PREDICTIVE DENSITY ESTIMATION IN ACCELERATED LIFE TESTING FOR LOGNORMAL LIFE DISTRIBUTIONS

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ABSTRACT

This paper proposes a simple method of obtaining a predictive density for a future observation from a lognormal distribution, using data from an accelerated life test. It is based on the Maximum Likelihood Predictive Density method proposed by Lejeune and Faulkenberry in [1]. The resulting predictive density is a t-distribution. A Monte Carlo simulation study was carried out using parameter values motivated by the data from Crawford [2] which reported a temperature accelerated study of motorettes insulation and used lognormal distribution to model the data. Monte Carlo methods show that the predictive density provides liberal prediction bounds but a simple modification to the degrees of freedom yields slightly conservative coverage probabilities.

I. INTRODUCTION

Life testing of components designed to last a relatively long time under field (design, nominal use) conditions may result in only a few, if any, failures within a reasonable amount of time. Under field conditions, products are subjected to stresses such as humidity, mechanical load, pressure, temperature, voltage, vibration, and use rate. Subjecting the components to higher than nominal stresses can result in more failures and such data from an “accelerated life test” can be employed to estimate one or more attributes of the life distribution under nominal use if a reasonable model relating one or more parameters of the life distribution to the stress can be assumed. In the following a methodology for employing data from an accelerated life test to obtain prediction intervals for a future log-normal observation under design stress is proposed.

A common objective in accelerated life testing is to estimate one or more quantiles of the life distribution under the design stress (see [3], [4], and [5]). For example, lower quantiles are important for purposes such as warranty assurance. Traditionally, the point estimate for the quantile under investigation is obtained by substituting parameter estimates of the life distribution into the equations which express the quantiles in terms of these parameters. In contrast, estimating a predictive density for a future observation at the design stress level may be a better approach. This density can then be used not only to estimate quantiles of the associated life distribution, but also to obtain prediction bounds.

Parametric accelerated life testing models have two components: 1) a parametric distribution for the lifetime of a unit; and 2) an assumed relationship between one or more of the parameters of the distribution and the stress. The proposed method is developed for lognormal life distributions with the mean and the log of the standard deviation of the normal distribution assumed to be linear functions of the stress. A modified version of maximum likelihood predictive density (MLPD) first proposed in [1] is used to estimate the predictive distribution of a future observation under nominal use. These modifications are a direct result of our efforts to arrive at a predictive density with readily available percentile points. The proposed method is derived from procedures based on the concept of "predictive likelihood" (see [6]) and was introduced as a non-Bayesian method. It does have a Bayesian interpretation but we investigate its properties in the frequentist sense.

The field of accelerated life testing is quite extensive with many authors assuming exponential or Weibull life distributions with the scale parameter expressed as some function of stress. Others assume a lognormal (normal) life distribution with the scale (location) parameter expressed as a function of the stress level. A sample of these formulations are discussed in the following.

Reference [7] considers maximum likelihood estimation of the failure rate of an exponential distribution at the design stress and assumes that the exponential mean is a quadratic function of the stress and that failure rate is an exponential function of the stress. Reference [8] considers the least squares estimation of the mean of a normal distribution at the design stress and assumes this mean to be a simple linear regression function of the stress while the standard deviation is a constant. Reference [3] considers the estimation of percentiles of a Weibull distribution at the design stress and assumes that the scale parameter is a polynomial function of the reciprocal of the stress level and the shape parameter to be a constant function of stress. Reference [4] considers maximum likelihood estimation of a percentile of an extreme-value distribution at the design stress with the mean a linear function of the stress and while the scale parameter is constant. Reference [9] considers maximum likelihood estimation of the mean of a normal distribution at the design stress and assumes the mean to be a linear function of the stress and the standard deviation to be independent of stress. Reference [5] considers maximum likelihood estimation of a given percentile of a Weibull distribution at a design stress and assumes that the scale parameter is an inverse power function of the stress and the shape parameter to be a constant. Reference [10] provides an extensive and comprehensive source for the research work done on accelerated life testing prior to 1990. Reference [11] is an excellent additional reference. A predictive density approach to obtaining prediction bounds for Weibull distribution under accelerated life-testing scenario in considered in [12].

II. MAXIMUM LIKELIHOOD PREDICTIVE DENSITY

A brief outline of the MLPD method of deriving the predictive density is given in this section. Consider a set of observations $X_1, X_2, \ldots, X_n$ from a distribution $f(x;\theta)$ and a set of future observations $Y_1, Y_2, \ldots, Y_m$ independent of $\tilde{X} = (X_1, X_2, ..., X_n)'$ from the same distribution. Let Z be some statistic based on $\tilde{Y} = (Y_1, Y_2, ..., Y_m)'$. Suppose one wishes to obtain an estimate of the density of Z based on the observed value $\tilde{x}$ of $\tilde{X}$. Reference [1] proposed $\hat{f}(z \mid \tilde{x}) = k(\tilde{x}) \sup_{\theta\in\Theta} f(\tilde{x};\theta) g(z;\theta)$ as a "predictive probability density" function for Z, where $f(\tilde{x};\theta)$ is the joint probability density function (pdf) of the X's and $g(z;\theta)$ is the pdf of the statistic Z, Θ is the parameter space of the unknown parameter θ, and $k(\tilde{x})$ is a normalizing constant. The mechanics of this procedure amounts to computing $\hat{f}$ by replacing θ in likelihood equation by its maximum likelihood estimate (MLE), $\hat{\theta}$, based on the data $(\tilde{x}, z)$ and then normalizing the resulting function. It is important to note that $\hat{\theta}$ is based on both $\tilde{x}$ and z rather than on the past data $\tilde{x}$ alone. This procedure of obtaining predictive densities is known as the profile predictive likelihood method (see [6]) and the probability density thus obtained is called the maximum likelihood predictive density (MLPD) of Z.

III. THE PROPOSED METHOD

Let D, L, and H denote the design, low, and high stress levels respectively. We make the following model assumptions:

A. The logarithm of the (component) life has a normal distribution with mean $\mu(V)$ and standard deviation $\sigma(V)$ at stress V (possibly transformed).

