# A Bayesian Approach to Modeling Measurement Errors

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**Abstract**
The impacts of ignoring covariate measurement error in regression models have been well documented and include bias in parameter estimates and a loss of power. Measurement error in the response variable has received less attention and correlated measurement errors (between the response and explanatory variables) even less. A Bayesian model that accounts for measurement errors is implemented in the simple linear regression setting, with extensions to the correlated measurement error setting and multiple linear regression setting. A simulation study is used to explore this approach, and allows for comparison to other popular measurement error correction methods (SIMEX, method of moments, ordinary least squares-no correction). The Bayesian measurement error model provided approximately unbiased results in all cases considered and directly provides a corrected estimate of unexplainable random variation.

**Key Words:** Measurement Error, Errors-In-Variables, Response Measurement Error, Bayesian Methods

## 1. Introduction

In many research situations a variable of interest cannot be observed exactly, but instead is observed with error. When this occurs, the variable is said to contain *measurement error*. Measurement error, also referred to as *errors-in-variables* (Stefanski, 2000), can arise in a variety of circumstances, including, but not limited to: miscalibration of a measuring instrument, sampling error, misclassification, abundance modeling, and anytime responses are estimated. The consequences of ignoring measurement error in analyses have been well documented and include bias in parameter estimation and a loss of power (Carroll et al., 2006).

There are two main types of measurement error, Berkson (Berkson, 1950) and classical. This work focuses on classical measurement error present in either the response or explanatory variables, working towards modeling situations where measurement error is present in both. Classical measurement error occurs when the true value of the continuous random variable ($X_i$) is measured with error, resulting in an observed value ($w_i$). In this work, the additive error structure will be used, however other error structures exist (linear, multiplicative, etc.). The classical additive measurement error model for the mismeasured random variable $W_i$ given $x_i$ is

$$W_i|x_i = x_i + u_i, \tag{1}$$

where $u_i$ represents the measurement error, $E(u_i|x_i) = 0$, and $Var(u_i|x_i) = \sigma_u^2$. A result of the mean 0, additive error structure is that $W$ is an unbiased estimator of $x$ since $E(W|x) = x$.

In order to explore measurement error modeling, we start with the simplest situation where we have a quantitative response and single quantitative explanatory variable, the simple linear regression (SLR) model. If we let Y and X denote the response and explanatory variables respectively, the SLR model is

$$Y_i \sim N(\beta_0 + \beta_1 X_i, \sigma_\epsilon^2). \tag{2}$$

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Often it is not possible to measure the true value of the response variable, explanatory variable, or sometimes both. If we let $(D_i, W_i)$ represent the mismeasured response and explanatory variables containing classical, additive measurement error, the SLR model becomes

$$Y_i + q_i = \beta_0 + \beta_1(x_i + u_i) + \epsilon_i \tag{3}$$

$$\Rightarrow D_i = \beta_0 + \beta_1 w_i + \epsilon_i \tag{4}$$

where $q_i \sim N(0, \sigma_q^2)$, $u_i \sim N(0, \sigma_u^2)$, $Cov(q_i, u_i) = \sigma_{uq}$, and $\epsilon_i \sim N(0, \sigma_\epsilon^2)$. Here, the measurement error variances and covariance are constant across observations, however the model and notation above can be altered to allow them to vary across observations (see Section 2.4 for a discussion of this scenario).

To account for measurement errors when present, the measurement error variance must either be assumed known or estimated from replicate data (repeated observations on the same subject at the same time). The measurement error variance is often unknown and replicate data are not available to estimate it, therefore measurement errors often go unaccounted for. Biased parameter estimates and/or a loss of power are a known result of ignoring measurement errors when present in analyses (Fuller, 1987). In the simple linear regression setting, the naive (ordinary least squares regression of $D_i$ on $W_i$) estimator of the slope is biased towards 0 because

$$\beta_{1,naive} = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_u^2}\beta_1 = k\beta_1, \tag{5}$$

where $k = \frac{\sigma_X^2}{\sigma_X^2+\sigma_u^2}$ and is referred to as the reliability ratio, $\beta_1$ is the true slope, $\sigma_X^2$ is the variance of the true random variable X, and $\sigma_u^2$ is the measurement error variance associated with the mismeasured explanatory variable (Buonaccorsi, 2010). Larger measurement error variance associated with the mismeasured explanatory variable ($\sigma_u^2$) results in $\beta_{1,naive}$ being a more biased estimator, as can be seen in Equation 5 and Figure 1. When measurement error is present in the response variable only, the estimator of $\beta_{1,naive}$ is unbiased as Equation 5 does not depend on $\sigma_q^2$. Although the $\beta_{1,naive}$ estimator is unbiased when measurement error is present only in the response variable, $\sigma_{\epsilon,naive}^2$ is inflated resulting in less precise inferences for $\beta_1$ and a loss of power for detecting effects.

