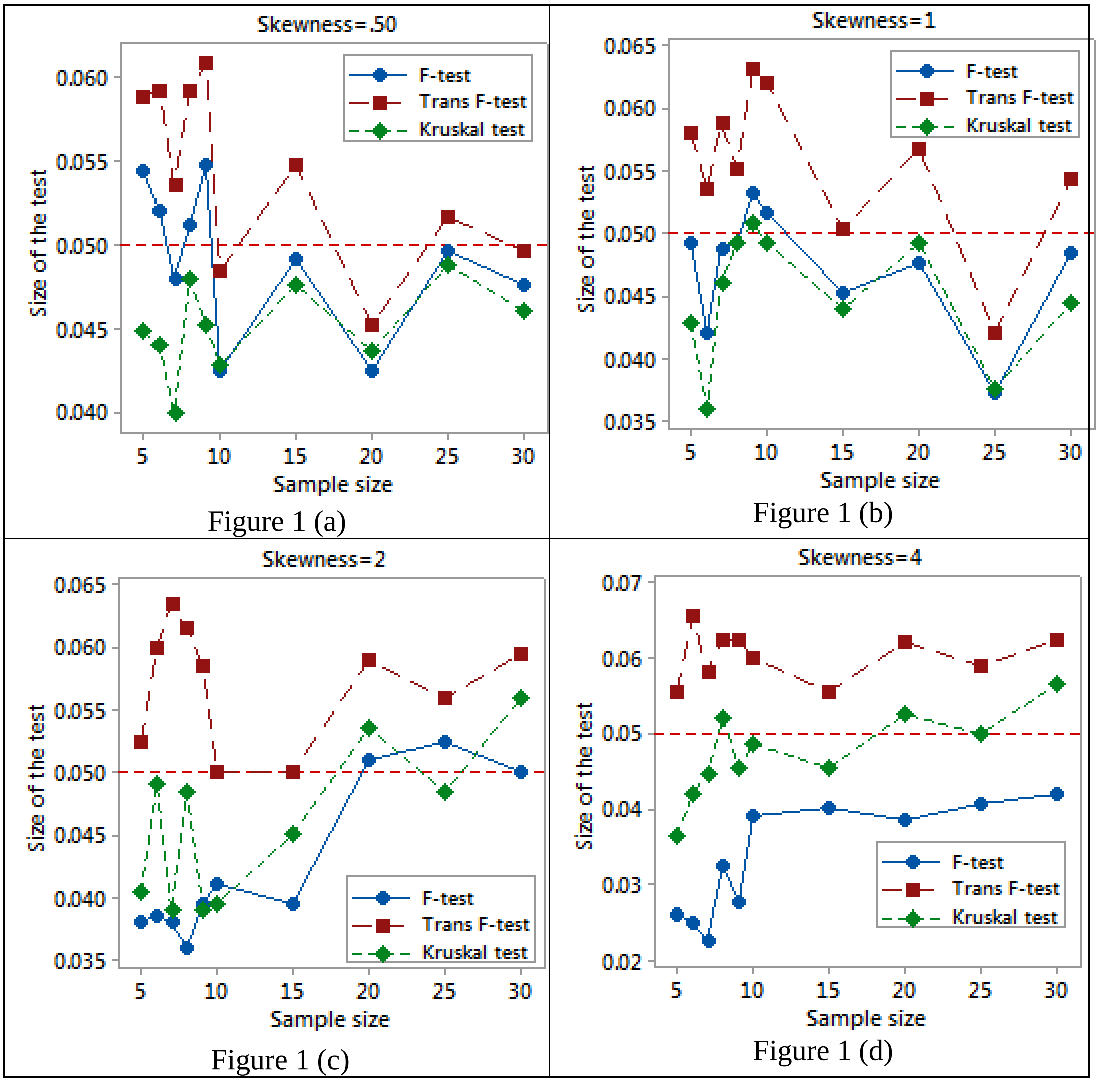
Note that the skewness of $G(\theta, \beta)$ distribution is $\gamma_1 = 2/\sqrt{\theta}$. In simulations, we choose different values of the parameters $\theta$ and $\beta$ to allow varying levels of skewness at 0.5, 1, 2 and 4 chosen arbitrarily keeping means of all simulated distributions fixed at 1 under the null model. The mean difference ($\Delta$) of 0.15, 0.30, 0.45 and 0.60 are considered under the alternative models to ensure a testing power away from 0 and 1 for the purpose of the comparisons. In all simulations, the Monte Carlo size is 2,000. The power of a test is estimated from the proportion of rejection of null hypothesis under alternative models over a Monte Carlo simulation of size 2,000 at 5% level of significance. In a similar manner, the level of significance is estimated from the proportion of rejection of the null hypothesis over a Monte Carlo simulation of size 2,000 at 5% level of significance when the null hypothesis is true. Table 1 provides the values of the parameters $\theta$ and $\beta$ used in the simulation of samples to allow varying values of the skewness with fixed mean.