B. The mean of the log life is a linear function of the stress V. That is

$$\mu(V) = \beta_0 + \beta_1 V. \tag{1}$$

C. The log of the standard deviation of the log life is a linear function of the stress given by

$$\ln \sigma(V) = \alpha_0 + \alpha_1 V. \tag{2}$$

D. The lifetimes of the components are independent of each other.

The high stress is chosen as high as possible but not so high to invalidate the assumed physical model. The low stress level is chosen to be between the design stress level and the high stress level. It is preferable for it to be close to the design level but one has to make sure that it is high enough to ensure the failure of all of the components during the test (see [13]). Let μ_D, μ_L, and μ_H be the mean life at the design, low, and high stress levels respectively. Without loss of generality let $V_D = 0$, $0 < V_L < 1$, and $V_H = 1$, because otherwise, we can reparameterize so that $V^* = (V - V_D)(V_H - V_D)^{-1}$. Therefore, with the added assumption $\beta_1 < 0$ the mean life under each of the three levels satisfy $\mu(V_D) > \mu(V_L) > \mu(V_H)$ where $\mu(V_D) = \beta_0$. Let $\sigma_D = \sigma(V_D)$, $\sigma_L = \sigma(V_L)$, and $\sigma_H = \sigma(V_H)$ be the standard deviations at the design, low, and high stress levels. Then $\ln \sigma_D = \alpha_0$, $\ln \sigma_L = \alpha_0 + \alpha_1 V_L$, and $\ln \sigma_H = \alpha_0 + \alpha_1$. We further assume that $\alpha_1 < 0$. Let n_L and n_H be the number of components subjected to the low and high stress levels respectively and $N = n_L + n_H$ be the total number of components on test. The experiment is continued until all the components fail. Let x_{L1}, $x_{L2}, \ldots$, x_{Ln_L} and x_{H1}, $x_{H2}, \ldots$, x_{Hn_H} be the natural log failure times for the low and high stress levels respectively. Let Z be the log of a future observation from the design level of stress and let $\tilde{X} = \left(X_{L1}, X_{L2}, \ldots, X_{Ln_L}, X_{H1}, X_{H2}, \ldots, X_{Hn_H}\right)'$. Then the proposed prediction bound can be found in the following manner.

First we find the maximum likelihood estimates of σ_L and σ_H using only the data from low and high levels of stress (see equations A19 and A20). Then we estimate $\tilde{\alpha}_1$ using equation

A21. Using the equations A11, A12 ..., and A17 and replacing $\hat{\alpha}_1$ by $\tilde{\alpha}_1$, we evaluate quantities defined by C_1, C_2, …, and C_7 respectively. Next using the evaluated values of C_1, C_2, …, C_7 and equations A23, A24, …, A28, we estimate $\tilde{W}_{Lj}$, $\tilde{k}_L$, $\tilde{W}_{Hj}$, $\tilde{k}_H$, $\tilde{W}_D$, and $\tilde{k}_D$ respectively. Then the quantities A_0, A_1, and A_2 are evaluated using the equations A29, A30, and A31 respectively. Since the predictive density given by equation A33 is a t-distribution with N degrees of freedom, we estimate the 100 p th percentile of the predictive density z_p by

$$z_p = N^{-1/2}\left(A_0A_2 - A_1^2\right)^{1/2} A_2^{-1} t_{p,N} - A_1A_2^{-1}. \quad (3)$$

Then the 100 p th percentile of the lognormal distribution at the nominal stress level is

$$\tau_p = anti\log\left(z_p\right). \quad (4)$$

IV. MONTE CARLO SIMULATION

All simulation work was done on an HP9000/735 machine with HP-UX operating system in double precision and the simulation programs were written in FORTRAN. These programs called International Mathematical and Statistical Libraries (IMSL) subroutines, whenever applicable. They were run on a FORTRAN 77 compiler. Each simulation was started by setting the initial seed to 123457.

We investigated the coverage probabilities of the first, fifth, tenth, ninetieth, ninety-fifth, and ninety-ninth percentiles of the estimated predictive density for a future observation at the design stress level. Specifically we estimated $E\left[P\left(Z \geq \hat{Z}_p\right)\right]$ for the lower percentiles and $E\left[P\left(Z \leq \hat{Z}_p\right)\right]$ for the upper percentiles where $\hat{Z}_p$ is the 100 p th percentile where of the fitted predictive density. We also estimated $E\left(\hat{Z}_p^*\right)$ where $\hat{Z}_p^*$ is the standardized estimated percentile given by $\hat{Z}_p^* = \sigma_D^{-1}\left\{\hat{Z}_p - \mu\left(V_D\right)\right\}$.

Simulation parameters were motivated by the data given in reference [2] from a life test with several levels of acceleration. Reference [2] fitted the Arrhenius model $\mu(T) = \beta_0 + \beta_1 T^{-1}$ where $\mu(T)$ was the logarithmic (base 10) mean life and T was the absolute temperature in K^0. The design temperature was $130\,^0C$ and the other levels of temperature were $150\,^0C$, $170\,^0C$, $190\,^0C$, and $220\,^0C$. According to our parameterization, let $\mu(V) = \beta_0 + \beta_1 V$ and $\ln\sigma(V) = \alpha_0 + \alpha_1 V$, where

$$V = \left[\frac{1}{273.16+T} - \frac{1}{273.16+130}\right]\left[\frac{1}{273.16+220} - \frac{1}{273.16+130}\right]^{-1}.$$