There are many methods available to correct for measurement error, however each method is not appropriate in every situation and performance of the methods vary greatly. Common correction methods include the Method of Moments correction (Buonaccorsi, 2010), the SIMEX correction (Cook & Stefanski, 1994), the GSIMEX correction (Ronning & Rosemann, 2008), Regression Calibration (Ch 4. of Carroll et al., 2006), and Bayesian Methods (Gilks et al., 1996). The method of moments correction is a formula-based correction method that uses a formula to correct parameters and is specific to each modeling situation. SIMEX uses simulation to obtain data containing more measurement error than was observed, then fits the original model and obtains estimates of the parameters. This process is repeated multiple times for a vector of increasing (*additional*) measurement error, and the algorithm then uses extrapolation in an attempt to obtain an estimate of the parameter in the case of no measurement error. It is a general framework applicable to many situations, including measurement errors in response and explanatory variables, however it does not directly account for correlated measurement errors. GSIMEX is a generalization of the SIMEX method that attempts to account for correlated measurement errors. Regression calibration requires fitting a calibration model that requires replication, validation, or instrumental data. Bayesian methods exist to correct for measurement error in the explanatory variable, however are generally developed for specific applications.

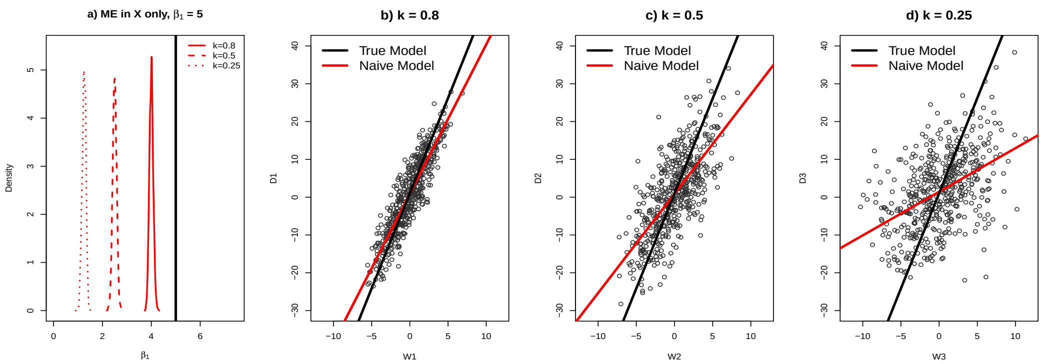


**Figure 1**: Data were generated with a true slope of $\beta_1 = 5$ containing measurement error in the explanatory variable. Three values of the reliability ratio, $k = (0.8, 0.5, 0.25)$, were considered to show the effect of increasing measurement error on the estimate of $\beta_1$. 500 simulations were used for each value of k, with each data set consisting of $n = 500$ observations. a) Distributions of $\hat{\beta}_{1,naive}$ for each value of k. Vertical bold line represents the true value of $\beta_1$. b-d) True model versus naive estimate for one realization for each value of k.

Here, a general framework for a Bayesian measurement error model that corrects for measurement error when present in the explanatory and/or response variable in the SLR setting is presented. Bayesian model specification and prior distributions are discussed in Section 2. A simulation study was used to compare the performance of the Bayesian measurement error model to other common correction methods in the SLR setting and results are presented in Section 3.

## 2. Bayesian Simple Linear Regression Measurement Error Model

Bayesian methods are quickly becoming a popular and effective method for statisticians in every field, in part due to their ability to handle complex modeling situations. When present, measurement error can change a very basic modeling problem into a complex one. There are four distinct modifications of the naive SLR model to consider when measurement error is present: Case 1 - measurement error present in the explanatory variable only; Case 2 - measurement error present in the response variable only; Case 3 - uncorrelated measurement errors present in both the explanatory and response variables; and, Case 4 - correlated measurement errors between the explanatory and response variables. A Bayesian measurement error model is given below for Case 1 and Case 2. The model for Case 3 can be obtained by combining the two models given, while Case 4 is a topic for future research.

