# A Study of the Quantile Control Chart

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## Abstract

Traditional control charts, such as Shewhart, CUSUM, and EWMA charts, typically monitor the mean and/or the standard deviation, but in many cases, monitoring the shape of the distribution is necessary. Grimshaw and Alt (1997) proposed a quantile control chart to monitor the quantile function of a continuous distribution by monitoring a selected set of quantiles. Their method is based on an asymptotic chi-square statistic. The main goal of this article is to examine the robustness of the quantile control chart under the situations where 1) the underlying distributions are of different shapes, 2) the number of chosen quantiles varies, and 3) the positions of the quantiles vary. By summarizing the simulation results, we provide recommendations for the implementation of the quantile control chart. We also examine an adjustment to the control limit to improve the performance of the quantile control chart.

**Key Words:** quantile values; control chart; asymptotic chi-square statistic; robustness.

## 1. INTRODUCTION

The quantile function, QF, of a probability distribution is the inverse of its cumulative distribution function (CDF). The CDF describes the distributional shape, hence the QF, the inverse of the CDF, does the same: a change in the quantile function indicates a change in the shape of the distribution. The median, which is also the 50$^{th}$ percentile, is one of the most well-known quantiles because of its resistance to outliers as a measure of the location of a distribution. The first and the third quartiles, which are the 25$^{th}$ and 75$^{th}$ percentile, respectively, provide a more robust description of the spread, via the interquartile range, than the range and the standard deviation. The upper and the lower 5$^{th}$ percentiles are useful to describe the upper and the lower tail of the distribution. Thus, monitoring the quantiles and the quantile function provides meaningful information about the underlying distribution.

Alt (1985) proposed a chi-square test statistic to monitor the quantiles of an unknown distribution. Grimshaw and Alt (1997) further developed this test statistic into a quantile control chart (QCC), to better detect distributional (shape) changes of a random variable, which cannot be achieved by the traditional control charts for the mean and the standard deviation. Moreover, traditional control charts, such as the Shewhart and the CUSUM control charts, require the normality assumption for the underlying distribution, however many situations exist in practice where this assumption cannot be met. In fact in some applications it is more common to have a skewed underlying distribution. To this end, Grimshaw and Alt (1997) cite an example from the Brigham Young University (BYU)

Creamery. The gross weight for a half pint bottle of milk is being monitored. Unreasonable errors for the half pint of milk produced at the BYU creamery are defined as 1) minus errors: greater than or equal to ¼ ounce, and 2) plus errors: greater than or equal to ½ ounce. Because the unreasonable errors differ in magnitude, it was natural to expect the distribution of weights to be skewed. In fact, the authors stated that “the ideal process would have a quickly decaying left tail and a more slowly decaying right tail”, and felt that the QCC would be more helpful since it monitors the distributional shape by monitoring the quantiles. To emphasize the point that monitoring the mean and variance might not be enough to monitor the shape of the distribution, note that for example, a normal distribution with mean one and standard deviation one is fundamentally different from an exponential distribution with rate 1 but they both have the same mean and variance. Thus, traditional control charts, monitoring the mean and the variance, would most likely not pick up the difference between the shapes of the two distributions. The QCC, on the other hand, is expected to be useful in detecting such a more drastic, distributional change.

In the second section of this paper, we give a detailed description of the QCC and uncover some questions in the implementation of this methodology that need to be addressed. These questions relate to the performance of the QCC with regard to (i) the shape of the underlying distribution (ii) the number of the chosen quantiles and (iii) the positions of the chosen quantiles (where the quantile function is evaluated). We follow up on these questions in section three to provide some answers based on extensive simulation results. We conclude in section four with some conclusions and recommendations.

# 2 METHODOLOGY

## 2.1 QCC: Quantile Control Chart for monitoring an unknown continuous distribution

In reality, it is more common that the normality assumption for the underlying distribution cannot be met. Thus Grimshaw and Alt (1997) proposed the QCC to monitor some specified quantiles of an unknown distribution. The proposed control chart is based on plotting and monitoring a statistic that has an asymptotic chi-square distribution. In this section, we give the description of this method. First we define the quantile function.