The means of the natural log of data after 33 months at $170\,^0C$, $190\,^0C$ and $220\,^0C$ were 8.36, 7.27, and 6.30 respectively and the standard deviations of the natural log of data after 33 months at $170\,^0C$, $190\,^0C$ and $220\,^0C$ were 0.469, 0.729, and 0.189 respectively. Although we expect the standard deviation to decrease as temperature increases, in this data standard deviation has

increased from $170\,^0C$ to $190\,^0C$ before it decreases again at $220\,^0C$. Therefore we use the information from $170\,^0C$ and $220\,^0C$ to estimate the following linear relationships for the mean and the standard deviation.
Using the data from levels $170\,^0C$ and $220\,^0C$ after 33 months, a linear relationship between $\ln\hat{\sigma}(V)$ and V was found to be $\ln\hat{\sigma}(V)=0.13-1.79\,V$ and a linear relationship between $\hat{\mu}$ and V was found to be $\hat{\mu}=10.43-4.13V$. Consider the transformation $y=\frac{1}{10.43}\ln(Time)$. Then we get $\hat{\mu}(y\,|\,V)=1.00-0.4V$. Since $\sigma[\ln(Time)\,|\,V]=\exp\{0.13-1.79V\}$ and $\sigma(y\,|\,V)=\frac{1}{10.43}\,\sigma[\ln(Time)\,|\,V]$ we get $\ln\sigma(y\,|\,V)=\ln\sigma(\ln(Time)\,|\,V)-\ln(10.43)=$ $0.13-1.79\,V-\ln(10.43)$ = $-2.21-1.79V$.

For the simulation we chose the following parameter values: β_0 = 1, β_1 = –0.8 and –0.4, α_0 = –5.0, –3.5, and –2.0, and α_1 = –0.30, –0.15, and 0.00. The values of V_L were changed according to V_L = 0.125 (0.125) 0.5, 0.75. Since changing α_0 and β_1 did not change the coverage probabilities significantly we report the results only for β_0 = 1, β_1 = –0.4, α_0 = –3.5, and α_1 = –0.30, –0.15, and 0.00. Only limited results of the simulation study are reported in this paper.

Using the IMSL subroutine DRNNOR two samples of size n_L and n_H were generated with means $\mu(V_L)$ and $\mu(V_H)$ and standard deviations σ_L and σ_H respectively. Sample sizes were selected according to (i) n_L =10 and n_H =10, (ii) n_L =20 and n_H =20, (iii) n_L =30 and n_H =30, (iv) n_L =40 and n_H =40. Since our preliminary simulations showed that using unequal sample sizes did not work as well as using equal sample sizes, results for only equal sample sizes case are reported. The 100 p th percentile of the predictive density Z_p was calculated using the equation (3) for p = 0.05, 0.10, 0.90, and 0.95. Using the theoretical distribution, a normal random variable with mean β_0 and standard deviation σ_D, we obtained the probability that Z^* will be greater than the estimates of the three lower percentiles and being smaller than the estimates of the three upper percentiles using the IMSL subroutine DNORDF. Percentiles for the t-distribution were calculated with 5 significant digits using Statistical Analysis System (SAS) and used for simulations. This procedure was repeated 10,000 times and the coverage probabilities were averaged. Simulation results show that the coverages are slightly liberal. That is the average of the probabilities of a normal random variable from the theoretical distribution being greater than the estimated lower prediction limits or being smaller than the estimated upper prediction limits is slightly lower than the nominal value. Therefore an *ad-hoc* adjustment motivated by the preliminary simulation results was made in equation (3). Replacing N by $(N-\sqrt{N}-2)$ in equation (3) (including the degrees of freedom of the t-distribution) improved the coverage. Simulation results show that changes in β_1 and α_0 do not change the expected values of percentile points and the coverages to the third decimal place. As such simulation results for only a subset of parameter combinations are given in Tables 3 & 4.

V. EXAMPLE

Zahang et al. [14] provide data from an accelerated life test of white organic light-emitting diodes (Table 1). In this research, stress factor is the current measured in mA, the design level of stress is 3.20mA and the three levels of acceleration are 9.64 mA, 17.09 mA, and 22.58 mA respectively. They use the lognormal model to fit the data and assume the standard deviation of the corresponding normal distribution to be constant. Our model assumes that $\ln\sigma_L = \alpha_0 + \alpha_1 V_L$. For this example, we assume that $\ln\sigma_L = \alpha_0$, or $\alpha_1 = 0$.

Table 1. Data from Example.

Failure Times in Hours	Current Stress/ mA		
	9.64 mA	**17.09 mA**	**22.58 mA**
t_1	1691.50	601.50	406.00
t_2	2084.67	689.67	440.50
t_3	2100.32	697.33	463.50
t_4	2374.50	716.50	532.50
t_5	2421.50	785.50	555.50
t_6	2586.00	854.50	643.67
t_7	2621.50	889.50	651.33
t_8	2680.50	1115.67	716.50
t_9	2868.00	1131.33	762.50
t_{10}	2879.50	1251.50	-

Using the data and equations A29, A30, and A31 it can be shown that $A_0 = 55.16703$, $A_1 = -5.64066$, and $A_2 = 0.58359$. Plugging these quantities into equation 3 results in $t_p = 4.13602\left(-9.66533 + z_p\right)$.

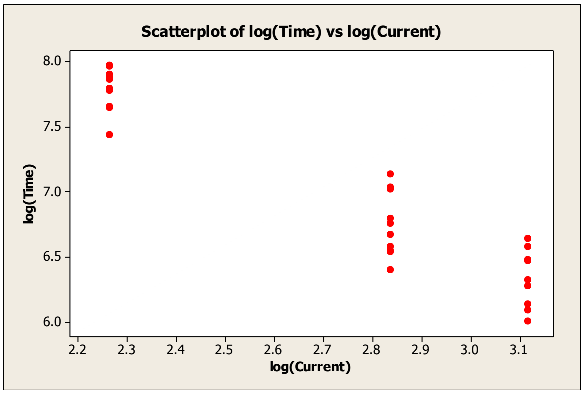

Figure 1. Log(Time) vs. Log(Current)

Table 2. Comparison of Percentiles

Comparison of Percentiles from Our Method to Zhang et al. Method			
Percentile	**Zhang et al.**	**Our Method**	**Our Method with Adjustment**
Fifth	11182.12	10386.49	10290.49
Tenth	12077.47	11441.17	11385.98
Fiftieth	15838.04	15781.57	15781.57
Ninetieth	20769.53	21713.43	21818.68
Ninety Fifth	22432.55	23918.26	24127.31

Table 3: **Approximate coverage probabilities** $E\left[P\left(Z \geq \hat{Z}_p\right)\right]$ **and** $E\left[\hat{Z}_p^*\right]$ **for lower percentiles.** Standard deviations of the percentile points are given within parenthesis.