In the following models that account for the different cases of measurement error in SLR, the explanatory variable is first *centered* to have mean 0. Centering explanatory variables is commonly done in Bayesian analyses, and results in $\beta_0$ and $\beta_1$ being independent. In all of the cases presented, the Gibbs sampling algorithm (Geman & Geman, 1984) is used to obtain samples from the joint posterior distribution of the parameters, and it is most efficient when parameterized in terms of independent components.

### 2.1 Case 1: Measurement Error in the Explanatory Variable

When measurement error is present in the explanatory variable only, Equation 3 becomes

$$Y_i = \beta_0 + \beta_1 w_i + \epsilon_i, \tag{6}$$

where $w_i$ is the mismeasured explanatory variable. In a Bayesian analysis where measurement error is present in the explanatory variable, the true *unobserved* value ($X_i$) is treated like a latent variable and given a prior distribution. Therefore, the Bayesian SLR measurement error model when measurement error is present in the explanatory variable is described using three pieces: an outcome model, a measurement model, and a latent variable model. The outcome model is a model for the outcomes that you would obtain if measurement error was not present and is

$$Y_i \sim N(\beta_0 + \beta_1 X_i, \sigma_\epsilon^2). \tag{7}$$

The measurement model is a model for the mismeasured variable given the true variable and is

$$W_i | X_i \sim N(X_i, \sigma_u^2). \tag{8}$$

This is a direct result of Equation 1, which implies that when classical additive measurement error is present in the explanatory variable, $E(W_i|x_i) = x_i$ and $Var(W_i|x_i) = \sigma_u^2$. The latent variable model is a model for the true unobserved variable ($X_i$) and, by centering the observed variable, is

$$X_i \sim N(0, \sigma_x^2). \tag{9}$$

In this model, $\sigma_u^2$ is assumed known and all other parameters ($\beta_0$, $\beta_1$, $\tau_X = \frac{1}{\sigma_x^2}$, $\tau_\epsilon = \frac{1}{\sigma_\epsilon^2}$) are given prior distributions, which depend on the particular situation and are often selected to be non-informative. For example, a diffuse Normal distribution is used below for the $\beta' s$ and a $Gamma(0.5, 2)$ distribution is used for the $\tau' s$.

## 2.2 Case 2: Measurement Error in the Response Variable

When measurement error is present in the response variable only, Equation 3 becomes

$$D_i = \beta_0 + \beta_1 x_i + \epsilon_i, \tag{10}$$

where $D_i$ is the mismeasured response variable. In this scenario, the naive estimator of $\beta_1$ is unbiased (Equation 5), however the measurement error causes a loss of power to detect effects. The Bayesian SLR measurement error model for measurement error in the response variable only is described using two pieces: an outcome model and a measurement model. The outcome model is

$$Y_i \sim N(\beta_0 + \beta_1 X_i, \sigma_\epsilon^2). \tag{11}$$

When classical additive measurement error is present in the response variable, $E(D_i|y_i) = y_i$ and $Var(D_i|y_i) = \sigma_q^2$ as a result of Equation 1. The measurement model is a model for the mismeasured variable given the true variable and is

$$D_i | Y_i \sim N(Y_i, \sigma_q^2). \tag{12}$$

Similar to Case 1, $\sigma_q^2$ is assumed known and all other parameters ($\beta_0$, $\beta_1$, $\tau_\epsilon = \frac{1}{\sigma_\epsilon^2}$) are given prior distributions.

## 2.3 Case 3: Measurement Error in both the Response Variable and the Explanatory Variable

When measurement error is present in both the explanatory variable and the response variable, the naive estimator of $\beta_1$ is biased and a loss of power occurs. Combining the models for Case 1 and Case 2 results in the model for Case 3. Here, the outcome model is

$$Y_i \sim N(\beta_0 + \beta_1 X_i, \sigma_\epsilon^2). \tag{13}$$

**Table 1**: Simulation study parameter values. $\sigma_q^2 = 0$ in Case 1 and $\sigma_u^2 = 0$ in Case 2. $(\sigma_u^2,\sigma_q^2)^*$ only apply to Case 3.

| $\beta_0$ | $\beta_1$ | $\sigma_\epsilon^2$ | $\sigma_X^2$ | $\sigma_u^2$ | $\sigma_q^2$ | $(\sigma_u^2,\sigma_q^2)^*$ |
|---|---|---|---|---|---|---|
| 1 | 5 | 4 | 4 | (1, 4, 12) | (1, 4, 12) | (1,1), (4,4), (12,12) |