**Table 1**: Values of $\theta$, $\beta$ and $\gamma_1$ used in simulations of samples under four null models (M1-M4)

| Models | $\theta$ | $\beta$ | $\gamma_1$ | mean |
|---|---|---|---|---|
| M1 | 16 | 0.0625 | 0.5 | 1 |
| M2 | 4 | 0.25 | 1 | 1 |
| M3 | 1 | 1 | 2 | 1 |
| M4 | 0.25 | 4 | 4 | 1 |

Tables 2 provides estimated size of the test from the simulated samples for varying values of skewness ($\gamma_1$) and sample size ($n$) under the four null models. The estimated simulated size reported in Table 2 has been shown in Figure 1 for better understand of the performance of the three tests in controlling the size of the test at $\alpha = 0.05$. Tables 3-6 provide estimated power from the simulated samples for varying values of skewness ($\gamma_1$), mean difference ($\Delta$) and sample size under alternative hypotheses.

**Figure 1**: Estimated size at 5% significance level for F, transformed F and Kruskal-Wallis tests

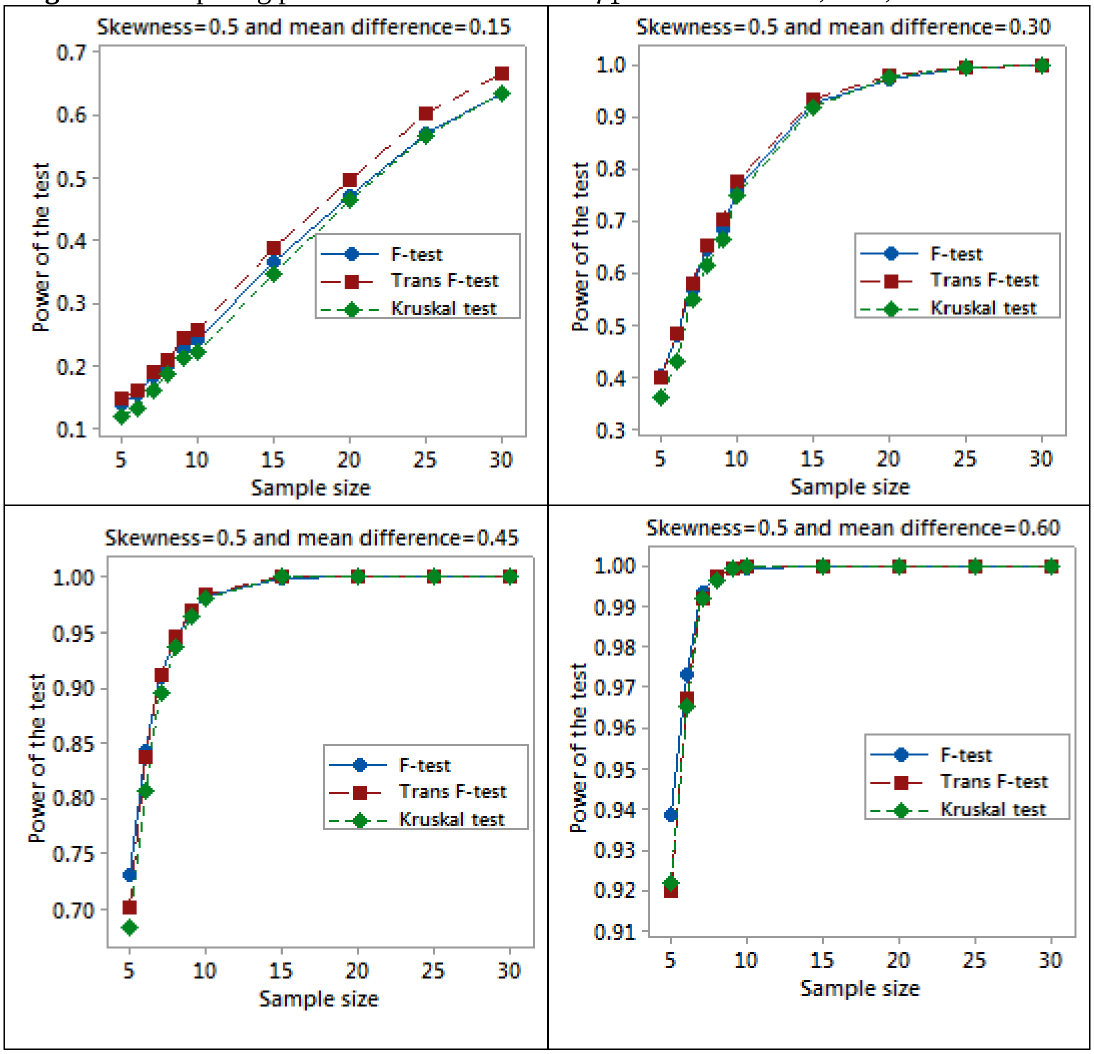



As we see from the Figure 1(a)-(d), for a skewness of 0.5 with small sample size ($n \leq 10$), all tests are far from the desired level of 5%. The Kruskal-Wallis test seems to be more conservative among all three tests in rejecting the null hypothesis. When skewness is 4 as shown in Figure 1 (d), F test is the most conservative in the rejection of the null hypothesis. The Kruskal-Wallis test seems to be more robust among all three tests in rejecting the null hypothesis. Overall, all tests are about 3% in either side of the desired nominal level of 5%, with transformed F test having about 1.25% over estimating $\alpha$, while others underestimating $\alpha$ ranging from 1.5% to 2.25%. In all consideration, the performance of the transformed F test is as good or as bad as ANOVA F or Kruskal-Wallis test.