n_L	n_H	$E\left[\hat{Z}_{.1}^*\right]$	$E\left[\hat{Z}_{.05}^*\right]$	$E\left[P\left(Z \geq \hat{Z}_{.1}\right)\right]$ Unadjusted	$E\left[P\left(Z \geq \hat{Z}_{.1}\right)\right]$ Adjusted	$E\left[P\left(Z \geq \hat{Z}_{.05}\right)\right]$ Unadjusted	$E\left[P\left(Z \geq \hat{Z}_{.05}\right)\right]$ Adjusted
α_1 = -0.30, V_L = 0.125							
10	10	-1.297 (0.491)	-1.690 (0.570)	0.878	0.923	0.929	0.963
20	20	-1.296 (0.345)	-1.675 (0.513)	0.890	0.917	0.940	0.961
30	30	-1.290 (0.278)	-1.664 (0.320)	0.893	0.913	0.944	0.958
40	40	-1.290 (0.241)	-1.662 (0.277)	0.895	0.911	0.945	0.957
α_1 = -0.30, V_L = 0.75							
10	10	-3.092 (3.802)	-4.028 (4.028)	0.895	0.932	0.937	0.966
20	20	-2.176 (1.806)	-2.814 (2.219)	0.896	0.918	0.938	0.955
30	30	-1.860 (1.257)	-2.399 (1.516)	0.895	0.911	0.937	0.950
40	40	-1.722 (1.025)	-2.216 (1.226)	0.896	0.909	0.938	0.948
α_1 = -0.15, V_L = 0.125							
10	10	-1.299 (0.497)	-1.692 (0.576)	0.878	0.923	0.929	0.963
20	20	-1.297 (0.348)	-1.677 (0.401)	0.890	0.917	0.940	0.961
30	30	-1.291 (0.281)	-1.665 (0.322)	0.893	0.913	0.944	0.958
40	40	-1.292 (0.244)	-1.663 (0.279)	0.895	0.911	0.945	0.957
α_1 = -0.15, V_L = 0.75							
10	10	-3.235 (3.824)	-4.216 (4.844)	0.899	0.935	0.940	0.969
20	20	-2.270 (1.845)	-2.936 (2.240)	0.899	0.921	0.941	0.958
30	30	-1.932 (1.296)	-2.492 (1.540)	0.897	0.914	0.940	0.953
40	40	-1.780 (1.062)	-2.219 (1.250)	0.898	0.912	0.941	0.951
α_1 = 0.00, V_L = 0.125							
10	10	-1.302 (0.502)	-1.696 (0.581)	0.878	0.923	0.929	0.963
20	20	-1.299 (0.352)	-1.679 (0.404)	0.890	0.917	0.940	0.961
30	30	-1.292 (0.284)	-1.667 (0.325)	0.893	0.913	0.944	0.958
40	40	-1.292 (0.246)	-1.664 (0.282)	0.895	0.912	0.945	0.958
α_1 = 0.00, V_L = 0.75							
10	10	-3.406 (3.858)	-4.438 (4.852)	0.899	0.938	0.943	0.971
20	20	-2.384 (1.898)	-2.083 (2.272)	0.901	0.924	0.944	0.961
30	30	-2.019 (1.349)	-2.604 (1.575)	0.900	0.917	0.943	0.956
40	40	-1.850 (1.111)	-2.382 (1.285)	0.901	0.915	0.944	0.954

Table 4: **Approximate coverage probabilities** $E\left[P\left(Z \geq \hat{Z}_p\right)\right]$ **and** $E\left[\hat{Z}_p^*\right]$ **for upper percentiles.** Standard deviations of the percentile points are given within parenthesis.

n_L	n_H	$E\left[\hat{Z}_{.90}^*\right]$	$E\left[\hat{Z}_{.95}^*\right]$	$E\left[P\left(Z \leq \hat{Z}_{.90}\right)\right]$ Unadjusted	Adjusted	$E\left[P\left(Z \leq \hat{Z}_{.95}\right)\right]$ Unadjusted	Adjusted
α_1 = -0.30, V_L = 0.125							
10	10	1.305 (0.491)	1.697 (0.571)	0.879	0.924	0.930	0.964
20	20	1.301 (0.345)	1.681 (0.398)	0.891	0.916	0.941	0.961
30	30	1.293 (0.284)	1.666 (0.327)	0.893	0.913	0.943	0.958
40	40	1.291 (0.243)	1.662 (0.279)	0.895	0.911	0.945	0.957
α_1 = -0.30, V_L = 0.75							
10	10	3.124 (3.825)	4.061 (4.869)	0.901	0.937	0.941	0.969
20	20	2.191 (1.792)	2.829 (2.206)	0.898	0.920	0.939	0.956
30	30	1.865 (1.275)	2.404 (2.536)	0.893	0.909	0.935	0.948
40	40	1.714 (1.031)	2.209 (1.233)	0.894	0.908	0.937	0.948
α_1 = -0.15, V_L = 0.125							
10	10	1.308 (0.496)	1.701 (0.576)	0.879	0.925	0.930	0.964
20	20	1.303 (0.349)	1.683 (0.401)	0.891	0.918	0.941	0.961
30	30	1.294 (0.287)	1.668 (0.329)	0.893	0.913	0.943	0.958
40	40	1.291 (0.245)	1.663 (0.281)	0.895	0.911	0.945	0.957
α_1 = -0.15, V_L = 0.75							
10	10	3.273 (3.850)	4.254 (4.871)	0.903	0.940	0.944	0.972
20	20	2.287 (1.830)	2.953 (2.225)	0.901	0.923	0.942	0.959
30	30	1.937 (1.317)	2.497 (1.563)	0.896	0.912	0.938	0.951
40	40	1.771 (1.069)	2.282 (1.258)	0.897	0.911	0.940	0.951
α_1 = 0.00, V_L = 0.125							
10	10	1.311 (0.502)	1.706 (0.580)	0.879	0.925	0.930	0.964
20	20	1.305 (0.353)	1.685 (0.405)	0.891	0.918	0.941	0.961
30	30	1.295 (0.291)	1.669 (0.332)	0.893	0.913	0.943	0.958
40	40	1.292 (0.248)	1.664 (0.283)	0.895	0.912	0.945	0.958
α_1 = 0.00, V_L = 0.75							
10	10	3.448 (3.887)	4.481 (4.881)	0.906	0.942	0.947	0.974
20	20	2.402 (1.882)	3.102 (2.254)	0.904	0.926	0.945	0.961
30	30	2.024 (1.371)	2.609 (1.599)	0.898	0.915	0.941	0.954
40	40	1.841 (1.119)	2.372 (1.293)	0.899	0.913	0.943	0.953

REFERENCES

[1] M. Lejeune and G. D. Faulkenberry, "A Simple Predictive Function", Journal of the American Statistical Association, 77, pp 654-657, 1982.