A shifted, piecewise linear sample quantile function defined by Parzen (1979) is used to estimate the quantiles of a random sample of size $n$:

$$\tilde{Q}(u) = n\left(\frac{2i+1}{2n} - u\right) X(i;n) + n\left(u - \frac{2i-1}{2n}\right) X(i+1;n)$$

for $\frac{2i-1}{2n} < u \leq \frac{2i+1}{2n}, i = 1,2, \dots n-1,$ where $X(1;n) \leq X(2;n) \leq \cdots \leq X(n;n)$ denote the order statistics. This function is claimed to behave well when the sample size is small. It is also the function that Grimshaw and Alt (1997) used. However, we use the more traditional definition of a sample quantile as an order statistic. Hence we take $\tilde{Q}(u) = X([un];n)$ to be the estimator, where $[un]$ denotes the largest integer not exceeding $un$ and $X([un];n)$ denotes the order statistic.

Regardless of the true underlying distribution, the quantile values are estimated from historical reference (Phase I) data. Let N subgroups of Phase I data, each of size $n$, are available. The estimator $\tilde{Q}_j(u_i)$, for a specified $u_i$, is calculated for each subgroup $j = 1,2,\dots,N$ and then averaged over $N$. Let $Q_0$ be a column vector containing $k$ elements,

$$Q_0 = \begin{bmatrix} Q_0(u_1) \\ Q_0(u_2) \\ \vdots \\ Q_0(u_k) \end{bmatrix}$$

where $Q_0(u_i) = \frac{1}{N}\sum_{j=1}^{N} \tilde{Q}_j(u_i)$ and $0 < u_1 < \cdots < u_k < 1$. Now let $\tilde{Q}$ denote a $k$ by 1 vector of sample quantiles constructed at the same $0 < u_1 < \cdots < u_k < 1$, from the Phase II data that is being monitored. Let $u = (u_1, \dots, u_k)'$ denote the $k$ by 1 vector of positive fractions. A one-sided control chart is constructed to monitor the chi-square statistics below.

$$W = (\tilde{Q} - Q_0)^T \Sigma_0^{-1} (\tilde{Q} - Q_0)$$

where the asymptotic covariance matrix $\sum_0$ is given by $\sigma_{ij} = \frac{u_i(1-u_j)}{nf(Q_0(u_i))f(Q_0(u_j))}$, for $i \leq j$. Note that this covariance matrix is symmetric and $fQ_0(u_i)$ is the density-quantile function of the in-control distribution. For a sufficiently large sample size $n$, it can be shown that W approximately follows a chi-square distribution with $k$ degrees of freedom when the monitored process is in-control. Thus the computed value of W is compared to $\chi^2_\alpha(k)$ which is the 100(1-α) percentile of the chi-square distribution with k degrees of freedom. This percentile serves as the upper control limit (UCL) of the QCC. If W > UCL, the process is declared to be out-of-control.

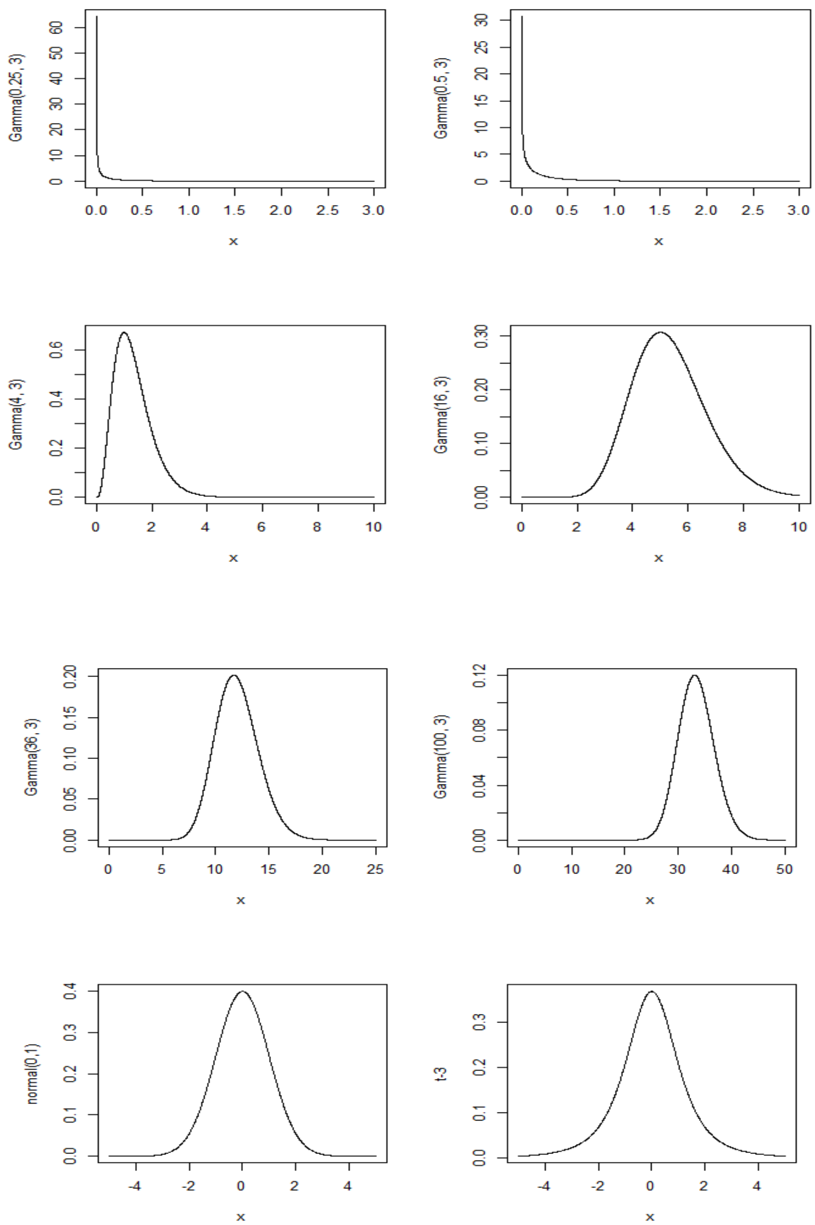
Next, we examine the approximate (large sample) in-control distribution of the plotting statistic $W$. How large does the sample size $n$ have to be for this approximation to be satisfactory? In practice, taking large samples in statistical process control is inefficient and uneconomical, so there are concerns about the performance of the chi-square critical value based chart under a "limited" sample size. In the following section, we dig into these questions via simulations.