However, when the power of the three tests is of concern, let us have a look at the estimated testing power of the three tests reported in Tables 3-6. For better understand the power of the three tests, the estimated power reported in Table 3 (skewness=0.5) and

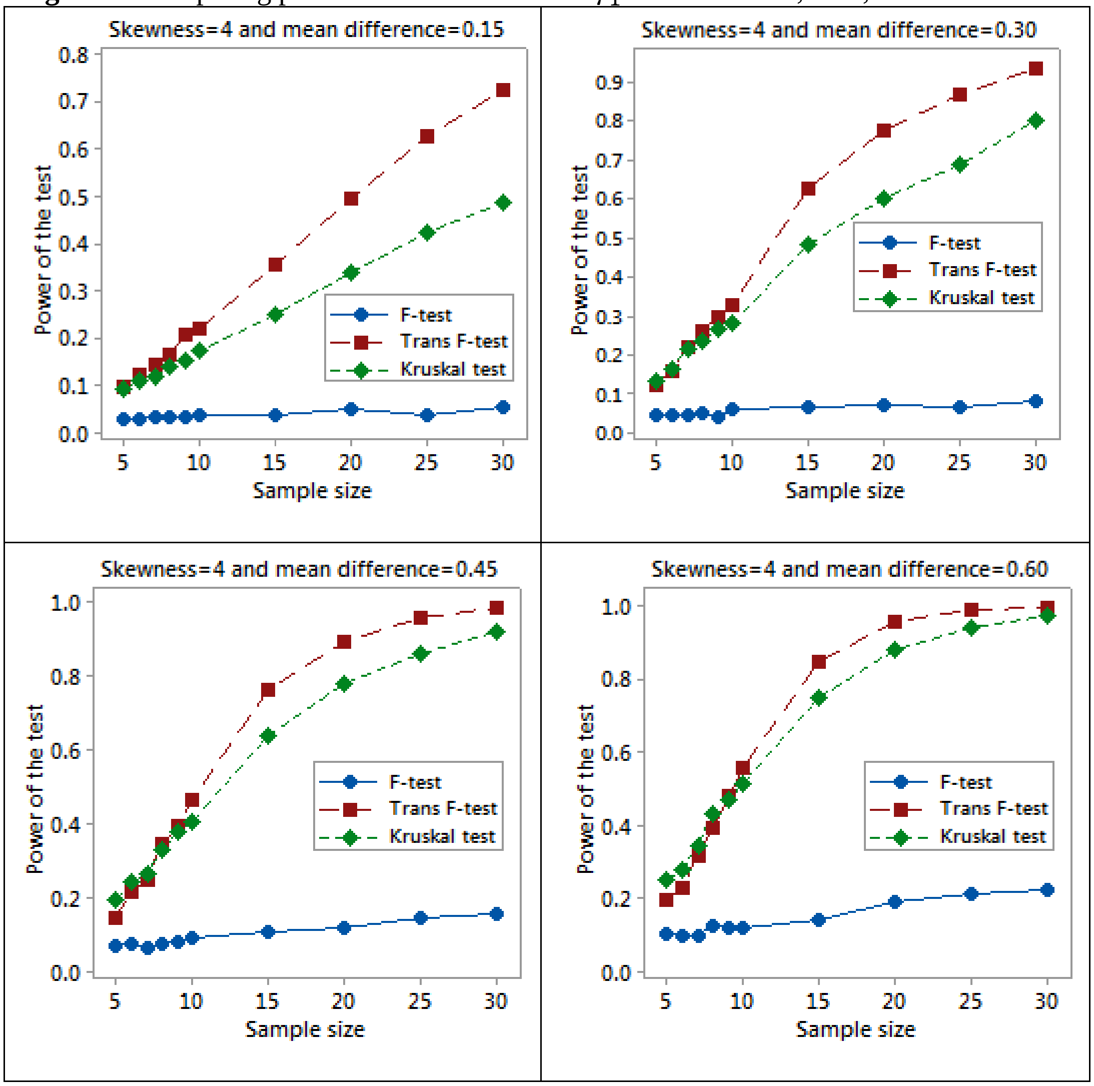
Table 6 (skewness=4) are shown in Figures 2 and 3, respectively. Figure 2 suggests that when the mean difference Delta is 0.15, the transformed F test (trans F) has highest power compared to F and Kruskal-Wallis tests. As difference of the means goes higher, the difference in the power of all tests get reduced. The power of all tests increases as the sample size gets larger.

**Figure 2**: Comparing power of three tests when $\gamma_1$=0.5 and Δ=0.15, 0.30, 0.45 and 0.60




Looking at Figure 3, again, the trans F test performs the best with respect to the power as compared to the F or Kruskal-Wallis test. Note that, the traditional ANOVA F test performs poorly as skewness reaches 4, with apparently insignificant improvement of the power with the increase of the sample size. However, the trans F and Kruskal-Wallis tests show significant improvement in power as the sample size gets larger, with trans F showing the best performance.