[2] D. E. Crawford, "Analysis of Incomplete Life Test Data on Motorettes", Insulation/Circuits, 16, pp 43-48, 1970.

[3] N. R. Mann, "Design of Over-Stress Life-Test Experiments When Failure Times Have a Two Parameter Weibull Distribution", Technometrics, 14, pp 437-451, 1972.

[4] W. Q. Meeker and W. Nelson, "Optimum Accelerated Life-Tests for the Weibull and Extreme Value Distributions", IEEE Transactions of Reliability, R-24, 5, pp 321-332, 1975.

[5] W. Nelson and W. Q. Meeker, "Theory for Optimum Accelerated Life Tests for Weibull and Extreme Value Distributions, Technometrics, 20, pp 171-177, 1978.

[6] J. F. Bjornstad, "Predictive Likelihood: A Review", Statistical Science, 5, pp 242-265, 1990.

[7] H. Chernoff, "Optimal Accelerated Life Designs for Estimation", Technometrics, 4, pp 381-408, 1962.

[8] R. E. Little and E. H. Jebe, "A Note on the Gain in Precision for Optimal Allocation in Regression as Applied to Extrapolation in S-R Fatigue Testing", Technometrics, 11, pp 389-392, 1969.

[9] W. Nelson and T. J. Keilpinski, "Theory for Optimum Accelerated Life Tests for Normal and Lognormal Life Distributions", Technometrics, 18, pp 105-114, 1976.

[10] W. Nelson, Accelerated Testing: Statistical Models, Test Plans, and Data Analysis. New York: John Wiley, 1990.

[11] W. Q. Meeker and L A. Escobar: Statistical Methods for Reliability Data. New York: John Wiley, 1998.

[12] A. A. Jayawardhana and V. A. Samaranayake, "A Prediction Bounds in Accelerated Life Testing: Weibull Models with Inverse Power Relationship". Journal of Quality Technology, 35, No.1, pp 89-103, 2003.

[13] W. Q. Meeker and G. J. Hahn, "How to Plan Accelerated Life Tests: Some Practical Guidelines", Volume 10 of the ASQC Basic References in Quality Control: Statistical Techniques. American Society of Quality Control, Milwaukee, Wisconsin, 1985.

[14] J. Zhang, F. Liu, Y. Liu, H. Wu, and A. Zhou, “A Study of Accelerated Life Test of White OLED Based on Maximum Likelihood Estimation Using Lognormal Distribution”, IEEE Transactions on Electronic Devices, Vol. 59, No. 12, December 2012.

APPENDIX
DERIVATION OF THE PREDICTIVE DENSITY

We assume the model formulation given in the section on proposed method. Then the joint probability density function of $\tilde{X}$ and Z is

$$f\left(\tilde{x},z\right) = \left(\sqrt{2\pi}\right)^{-(N+1)} \sigma_L^{-n_L}\ \sigma_H^{-n_H}\ \sigma_D^{-1}\ \exp\left\{-\left(2\sigma_L^2\right)^{-1}\sum_{j=1}^{n_L}\left(x_{Lj}-\beta_0-\beta_1 V_L\right)^2\right\}\times$$
$$\exp\left\{-\left(2\sigma_H^2\right)^{-1}\sum_{j=1}^{n_H}\left(x_{Hj}-\beta_0-\beta_1\right)^2\right\}\ \exp\left\{-\left(2\sigma_D^2\right)^{-1}\left(z-\beta_0\right)^2\right\}$$

$$= \left(\sqrt{2\pi}\right)^{-(N+1)} \exp\left[-\left\{n_L\left(\alpha_0+\alpha_1 V_L\right)+n_H\left(\alpha_0+\alpha_1\right)+\alpha_0\right\}\right]\times$$
$$\exp\left\{-2^{-1}e^{-2\left(\alpha_0+\alpha_1 V_L\right)}\sum_{j=1}^{n_L}\left(x_{Lj}-\beta_0-\beta_1 V_L\right)^2\right\}\times$$
$$\exp\left\{-2^{-1}e^{-2\left(\alpha_0+\alpha_1\right)}\sum_{j=1}^{n_H}\left(x_{Hj}-\beta_0-\beta_1\right)^2\right\}\times$$
$$\exp\left\{-2^{-1}e^{-2\alpha_0}\left(z-\beta_0\right)^2\right\}. \qquad \text{(A1)}$$

Then the log likelihood function given observations $\tilde{x} = \left(x_{L1},\ x_{L2},\ldots,\ x_{Ln_L},\ x_{H1},\ x_{H2},\ \ldots,\ x_{Hn_H}\right)'$ and z is

$$\ln\left\{f\left(\alpha_0,\alpha_1,\beta_0,\beta_1 \mid \tilde{x},z\right)\right\} = -\left(N+1\right)\ln\left(\sqrt{2\pi}\right)-n_L\left(\alpha_0+\alpha_1 V_L\right)-n_H\left(\alpha_0+\alpha_1\right)$$
$$-\alpha_0-2^{-1}e^{-2\left(\alpha_0+\alpha_1 V_L\right)}\sum_{j=1}^{n_L}\left(x_{Lj}-\beta_0-\beta_1 V_L\right)^2$$
$$-2^{-1}e^{-2\left(\alpha_0+\alpha_1\right)}\sum_{j=1}^{n_H}\left(x_{Hj}-\beta_0-\beta_1\right)^2$$
$$-2^{-1}e^{-2\alpha_0}\left(z-\beta_0\right)^2. \qquad \text{(A2)}$$