### 2.2 Simulation results and comparison

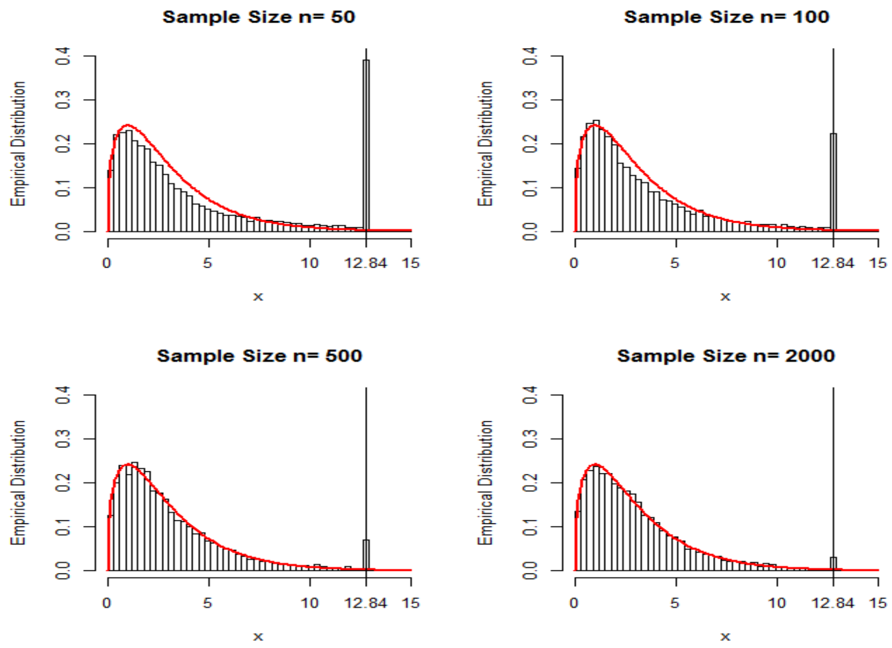
As stated in the beginning of this section, the performance of the QCC is examined from three practical aspects: (i) the impact of the shape of the underlying distribution (ii) the number of chosen quantiles and (iii) the positions of the chosen quantiles, where the quantile function is evaluated, with the nominal false alarm rate equal to 0.05 and 0.005.

The eight figures below show the eight different distributions used in our study. The gamma(0.5,3) and gamma(0.25,3) are highly right skewed distributions but gamma (4,3), gamma(16,3), gamma(36,3) and gamma(100,3) are only slightly right skewed with decreasing amount of skewness. The normal (0,1) and t(3) are symmetric distributions with different thickness of tails.

The number of chosen quantiles was varied from $k$ = 3 to $k$ = 5. The positions used are as follows: for $k$ = 3, u = (0.05, 0.5, 0.95)', u = (0.1, 0.5, 0.9)' and u = (0.25, 0.5, 0.75)'. For $k$ = 5, u = (0.05, 0.25, 0.5, 0.75, 0.95)' and u = (0.25, 0.375, 0.5, 0.625, 0.75)'.