**Figure 3**: Comparing power of three tests when $\gamma_1$=4 and $\Delta$=0.15, 0.30, 0.45 and 0.60



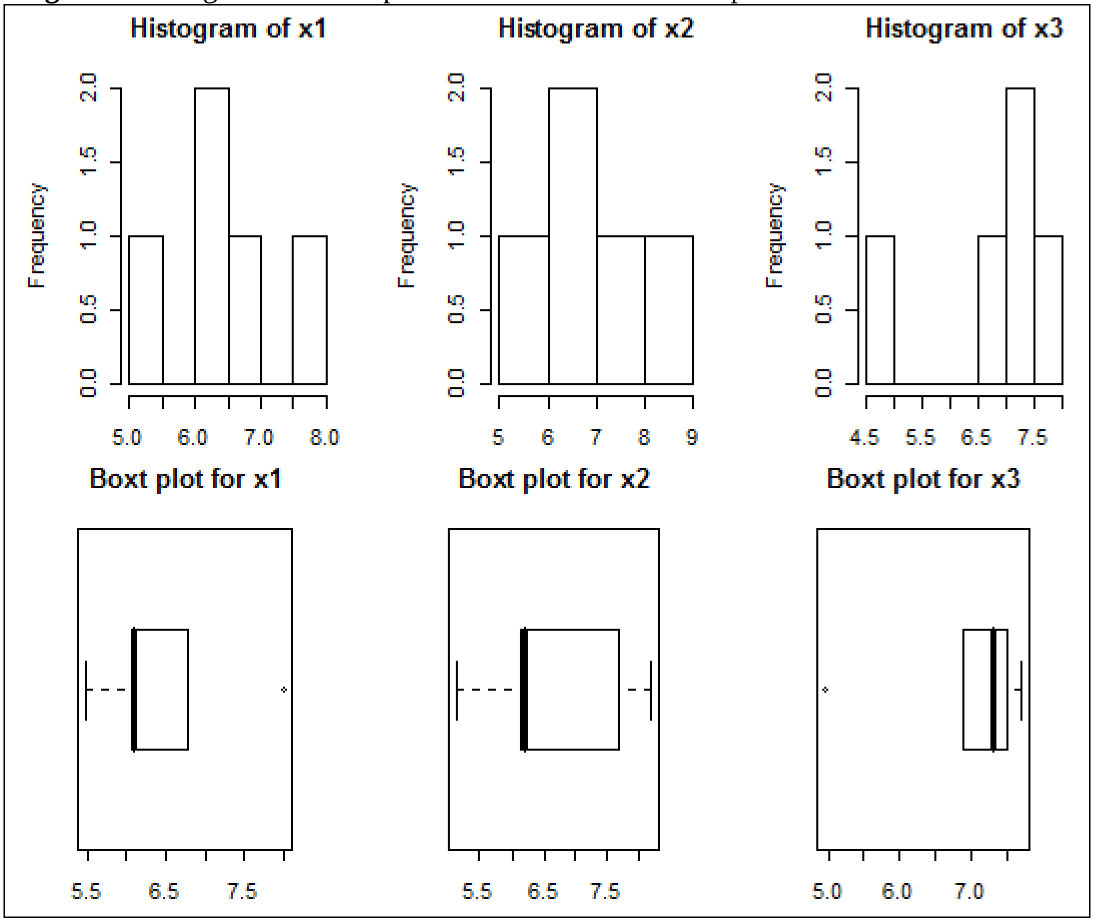

## 4. Example

The prices (in dollars) for 30-count packages of randomly selected store-band vitamin/mineral supplements are listed from three different sources as appeared in Bluman, A. G. (2014):

| | | | | | |
|---|---|---|---|---|---|
| Grocery store (x1) | 6.79 | 6.09 | 5.49 | 7.99 | 6.10 |
| Drugstore (x2) | 7.69 | 8.19 | 6.19 | 5.15 | 6.14 |
| Discount store (x3) | 7.49 | 6.89 | 7.69 | 7.29 | 4.95 |

From the histograms and the box plots in Figure 4, it follows that x1, x2 and x3 are skewed.

**Figure 4**: Histograms and box plots for datasets in the Example




The sample mean and skewness are as follows:

$$\bar{x}_1 = 6.49\text{, skewness = 1.08}$$
$$\bar{x}_2 = 6.67\text{, skewness = 0.17}$$
$$\bar{x}_3 = 6.86\text{, skewness = -1.86}$$

We want to test

$$H_0: \mu_1 = \mu_2 = \mu_3 \text{ versus } H_1: \mu_j \neq \mu_{j'} \text{ for some } j \neq j'$$

For transformed test, the search for $\hat{\lambda}_n$ is made over the interval [-2, 2] with an increment of 0.1 written hereafter as $\lambda \in \{-2: 0.1: 2\}$. The estimated value of $\lambda = 1.93$.

| Tests | Value of test statistic and $P$-value |
|---|---|
| F test | F = 0.1393; $df(n; d)^* = (2,12)$; $p$-value = 0.8713 |
| Kruskal-Wallis test | Chi-squared = 0.546, df = 2; $p$-value = 0.7611 |
| Trans F test | F = 0.1565; $df(n, d) = (2,12)$; $p$-value = 0.8568 |

$*\, df(n; d)$ refers to the numerator and denominator degrees of freedom for F distribution.

From the summary of the analysis results presented in the table above, it follows that all three tests failed to reject the null hypothesis of equality of average price for the three store-band vitamin/mineral supplements at 5% level of significance.