Taking partial derivatives of $\ln\left\{f\left(\alpha_0,\alpha_1,\beta_0,\beta_1 \mid \tilde{x},z\right)\right\}$ with respect to α_0 and α_1, setting them equal to zero we obtain

$$N+1 = \hat{\sigma}_D^{-2}e^{-2\hat{\alpha}_1 V_L}\sum_{j=1}^{n_L}\left(x_{Lj}-\hat{\beta}_0-\hat{\beta}_1 V_L\right)^2 + \hat{\sigma}_D^{-2}e^{-2\hat{\alpha}_1}\sum_{j=1}^{n_H}\left(x_{Hj}-\hat{\beta}_0-\hat{\beta}_1\right)^2$$
$$+\ \hat{\sigma}_D^{-2}\left(z-\hat{\beta}_0\right)^2 \qquad \text{(A3)}$$

and

$$n_L V_L + n_H = V_L \hat{\sigma}_D^{-2} e^{-2\hat{\alpha}_1 V_L} \sum_{j=1}^{n_L} \left(x_{Lj} - \hat{\beta}_0 - \hat{\beta}_1 V_L \right)^2 + \hat{\sigma}_D^{-2} e^{-2\hat{\alpha}_1} \sum_{j=1}^{n_H} \left(x_{Hj} - \hat{\beta}_0 - \hat{\beta}_1 \right)^2 . \qquad \text{(A4)}$$

By definition $\sigma_L^2 = e^{2\alpha_0 + 2\alpha_1 V_L} = \sigma_D^2 e^{2\alpha_1 V_L}$ and $\sigma_H^2 = e^{2\alpha_0 + 2\alpha_1} = \sigma_D^2 e^{2\alpha_1}$ which implies that
$\sigma_L^{-2} \sigma_H^2 = e^{2\alpha_1 (1 - V_L)}$. Then the maximum likelihood estimates of σ_L, σ_H, and α_1 satisfy

$$e^{2\hat{\alpha}_1} = \left(\hat{\sigma}_L^{-2} \hat{\sigma}_H^2 \right)^{1/(1-V_L)} . \qquad \text{(A5)}$$

Substituting this in equation (A3) yields

$$(N+1)\hat{\sigma}_D^2 = \left(\hat{\sigma}_L^2 \hat{\sigma}_H^{-2} \right)^{V_L/(1-V_L)} \sum_{j=1}^{n_L} \left(x_{Lj} - \hat{\beta}_0 - \hat{\beta}_1 V_L \right)^2 + \left(\hat{\sigma}_L^2 \hat{\sigma}_H^{-2} \right)^{1/(1-V_L)} \sum_{j=1}^{n_H} \left(x_{Hj} - \hat{\beta}_0 - \hat{\beta}_1 \right)^2 + \left(z - \hat{\beta}_0 \right)^2 . \qquad \text{(A6)}$$

Taking partial derivatives of $\ln\left\{ f\left(\alpha_0, \alpha_1, \beta_0, \beta_1 \mid \tilde{x}, z \right) \right\}$ in equation (A2) with respect to β_0 and β_1 and setting them to zero we obtain

$$e^{-2\hat{\alpha}_1 V_L} \sum_{j=1}^{n_L} x_{Lj} + e^{-2\hat{\alpha}_1} \sum_{j=1}^{n_H} x_{Hj} + z = \left(n_L e^{-2\hat{\alpha}_1 V_L} + n_H e^{-2\hat{\alpha}_1} + 1 \right) \hat{\beta}_0 + \left(n_L V_L e^{-2\hat{\alpha}_1 V_L} + n_H e^{-2\hat{\alpha}_1} \right) \hat{\beta}_1 \qquad \text{(A7)}$$

and

$$V_L e^{-2\hat{\alpha}_1 V_L} \sum_{j=1}^{n_L} x_{Lj} + e^{-2\hat{\alpha}_1} \sum_{j=1}^{n_H} x_{Hj} = \left(n_L V_L e^{-2\hat{\alpha}_1 V_L} + n_H e^{-2\hat{\alpha}_1} \right) \hat{\beta}_0 + \left(n_L V_L^2 e^{-2\hat{\alpha}_1 V_L} + n_H e^{-2\hat{\alpha}_1} \right) \hat{\beta}_1 . \qquad \text{(A8)}$$

From equations (A7) and (A8), the maximum likelihood estimates of β_0 and β_1 based on $\tilde{x}$ and z are

$$\hat{\beta}_1 = \left[C_4 + C_5 \right]^{-1} \left[C_1 \sum_{j=1}^{n_L} x_{Lj} + C_2 \sum_{j=1}^{n_H} x_{Hj} - C_3 z \right] \qquad \text{(A9)}$$

$$\hat{\beta}_0 = \left[C_4 + C_5 \right]^{-1} \left[C_6 \sum_{j=1}^{n_L} x_{Lj} + C_7 \sum_{j=1}^{n_H} x_{Hj} + C_5 z \right] \qquad \text{(A10)}$$

where

$$C_1 = V_L e^{-2\hat{\alpha}_1 V_L} + n_H \left(V_L - 1 \right) e^{-2\hat{\alpha}_1 (V_L + 1)} , \qquad \text{(A11)}$$

$$C_2 = e^{-2\hat{\alpha}_1} + n_L \left(1 - V_L \right) e^{-2\hat{\alpha}_1 (V_L + 1)} \qquad \text{(A12)}$$

$$C_3 = n_L V_L e^{-2\hat{\alpha}_1 V_L} + n_H e^{-2\hat{\alpha}_1} , \qquad \text{(A13)}$$

$$C_4 = n_L n_H \left(1 - V_L \right)^2 e^{-2\hat{\alpha}_1 (V_L + 1)} \qquad \text{(A14)}$$

$$C_5 = n_L V_L^2 e^{-2\hat{\alpha}_1 V_L} + n_H e^{-2\hat{\alpha}_1}, \tag{A15}$$

$$C_6 = n_H \left(1 - V_L\right) e^{-2\hat{\alpha}_1 (V_L + 1)} \tag{A16}$$

and $C_7 = n_L V_L \left(V_L - 1\right) e^{-2\hat{\alpha}_1 (V_L + 1)}$. (A17)