As an example consider the simulation from the gamma(0.25, 3) distribution. With 10,000 simulations, we got the empirical distribution of the values of the monitored statistic. In the graph below, $u$ = (0.05, 0.5, 0.95)$'$. If the computed value of the statistic is greater than 12.8, which is the upper 0.005 percentiles of $\chi^2_{(3)}$, we input 12.8 as the value of the statistic. The curve shows the true pdf of $\chi^2_{(3)}$. The height of the bar on the right indicates the probability that the values of the monitored statistic fall outside the UCL. As the sample size increases, we can see that the height of the bar gets closer to 0.005, the nominal false alarm rate, which indicates the robustness of the in control ARL gets better.




The steps of the simulation study are as follows

1. Choose one of the eight distributions, the number of quantiles *k*, the positions of quantiles $(u_1 < \cdots < u_k)$ and a nominal $\mathrm{ARL}_0$.
2. Calculate $Q_0$ and $\Sigma_0$ from the chosen in control distribution.
3. Randomly generate a sample of size *n* from the chosen distribution and compute the k by 1 quantile vector $\tilde{Q}$ at the same positions $u_i$, *i* =1, …, *k*.
4. Compute the value of the W and compare it to $\chi^2_\alpha(k)$.
5. Repeat steps 2 and 3 for 10,000 times and each time count if W > $\chi^2_\alpha(k)$.
6. Calculate the false alarm rate by using the total counts of W> $\chi^2_\alpha(k)$ divided by 10,000.
7. Increase *n* and study its impact on the false alarm rate

Same simulation experiments are run for all the cases. We provide full results table to show the results.

| | sample sizes | Number of positions | 3 | | |
|---|---|---|---|---|---|
| Skewness | Distributions | typr I error | u= 0.05,0.5,0.95 | u=0.1,0.5,0.9 | u=0.25, 0.5, 0.75 |
| 4 | gamma(0.25,3) | 0.05 | 3000+ | 3000+ | 1500 |
| | | 0.005 | 3000+ | 3000+ | 3000+ |
| 2.828 | gamma(0.5,3) | 0.05 | 2000 | 1000 | 400 |
| | | 0.005 | 2500 | 3000 | 1000 |
| 1 | gamma(4,3) | 0.05 | 100 | 50 | 45 |
| | | 0.005 | 12 | 14 | 11 |
| 0.5 | gamma(16,3) | 0.05 | 200 | 200 | 100 |
| | | 0.005 | 200 | 200 | 20 |
| 0.33 | gamma(36,3) | 0.05 | 400 | 200 | 100 |
| | | 0.005 | 200 | 50 | 19 |
| 0.2 | gamma(100,3) | 0.05 | 200 | 200 | 45 |
| | | 0.005 | 200 | 200 | 40 |
| 0 | Normal(0,1) | 0.05 | 200 | 200 | 200 |
| | | 0.005 | 100 | 100 | 40 |
| 0 | t with df=3 | 0.05 | 100 | 17 | 13 |
| | | 0.005 | 1000 | 200 | 200 |

| | | | Number of Positions | |
|---|---|---|---|---|
| | | sample sizes | 5 | |
| Skewness | Distributions | type I error | u =0.05,0.25,0.5,0.75,0.95 | u=0.25,0.375,0.5,0.625,0.75 |
| 4 | gamma(0.25,3) | 0.05 | 3000+ | 2500 |
| | | 0.005 | 3000+ | 3000+ |
| 2.828 | gamma(0.5,3) | 0.05 | 1500 | 500 |
| | | 0.005 | 3000+ | 16 |
| 1 | gamma(4,3) | 0.05 | 100 | 100 |
| | | 0.005 | 40 | 16 |
| 0.5 | gamma(16,3) | 0.05 | 300 | 200 |
| | | 0.005 | 100 | 11 |
| 0.33 | gamma(36,3) | 0.05 | 300 | 100 |
| | | 0.005 | 100 | 14 |
| 0.2 | gamma(100,3) | 0.05 | 300 | 300 |
| | | 0.005 | 200 | 13 |
| 0 | Normal(0,1) | 0.05 | 300 | 100 |
| | | 0.005 | 300 | 11 |
| 0 | t with df=3 | 0.05 | 11 | 13 |
| | | 0.005 | 1000 | 2000 |

From the results table, we can summarize the three conclusions below:

1. The closer the chosen quantiles, the smaller the required sample size.

2. More highly skewed distributions need a larger sample size to reach a robust type I error. The optimal number of quantiles is not obvious. The position plays an important role.