## 5. Conclusion

Under the assumption of normality of $k$ groups with a common variance, the traditional F test is the powerful test. However, in the violation of the normality of $k$ groups being far from the normality with higher skewness, the Kruskal-Wallis test is more robust than the F test. It is well known that gamma distribution is one of the popular choices for simulating data from the skewed distributions. For a gamma distribution $G(\theta, \beta)$, the skewness is given by $\gamma_1 = 2/\sqrt{\theta}$. To control the skewness efficiently, we consider this distribution for all simulation. In simulations, we choose different values of the parameters $\theta$ and $\beta$ to allow varying levels of skewness at 0.5, 1, 2 and 4 chosen arbitrarily keeping means of all simulated distributions fixed at 1 under the null model. The mean difference ($\Delta$) of 0.15, 0.30, 0.45 and 0.60 are considered under the alternative models to ensure a testing power away from 0 and 1 for the purpose of the comparisons.

On the basis of the simulation of Monte Carlo size of 2,000, we computed the level of significance at 5% and the testing power of the three tests discussed in this article. The power of a test is estimated from the proportion of rejection of null hypothesis under alternative models over a Monte Carlo simulation of size 2,000 at 5% level of significance. In a similar manner, the level of significance is estimated from the proportion of the rejection of the null hypothesis over a Monte Carlo simulation of size 2,000 at 5% level of significance when the null hypothesis is true.

From the simulated result, it appears that the new transformed test is more powerful than the traditional F test and the non-parametric Kruskal-Wallis test, and is appropriate for level of significance for the test. As skewness increases, the new transformed test is more powerful than other tests. It follows that as skewness decreases, the power of F test keeps increasing.

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**Table 2**: Estimated $\alpha$ at 5% significance level for F, transformed F and Kruskal tests.

| $\gamma_1$ | n | F | Trans F | Kruskal | mean ($\hat{\lambda}_n$) | sd($\hat{\lambda}_n$) |
|---|---|---|---|---|---|---|
| 0.5 | 5 | 0.054 | 0.059 | 0.045 | 0.118 | 0.838 |
| | 6 | 0.052 | 0.059 | 0.044 | 0.135 | 0.807 |
| | 7 | 0.048 | 0.054 | 0.040 | 0.163 | 0.756 |
| | 8 | 0.051 | 0.059 | 0.048 | 0.207 | 0.726 |
| | 9 | 0.055 | 0.061 | 0.045 | 0.226 | 0.690 |
| | 10 | 0.042 | 0.048 | 0.043 | 0.220 | 0.667 |
| | 15 | 0.049 | 0.055 | 0.048 | 0.264 | 0.560 |
| | 20 | 0.042 | 0.045 | 0.044 | 0.301 | 0.486 |
| | 25 | 0.050 | 0.052 | 0.049 | 0.305 | 0.419 |
| | 30 | 0.048 | 0.050 | 0.046 | 0.313 | 0.381 |
| 1 | 5 | 0.049 | 0.058 | 0.043 | 0.196 | 0.692 |
| | 6 | 0.042 | 0.054 | 0.036 | 0.251 | 0.610 |
| | 7 | 0.049 | 0.059 | 0.046 | 0.257 | 0.556 |
| | 8 | 0.049 | 0.055 | 0.049 | 0.271 | 0.495 |
| | 9 | 0.053 | 0.063 | 0.051 | 0.275 | 0.458 |
| | 10 | 0.052 | 0.062 | 0.049 | 0.293 | 0.424 |
| | 15 | 0.045 | 0.050 | 0.044 | 0.313 | 0.311 |
| | 20 | 0.048 | 0.057 | 0.049 | 0.317 | 0.257 |
| | 25 | 0.037 | 0.042 | 0.038 | 0.309 | 0.216 |
| | 30 | 0.048 | 0.054 | 0.044 | 0.307 | 0.194 |
| 2 | 5 | 0.038 | 0.053 | 0.041 | 0.237 | 0.438 |
| | 6 | 0.039 | 0.060 | 0.049 | 0.254 | 0.369 |
| | 7 | 0.038 | 0.064 | 0.039 | 0.271 | 0.302 |
| | 8 | 0.036 | 0.062 | 0.049 | 0.263 | 0.267 |
| | 9 | 0.040 | 0.059 | 0.039 | 0.270 | 0.243 |
| | 10 | 0.041 | 0.050 | 0.040 | 0.268 | 0.206 |
| | 15 | 0.040 | 0.050 | 0.045 | 0.273 | 0.142 |
| | 20 | 0.051 | 0.059 | 0.054 | 0.274 | 0.114 |
| | 25 | 0.053 | 0.056 | 0.049 | 0.271 | 0.098 |
| | 30 | 0.050 | 0.060 | 0.056 | 0.271 | 0.088 |
| 4 | 5 | 0.026 | 0.056 | 0.037 | 0.175 | 0.206 |
| | 6 | 0.025 | 0.066 | 0.042 | 0.172 | 0.152 |
| | 7 | 0.023 | 0.058 | 0.045 | 0.176 | 0.129 |
| | 8 | 0.033 | 0.063 | 0.052 | 0.169 | 0.102 |
| | 9 | 0.028 | 0.063 | 0.046 | 0.167 | 0.094 |
| | 10 | 0.039 | 0.060 | 0.049 | 0.167 | 0.083 |
| | 15 | 0.040 | 0.056 | 0.046 | 0.165 | 0.063 |
| | 20 | 0.039 | 0.062 | 0.053 | 0.166 | 0.055 |
| | 25 | 0.041 | 0.059 | 0.050 | 0.164 | 0.051 |
| | 30 | 0.042 | 0.063 | 0.057 | 0.166 | 0.049 |