The measurement model now consists of two pieces, a model for the mismeasured explanatory variable given the true explanatory variable and a model for the mismeasured response variable given the true response variable. The two pieces are

$$W_i|X_i \sim N(X_i, \sigma_u^2) \tag{14}$$

and

$$D_i|Y_i \sim N(Y_i, \sigma_q^2). \tag{15}$$

The true unobserved explanatory variable ($X_i$) is again treated like a latent variable and the latent variable model is

$$X_i \sim N(0, \sigma_x^2). \tag{16}$$

Similar to Case 1 and Case 2, $\sigma_u^2$ and $\sigma_q^2$ are assumed known while all other parameters ($\beta_0$, $\beta_1$, $\tau_X = \frac{1}{\sigma_x^2}$, $\tau_\epsilon = \frac{1}{\sigma_\epsilon^2}$) are given prior distributions.

### 2.4 Extensions

The models in Sections 2.1, 2.2, and 2.3 assume the measurement error variances are known, however if replicate data (referring to the situation where on unit $i$ there are $m_i$ replicate values of the error prone measure) are available, the measurement error variances can be given a prior distribution and estimated from the model. This feature is unique to the Bayesian measurement error methods, as other correction methods require the measurement error variances be known, or estimated from replicate data and then assumed known, in order to be used. Also, situations may arise where non-constant measurement error is present. The models presented here can easily extend to a non-constant measurement error setting by replacing $\sigma_u^2$ in Equation 8 and Equation 14 with $\sigma_u^2(i)$ and $\sigma_q^2$ in Equation 12 and Equation 15 with $\sigma_q^2(i)$.

## 3. Simulation Studies

Simulation studies were performed for Cases 1, 2, and 3 with constant measurement error to explore the Bayesian measurement error model results and to compare the results to some other common measurement error corrections. Within each case, three separate simulations were performed to see the effects of increasing measurement error. Overall, nine simulations were performed with each simulation consisting of $N = 500$ observations and $N_{sim} = 500$ simulations carried out. Data were generated with the parameter values given in Table 1. The simulation studies were conducted in R (R Core Team, 2014) and the R package R2jags (Su & Yajima, 2014) was used for the Bayesian analysis.

### 3.1 Case 1: Measurement Error in the Explanatory Variable Only

Three separate simulations were performed for Case 1, each with a different value of the measurement error variance ($\sigma_u^2$), resulting in reliability ratios of $k = 0.8, 0.5$, and $0.25$.

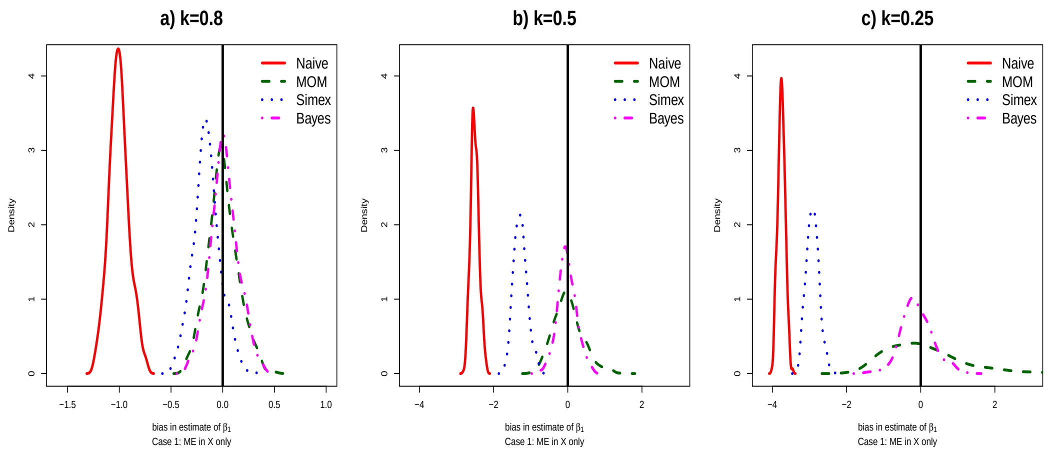


**Figure 2**: Resulting bias distributions of $\beta_1$ from the three simulations performed for Case 1. Bold vertical lines appear at a bias value of 0, which would indicate an unbiased estimator. The method of moments correction is labeled *MOM*.

Three measurement error correction methods (method of moments, SIMEX with quadratic extrapolation function, and the proposed Bayesian Method) were considered and compared to the naive (ordinary least squares - no correction) estimator.