3. Slightly skewed distributions need smaller sample size than symmetric distributions.

### 2.3 Improvement and results

From the results and graphs above, the empirical distribution of W from the simulations usually has a thicker tail than a chi-square distribution when the sample size is small. To improve the results, adjustment is made to the asymptotic chi-square critical value by multiplying a constant. Former results show that the performance of the asymptotic distribution is related to the sample size and the number of chosen quantiles, hence the multiplied constant is proposed as $1 + \frac{k^2}{n}$ . In this way, the modified critical value $(1 + \frac{k^2}{n}) \times \chi_\alpha^2(k)$ has the intuitive appeal that, for small sample sizes, the original sample sizes is increased by some function of the number of quantiles and the sample size. As $n \rightarrow \infty$, the proposed values will confirm to the original value. Below is the summary table of the modified critical value for different distributions.

| Adjusted CV | | sample sizes | 3 | | |
|---|---|---|---|---|---|
| Skewness | Distributions | typr I error | u= 0.05,0.5,0.95 | u=0.1,0.5,0.9 | u=0.25,0.5,0.75 |
| | | 0.05 | 3000+ | 3000+ | 1500 |
| 4 | gamma(0.25,3) | 0.005 | 3000+ | 3000+ | 3000+ |
| | | 0.05 | 2000 | 400 | 100 |
| 2.828 | gamma(0.5,3) | 0.005 | 3000+ | 3000+ | 1000 |
| | | 0.05 | 400 | 400 | 400 |
| 1 | gamma(4,3) | 0.005 | 200 | 400 | 200 |
| | | 0.05 | 500 | 300 | 1000 |
| 0.5 | gamma(16,3) | 0.005 | 500 | 1000 | 1000 |
| | | 0.05 | 500 | 500 | 300 |
| 0.33 | gamma(36,3) | 0.005 | 1000 | 400 | 300 |
| | | 0.05 | 400 | 400 | 400 |
| 0.2 | gamma(100,3) | 0.005 | 2000 | 500 | 500 |
| | | 0.05 | 500 | 400 | 300 |
| 0 | Normal(0,1) | 0.005 | 300 | 400 | 400 |
| | | 0.05 | 400 | 400 | 500 |
| 0 | t with df=3 | 0.005 | 1500 | 300 | 400 |

| Adjusted CV | | sample sizes | 5 | |
|---|---|---|---|---|
| Skewness | Distributions | typr I error | u =0.05,0.25,0.5,0.75,0.95 | u=0.25,0.375,0.5,0.625,0.75 |
| | | 0.05 | 3000+ | 3000+ |
| 4 | gamma(0.25,3) | 0.005 | 3000+ | 3000+ |
| | | 0.05 | 1000 | 3000+ |
| 2.828 | gamma(0.5,3) | 0.005 | 3000+ | 3000+ |
| | | 0.05 | 1000 | 3000+ |
| 1 | gamma(4,3) | 0.005 | 1500 | 500 |
| | | 0.05 | 1000 | 1500 |
| 0.5 | gamma(16,3) | 0.005 | 1000 | 2000 |
| | | 0.05 | 1500 | 1500 |
| 0.33 | gamma(36,3) | 0.005 | 1000 | 1000 |
| | | 0.05 | 1500 | 1000 |
| 0.2 | gamma(100,3) | 0.005 | 1500 | 2000 |
| | | 0.05 | 1000 | 1000 |
| 0 | Normal(0,1) | 0.005 | 2500 | 1500 |
| | | 0.05 | 1000 | 1000 |
| 0 | t with df=3 | 0.005 | 500 | 400 |

From the two summary tables, we can see that some of the cases require even larger sample sizes. However, some of the cases do show improvement, requiring less sample sizes. This is a problem for further study.

## 3 SUMMARY

The QCC with quantiles chosen to be close to each other requires a relatively smaller sample size to reach a robust in control ARL/FAR. Highly skewed distributions need a relatively larger sample size to reach a robust in control ARL/FAR. Slightly skewed distributions need a relatively smaller sample size than monitoring symmetric distributions to reach a robust in control ARL/FAR. The in control performance of the QCC chart, monitoring different number of quantiles, is affected by the positions of quantiles. Generally speaking, for highly skewed distribution, more than 1000 observations ($n$) is required; for slightly skewed distribution, less than 50 can work; for symmetric distribution, more than 100 observations ($n$) is needed.

The improvements for the above adjustment were not consistent in different cases. With the adjusted UCL, some cases need a smaller sample size ($n$) to reach a relatively robust ARL/FAR, while some need a larger sample size ($n$).

## REFERENCES

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