**Table 3**: Estimated power of F, transformed F and Kruskal tests when skewness $\gamma_1$=0.5

| Δ | n | F | Trans F | Kruskal | mean ($\hat{\lambda}_n$) | sd($\hat{\lambda}_n$) |
|---|---|---|---|---|---|---|
| 0.15 | 5 | 0.138 | 0.148 | 0.120 | 0.101 | 0.854 |
| | 6 | 0.154 | 0.160 | 0.132 | 0.151 | 0.808 |
| | 7 | 0.184 | 0.188 | 0.160 | 0.150 | 0.778 |
| | 8 | 0.198 | 0.210 | 0.185 | 0.172 | 0.738 |
| | 9 | 0.226 | 0.243 | 0.211 | 0.166 | 0.721 |
| | 10 | 0.240 | 0.256 | 0.223 | 0.215 | 0.677 |
| | 15 | 0.366 | 0.388 | 0.347 | 0.250 | 0.574 |
| | 20 | 0.471 | 0.496 | 0.465 | 0.254 | 0.507 |
| | 25 | 0.572 | 0.601 | 0.567 | 0.278 | 0.441 |
| | 30 | 0.633 | 0.667 | 0.634 | 0.277 | 0.393 |
| 0.3 | 5 | 0.404 | 0.398 | 0.362 | 0.063 | 0.860 |
| | 6 | 0.481 | 0.484 | 0.430 | 0.126 | 0.814 |
| | 7 | 0.568 | 0.580 | 0.550 | 0.138 | 0.782 |
| | 8 | 0.645 | 0.653 | 0.614 | 0.127 | 0.764 |
| | 9 | 0.683 | 0.702 | 0.664 | 0.153 | 0.735 |
| | 10 | 0.761 | 0.776 | 0.750 | 0.158 | 0.701 |
| | 15 | 0.924 | 0.935 | 0.918 | 0.243 | 0.575 |
| | 20 | 0.973 | 0.979 | 0.975 | 0.232 | 0.504 |
| | 25 | 0.994 | 0.996 | 0.995 | 0.249 | 0.454 |
| | 30 | 0.998 | 0.999 | 0.999 | 0.243 | 0.411 |
| 0.45 | 5 | 0.730 | 0.701 | 0.684 | 0.069 | 0.858 |
| | 6 | 0.843 | 0.838 | 0.806 | 0.078 | 0.833 |
| | 7 | 0.909 | 0.911 | 0.896 | 0.121 | 0.796 |
| | 8 | 0.945 | 0.947 | 0.938 | 0.163 | 0.762 |
| | 9 | 0.968 | 0.971 | 0.964 | 0.152 | 0.733 |
| | 10 | 0.983 | 0.985 | 0.981 | 0.148 | 0.697 |
| | 15 | 0.999 | 1.000 | 1.000 | 0.191 | 0.611 |
| | 20 | 1.000 | 1.000 | 1.000 | 0.210 | 0.520 |
| | 25 | 1.000 | 1.000 | 1.000 | 0.222 | 0.478 |
| | 30 | 1.000 | 1.000 | 1.000 | 0.245 | 0.425 |
| 0.6 | 5 | 0.938 | 0.920 | 0.922 | 0.069 | 0.865 |
| | 6 | 0.973 | 0.967 | 0.965 | 0.081 | 0.833 |
| | 7 | 0.994 | 0.992 | 0.992 | 0.099 | 0.798 |
| | 8 | 0.997 | 0.998 | 0.996 | 0.108 | 0.768 |
| | 9 | 1.000 | 1.000 | 0.999 | 0.128 | 0.750 |
| | 10 | 0.999 | 1.000 | 1.000 | 0.138 | 0.715 |
| | 15 | 1.000 | 1.000 | 1.000 | 0.175 | 0.607 |
| | 20 | 1.000 | 1.000 | 1.000 | 0.200 | 0.535 |
| | 25 | 1.000 | 1.000 | 1.000 | 0.204 | 0.495 |
| | 30 | 1.000 | 1.000 | 1.000 | 0.207 | 0.438 |