Using the method proposed in [1] the maximum likelihood predictive density function of z given $\tilde{x}$ is obtained by substituting the maximum likelihood estimates of the parameters in equation (A1), yielding

$$\begin{aligned} \tilde{f}\left(z\right) &= k\left(\tilde{x}\right) \hat{\sigma}_L^{-n_L} \hat{\sigma}_H^{-n_H} \hat{\sigma}_D^{-1} = k\left(\tilde{x}\right) \hat{\sigma}_D^{-(N+1)} e^{-(n_L V_L + n_H)\hat{\alpha}_1} \\ &= k\left(\tilde{x}\right) \left\{ e^{-2\hat{\alpha}_1 V_L} \sum_{j=1}^{n_L} \left(x_{Lj} - \hat{\beta}_0 - \hat{\beta}_1 V_L \right)^2 + e^{-2\hat{\alpha}_1} \sum_{j=1}^{n_H} \left(x_{Hj} - \hat{\beta}_0 - \hat{\beta}_1 \right)^2 \right. \\ &\qquad \left. + \left(z - \hat{\beta}_0 \right)^2 \right\}^{-(N+1)/2} \times e^{-(n_L V_L + n_H)\hat{\alpha}_1} \end{aligned} \tag{A18}$$

where $k\left(\tilde{x}\right)$ is a normalizing constant. Note that $\hat{\sigma}_L^2$, $\hat{\sigma}_H^2$, $\hat{\beta}_0$, and $\hat{\beta}_1$ are functions of the future observation Z. It is easily seen that $\tilde{f}$ is not in the form of any recognizable density and derivation of its percentile points would require numerical methods. Since our goal is to find an easy to use predictive density, we looked for approximations that would simplify the above function. A possible approximation is to replace $\hat{\sigma}_L^2$ and $\hat{\sigma}_H^2$ by the maximum likelihood estimates calculated using only the past observations. Note that

$$\tilde{\sigma}_L^2 = n_L^{-1} \sum_{J=1}^{n_L} \left(x_{Lj} - n_L^{-1} \sum_{i=1}^{n_L} x_{Lj} \right)^2 \tag{A19}$$

and $\tilde{\sigma}_H^2 = n_H^{-1} \sum_{J=1}^{n_H} \left(x_{Hj} - n_H^{-1} \sum_{i=1}^{n_H} x_{Hj} \right)^2$ (A20)

are the maximum likelihood estimates of σ_L^2 and σ_H^2. Now using equation (A5), let

$$e^{2\tilde{\alpha}_1} = \left(\tilde{\sigma}_H^2 \tilde{\sigma}_L^{-2} \right)^{1/(1-V_L)} . \tag{A21}$$

Then the predictive density in equation (A18) can be revised as

$$\begin{aligned} \tilde{f}_0\left(z\right) = k_1\left(\tilde{x}\right) &\left\{ e^{-2\tilde{\alpha}_1 V_L} \sum_{j=1}^{n_L} \left(x_{Lj} - \hat{\beta}_0 - \hat{\beta}_1 V_L \right)^2 + e^{-2\tilde{\alpha}_1} \sum_{j=1}^{n_H} \left(x_{Hj} - \hat{\beta}_0 - \hat{\beta}_1 \right)^2 \right. \\ &\qquad \left. + \left(z - \hat{\beta}_0 \right)^2 \right\}^{-(N+1)/2} \end{aligned} \tag{A22}$$

with $\tilde{\sigma}_L^2$, $\tilde{\sigma}_H^2$ replacing $\hat{\sigma}_L^2$, $\hat{\sigma}_H^2$. Observe that $k_1\left(\tilde{x}\right)$ is a revised normalizing constant. Substituting the maximum likelihood estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ from equations (A.9) and (A10) into each term in equation (A22) we obtain

$$
\begin{aligned}
x_{Lj} - \hat{\beta}_0 - \hat{\beta}_1 V_L &= x_{Lj} - \left[C_4 + C_5\right]^{-1}\left[C_6\sum_{j=1}^{n_L} x_{Lj} + C_7\sum_{j=1}^{n_H} x_{Hj} + C_5 z\right] \\
&\quad - \left[C_4 + C_5\right]^{-1}\left[C_1\sum_{j=1}^{n_L} x_{Lj} + C_2\sum_{j=1}^{n_H} x_{Hj} - C_3 z\right]V_L \\
&= x_{Lj} - \left[C_4 + C_5\right]^{-1}\left[\left(C_6 + C_1 V_L\right)\sum_{j=1}^{n_L} x_{Lj} + \left(C_7 + C_2 V_L\right)\sum_{j=1}^{n_H} x_{Hj}\right] \\
&\quad + \left[C_4 + C_5\right]^{-1}\left[C_3 V_L - C_5\right]z \\
&= W_{Lj} + k_L z,
\end{aligned}
$$

where $W_{Lj} = x_{Lj} - \left[C_4 + C_5\right]^{-1}\left[\left(C_6 + C_1 V_L\right)\sum_{j=1}^{n_L} x_{Lj} + \left(C_7 + C_2 V_L\right)\sum_{j=1}^{n_H} x_{Hj}\right]$ (A23)

and $k_L = \left[C_4 + C_5\right]^{-1}\left[C_3 V_L - C_5\right]$, (A24)

$$
\begin{aligned}
x_{Hj} - \hat{\beta}_0 - \hat{\beta}_1 &= x_{Hj} - \left[C_4 + C_5\right]^{-1}\left[C_6\sum_{j=1}^{n_L} x_{Lj} + C_7\sum_{j=1}^{n_H} x_{Hj} + C_5 z\right] \\
&\quad - \left[C_4 + C_5\right]^{-1}\left[C_1\sum_{j=1}^{n_L} x_{Lj} + C_2\sum_{j=1}^{n_H} x_{Hj} - C_3 z\right] \\
&= x_{Hj} - \left[C_4 + C_5\right]^{-1}\left[\left(C_6 + C_1\right)\sum_{j=1}^{n_L} x_{Lj} + \left(C_7 + C_2\right)\sum_{j=1}^{n_H} x_{Hj}\right] \\
&\quad + \left[C_4 + C_5\right]^{-1}\left[C_3 - C_5\right]z \\
&= W_{Hj} + k_H z,
\end{aligned}
$$