**Table 4**: Estimated power of F, transformed F and Kruskal tests when skewness $\gamma_1$=1

| Δ | n | F | Trans F | Kruskal | mean ($\hat{\lambda}_n$) | sd($\hat{\lambda}_n$) |
|---|---|---|---|---|---|---|
| 0.15 | 5 | 0.060 | 0.070 | 0.059 | 0.202 | 0.693 |
| | 6 | 0.075 | 0.087 | 0.066 | 0.208 | 0.636 |
| | 7 | 0.076 | 0.092 | 0.072 | 0.210 | 0.580 |
| | 8 | 0.082 | 0.098 | 0.085 | 0.245 | 0.521 |
| | 9 | 0.078 | 0.102 | 0.083 | 0.239 | 0.478 |
| | 10 | 0.092 | 0.106 | 0.090 | 0.256 | 0.436 |
| | 15 | 0.115 | 0.142 | 0.120 | 0.265 | 0.326 |
| | 20 | 0.157 | 0.194 | 0.168 | 0.275 | 0.268 |
| | 25 | 0.189 | 0.226 | 0.200 | 0.278 | 0.225 |
| | 30 | 0.188 | 0.238 | 0.215 | 0.273 | 0.198 |
| 0.3 | 5 | 0.138 | 0.151 | 0.136 | 0.151 | 0.718 |
| | 6 | 0.148 | 0.168 | 0.136 | 0.170 | 0.647 |
| | 7 | 0.185 | 0.208 | 0.180 | 0.209 | 0.589 |
| | 8 | 0.202 | 0.236 | 0.206 | 0.199 | 0.552 |
| | 9 | 0.222 | 0.249 | 0.218 | 0.230 | 0.499 |
| | 10 | 0.244 | 0.283 | 0.253 | 0.235 | 0.469 |
| | 15 | 0.353 | 0.428 | 0.386 | 0.251 | 0.343 |
| | 20 | 0.468 | 0.559 | 0.514 | 0.243 | 0.273 |
| | 25 | 0.575 | 0.683 | 0.634 | 0.247 | 0.237 |
| | 30 | 0.646 | 0.750 | 0.715 | 0.249 | 0.210 |
| 0.45 | 5 | 0.257 | 0.258 | 0.228 | 0.131 | 0.729 |
| | 6 | 0.305 | 0.320 | 0.282 | 0.157 | 0.674 |
| | 7 | 0.340 | 0.370 | 0.339 | 0.155 | 0.611 |
| | 8 | 0.413 | 0.446 | 0.411 | 0.192 | 0.565 |
| | 9 | 0.457 | 0.506 | 0.471 | 0.192 | 0.514 |
| | 10 | 0.496 | 0.556 | 0.526 | 0.214 | 0.481 |
| | 15 | 0.704 | 0.784 | 0.753 | 0.220 | 0.355 |
| | 20 | 0.821 | 0.896 | 0.878 | 0.226 | 0.283 |
| | 25 | 0.909 | 0.960 | 0.940 | 0.209 | 0.252 |
| | 30 | 0.955 | 0.986 | 0.979 | 0.224 | 0.215 |
| 0.6 | 5 | 0.402 | 0.377 | 0.374 | 0.111 | 0.736 |
| | 6 | 0.496 | 0.492 | 0.462 | 0.141 | 0.667 |
| | 7 | 0.567 | 0.588 | 0.570 | 0.127 | 0.620 |
| | 8 | 0.653 | 0.689 | 0.667 | 0.154 | 0.567 |
| | 9 | 0.723 | 0.772 | 0.750 | 0.155 | 0.523 |
| | 10 | 0.763 | 0.813 | 0.794 | 0.185 | 0.489 |
| | 15 | 0.917 | 0.959 | 0.948 | 0.192 | 0.360 |
| | 20 | 0.976 | 0.990 | 0.987 | 0.199 | 0.299 |
| | 25 | 0.992 | 0.999 | 0.998 | 0.204 | 0.250 |
| | 30 | 0.998 | 1.000 | 1.000 | 0.204 | 0.222 |

**Table 5**: Estimated power of F, transformed F and Kruskal tests when skewness $\gamma_1$=2

| Δ | n | F | Trans F | Kruskal | mean ($\hat{\lambda}_n$) | sd($\hat{\lambda}_n$) |
|---|---|---|---|---|---|---|
| 0.15 | 5 | 0.046 | 0.064 | 0.049 | 0.194 | 0.481 |
| | 6 | 0.049 | 0.075 | 0.050 | 0.191 | 0.400 |
| | 7 | 0.054 | 0.086 | 0.066 | 0.198 | 0.324 |
| | 8 | 0.052 | 0.080 | 0.059 | 0.214 | 0.292 |
| | 9 | 0.050 | 0.081 | 0.055 | 0.209 | 0.249 |
| | 10 | 0.053 | 0.084 | 0.063 | 0.210 | 0.230 |
| | 15 | 0.067 | 0.114 | 0.084 | 0.206 | 0.157 |
| | 20 | 0.058 | 0.119 | 0.090 | 0.209 | 0.126 |
| | 25 | 0.075 | 0.160 | 0.121 | 0.212 | 0.110 |
| | 30 | 0.081 | 0.186 | 0.140 | 0.213 | 0.093 |
| 0.3 | 5 | 0.067 | 0.105 | 0.089 | 0.156 | 0.511 |
| | 6 | 0.075 | 0.119 | 0.093 | 0.156 | 0.429 |
| | 7 | 0.090 | 0.139 | 0.115 | 0.185 | 0.363 |
| | 8 | 0.097 | 0.162 | 0.129 | 0.169 | 0.316 |
| | 9 | 0.089 | 0.170 | 0.135 | 0.168 | 0.270 |
| | 10 | 0.088 | 0.180 | 0.134 | 0.168 | 0.242 |
| | 15 | 0.120 | 0.257 | 0.203 | 0.182 | 0.170 |
| | 20 | 0.159 | 0.356 | 0.303 | 0.180 | 0.132 |
| | 25 | 0.185 | 0.427 | 0.335 | 0.184 | 0.109 |
| | 30 | 0.224 | 0.515 | 0.432 | 0.186 | 0.099 |
| 0.45 | 5 | 0.094 | 0.136 | 0.109 | 0.124 | 0.506 |
| | 6 | 0.122 | 0.184 | 0.142 | 0.120 | 0.438 |
| | 7 | 0.136 | 0.214 | 0.176 | 0.143 | 0.380 |
| | 8 | 0.142 | 0.238 | 0.208 | 0.131 | 0.323 |
| | 9 | 0.154 | 0.272 | 0.231 | 0.136 | 0.282 |
| | 10 | 0.168 | 0.304 | 0.269 | 0.144 | 0.255 |
| | 15 | 0.241 | 0.494 | 0.415 | 0.148 | 0.175 |
| | 20 | 0.296 | 0.626 | 0.548 | 0.163 | 0.137 |
| | 25 | 0.351 | 0.749 | 0.646 | 0.162 | 0.117 |
| | 30 | 0.415 | 0.830 | 0.761 | 0.166 | 0.104 |
| 0.6 | 5 | 0.151 | 0.179 | 0.174 | 0.085 | 0.535 |
| | 6 | 0.175 | 0.243 | 0.215 | 0.105 | 0.462 |
| | 7 | 0.204 | 0.287 | 0.272 | 0.092 | 0.386 |
| | 8 | 0.216 | 0.333 | 0.310 | 0.122 | 0.337 |
| | 9 | 0.264 | 0.403 | 0.389 | 0.127 | 0.293 |
| | 10 | 0.274 | 0.441 | 0.410 | 0.128 | 0.260 |
| | 15 | 0.374 | 0.675 | 0.615 | 0.136 | 0.176 |
| | 20 | 0.486 | 0.831 | 0.769 | 0.151 | 0.140 |
| | 25 | 0.582 | 0.917 | 0.864 | 0.155 | 0.119 |
| | 30 | 0.701 | 0.971 | 0.935 | 0.154 | 0.103 |

**Table 6**: Estimated power of F, transformed F and Kruskal tests when skewness $\gamma_1$=4

| Δ | n | F | Trans F | Kruskal | mean ($\hat{\lambda}_n$) | sd($\hat{\lambda}_n$) |
|---|---|---|---|---|---|---|
| 0.15 | 5 | 0.031 | 0.099 | 0.094 | 0.034 | 0.275 |
| | 6 | 0.031 | 0.120 | 0.111 | 0.046 | 0.214 |
| | 7 | 0.034 | 0.144 | 0.117 | 0.049 | 0.185 |
| | 8 | 0.034 | 0.165 | 0.138 | 0.059 | 0.139 |
| | 9 | 0.035 | 0.208 | 0.150 | 0.063 | 0.125 |
| | 10 | 0.037 | 0.220 | 0.172 | 0.069 | 0.108 |
| | 15 | 0.035 | 0.356 | 0.251 | 0.081 | 0.075 |
| | 20 | 0.051 | 0.496 | 0.338 | 0.089 | 0.056 |
| | 25 | 0.037 | 0.628 | 0.422 | 0.089 | 0.047 |
| | 30 | 0.055 | 0.726 | 0.489 | 0.093 | 0.039 |
| 0.3 | 5 | 0.042 | 0.123 | 0.131 | 0.009 | 0.293 |
| | 6 | 0.043 | 0.158 | 0.160 | 0.023 | 0.227 |
| | 7 | 0.045 | 0.222 | 0.213 | 0.028 | 0.184 |
| | 8 | 0.049 | 0.260 | 0.235 | 0.046 | 0.145 |
| | 9 | 0.040 | 0.298 | 0.264 | 0.051 | 0.122 |
| | 10 | 0.061 | 0.330 | 0.282 | 0.062 | 0.106 |
| | 15 | 0.066 | 0.626 | 0.483 | 0.072 | 0.074 |
| | 20 | 0.070 | 0.775 | 0.601 | 0.078 | 0.058 |
| | 25 | 0.065 | 0.867 | 0.690 | 0.085 | 0.049 |
| | 30 | 0.083 | 0.934 | 0.804 | 0.087 | 0.042 |
| 0.45 | 5 | 0.067 | 0.147 | 0.191 | -0.007 | 0.294 |
| | 6 | 0.075 | 0.215 | 0.241 | 0.007 | 0.237 |
| | 7 | 0.065 | 0.248 | 0.262 | 0.020 | 0.177 |
| | 8 | 0.072 | 0.343 | 0.332 | 0.032 | 0.148 |
| | 9 | 0.080 | 0.392 | 0.378 | 0.047 | 0.122 |
| | 10 | 0.090 | 0.467 | 0.408 | 0.052 | 0.106 |
| | 15 | 0.108 | 0.762 | 0.637 | 0.070 | 0.073 |
| | 20 | 0.118 | 0.896 | 0.781 | 0.077 | 0.058 |
| | 25 | 0.144 | 0.960 | 0.860 | 0.083 | 0.050 |
| | 30 | 0.157 | 0.985 | 0.923 | 0.087 | 0.042 |
| 0.6 | 5 | 0.102 | 0.196 | 0.249 | -0.017 | 0.302 |
| | 6 | 0.098 | 0.231 | 0.277 | -0.011 | 0.240 |
| | 7 | 0.097 | 0.316 | 0.346 | 0.007 | 0.189 |
| | 8 | 0.123 | 0.395 | 0.430 | 0.030 | 0.145 |
| | 9 | 0.117 | 0.481 | 0.472 | 0.037 | 0.126 |
| | 10 | 0.117 | 0.558 | 0.513 | 0.047 | 0.110 |
| | 15 | 0.142 | 0.848 | 0.749 | 0.068 | 0.073 |
| | 20 | 0.190 | 0.960 | 0.883 | 0.078 | 0.057 |
| | 25 | 0.214 | 0.990 | 0.941 | 0.084 | 0.047 |
| | 30 | 0.221 | 0.997 | 0.974 | 0.087 | 0.040 |