where $W_{Hj} = x_{Hj} - \left[C_4 + C_5\right]^{-1}\left[\left(C_6 + C_1\right)\sum_{j=1}^{n_L} x_{Lj} + \left(C_7 + C_2\right)\sum_{j=1}^{n_H} x_{Hj}\right]$ (A25)

and $k_H = \left[C_4 + C_5\right]^{-1}\left[C_3 - C_5\right]$, (A26)

and
$$
\begin{aligned}
z - \hat{\beta}_0 &= z - \left[C_4 + C_5\right]^{-1}\left[C_6\sum_{j=1}^{n_L} x_{Lj} + C_7\sum_{j=1}^{n_H} x_{Hj} + C_5 z\right] \\
&= -\left[C_4 + C_5\right]^{-1}\left[C_6\sum_{j=1}^{n_L} x_{Lj} + C_7\sum_{j=1}^{n_H} x_{Hj}\right] + \left[C_4 + C_5\right]^{-1} C_4 z \\
&= W_D + k_D z
\end{aligned}
$$

where $W_D = -\left[C_4 + C_5\right]^{-1}\left[C_6\sum_{j=1}^{n_L} x_{Lj} + C_7\sum_{j=1}^{n_H} x_{Hj}\right]$ (A27)

and $k_D = \left[C_4 + C_5\right]^{-1} C_4$. (A28)

Observe that W_{Lj}, W_{Hj}, W_D, k_L, k_H, and k_D are all functions of $\hat{\alpha}_1$; and $\hat{\alpha}_1$ is a function of both past and future observations. To simplify the function $\tilde{f}_0(z)$ we again replace $\hat{\alpha}_1$ (which satisfies (A5)) by $\tilde{\alpha}_1$ which is implicitly defined in (A21) and is based only on the past observations in W_{Lj}, W_{Hj}, W_D, k_L, k_H, and k_D. We denote estimates
using these modifications as $\tilde{W}_{Lj}$, $\tilde{W}_{Hj}$, $\tilde{W}_D$, $\tilde{k}_L$, $\tilde{k}_H$, and $\tilde{k}_D$ respectively. Then, the revised predictive density can be written as

$$\begin{aligned}\tilde{f}_1(z) &= k_2(\tilde{x}) \left[e^{-2\tilde{\alpha}_1 V_L} \sum_{j=1}^{n_L} \left(\tilde{W}_{Lj} + \tilde{k}_L z \right)^2 + e^{-2\tilde{\alpha}_1} \sum_{j=1}^{n_H} \left(\tilde{W}_{Hj} + \tilde{k}_H z \right)^2 + \left(\tilde{W}_D + \tilde{k}_D z \right)^2 \right]^{-(N+1)/2} \\ &= k_2(\tilde{x}) \left[A_0 + 2A_1 z + A_2 z^2 \right]^{-(N+1)/2},\end{aligned}$$

where

$$A_0 = e^{-2\tilde{\alpha}_1 V_L} \sum_{j=1}^{n_L} \tilde{W}_{Lj}^2 + e^{-2\tilde{\alpha}_1} \sum_{j=1}^{n_H} \tilde{W}_{Hj}^2 + \tilde{W}_D^2, \qquad \text{(A29)}$$

$$A_1 = \tilde{k}_L e^{-2\tilde{\alpha}_1 V_L} \sum_{j=1}^{n_L} \tilde{W}_{Lj} + \tilde{k}_H e^{-2\tilde{\alpha}_1} \sum_{j=1}^{n_H} \tilde{W}_{Hj} + \tilde{k}_D \tilde{W}_D, \qquad \text{(A30)}$$

$$A_2 = n_L \tilde{k}_L^2 e^{-2\tilde{\alpha}_1 V_L} + n_H \tilde{k}_H^2 e^{-2\tilde{\alpha}_1} + \tilde{k}_D^2 \qquad \text{(A31)}$$

and $k_2(\tilde{x})$ is a normalizing constant. Further simplification yields

$$\tilde{f}_1(z) = k_3(\tilde{x}) \left[1 + \left(A_0 A_2 - A_1^2 \right)^{-1} A_2^2 \left(A_1 A_2^{-1} + z \right)^2 \right]^{-(N+1)/2} \qquad \text{(A32)}$$

where $k_3(\tilde{x})$ is a normalizing constant. It can be shown that $A_0 A_2 - A_1^2 > 0$.

Using the transformation $t = \sqrt{N} \left(A_0 A_2 - A_1^2 \right)^{-1/2} A_2 \left(A_1 A_2^{-1} + z \right)$, in equation (A32) we obtain $\hat{f}(t) = k_4(\tilde{x}) \left[1 + N^{-1} t^2 \right]^{-(N+1)/2}$ (A33)

as a predictive density of Z. Observe that this is the probability density function of a t-distribution with N degrees of freedom. Furthermore, it can be shown that

$$k_4(\tilde{x}) = \Gamma\{(N+1)/2\} \{\Gamma(N/2)\}^{-1} \left\{ \sqrt{N\pi} \right\}^{-1}.$$

Let $t_{p,N}$ be the 100 p th percentile of the t-distribution with N degrees of freedom. Then the 100 p th percentile z_p of the predictive density can be written as

$$z_p = N^{-1/2} \left(A_0 A_2 - A_1^2 \right)^{1/2} A_2^{-1} t_{p,N} - A_1 A_2^{-1}.$$

As a special case, when σ is a constant function of the stress (i.e. when $\alpha_1 = 0$), it can be shown that the predictive density of Z is also a t-distribution.

The 100 p th percentile of the lognormal distribution at the nominal level is

$$\tau_p = anti\log\left(z_p \right).$$