For the Bayesian method, JAGS (Plummer, 2003) was used to obtain samples from the posterior distribution. Three independent chains, each with random starting values, were used and the first 1000 samples were discarded as *burn-in*. Each chain was run for 60,000 iterations and a thinning rate of 12 was applied, resulting in a final chain length of 5,000 iterations. The prior distributions used were $\beta_0 \sim N(0, 100000)$, $\beta_1 \sim N(0, 100000)$, $\tau_X \sim gamma(0.5, 2)$, and $\tau_\epsilon \sim gamma(0.5, 2)$. The $gamma(a, b)$ notation in JAGS specifies a Gamma distribution with mean $a/b$ and variance $a/b^2$.

Bias distributions for $\beta_1$ (simulated $\hat{\beta}_1$'s - $\beta_1$) resulting from the three simulations are shown in Figure 2. For the Bayesian method, the mean of the posterior distribution from each simulation is used as the estimate of $\beta_1$. The naive estimator produced the most biased results in all three cases. The SIMEX method is noticeably biased for all three cases, with the magnitude of the bias increasing as the measurement error variance increases. The method of moments correction method and the Bayesian correction method both appear approximately unbiased for all three cases, however the method of moments correction method becomes highly variable when the measurement errors are large ($k = 0.25$).

### 3.2 Case 2: Measurement Error in the Response Variable Only

Measurement error in the response variable is seldom taken into account since the naive estimator of $\beta_1$ is unbiased. In the naive models, the typical estimator of $\sigma^2_\epsilon$ includes random variation and variation in responses due to measurement error, so is positively biased and results in inflated standard errors for inferences on the slope coefficients. The naive estimator of $\sigma^2_\epsilon$ when measurement error is only present in the response variable is

$$\sigma^2_{\epsilon,naive} = \sigma^2_\epsilon + \sigma^2_q. \tag{17}$$

The bias in $\sigma^2_{\epsilon,naive}$ leads to bias in $SE(\hat{\beta}_1)$ since

$$SE(\hat{\beta}_1) = \sqrt{(\sigma^2_{\epsilon,naive})(X^TX)^{-1}_{[2,2]}} = \sqrt{(\sigma^2_\epsilon + \sigma^2_q)(X^TX)^{-1}_{[2,2]}}. \tag{18}$$

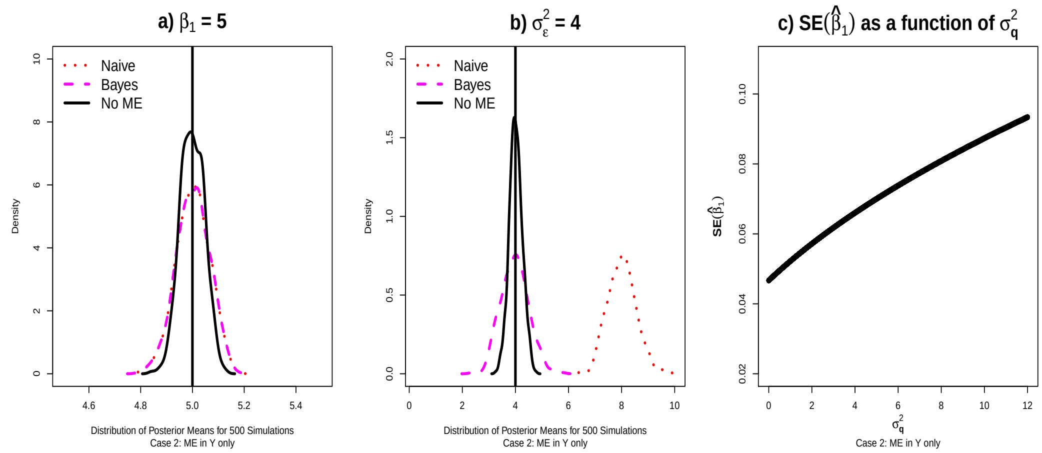


**Figure 3**: a-b) Distribution of posterior means for $\beta_1$ and $\sigma_\epsilon^2$ from 500 simulations. The solid black curve represents the case where $\sigma_q^2 = 0$ while the other two curves represent $\sigma_q^2 = 4$. Bold vertical lines appear at the true value of each parameter. c) Effects of increasing measurement error variance ($\sigma_q^2$) on $SE(\hat{\beta}_1)$ where $\sigma_\epsilon^2 = 4$.

To explore the performance of the Bayesian SLR measurement error model in Case 2, three situations were considered: a Bayesian SLR model using the responses observed without measurement error, a naive Bayesian SLR model that does not account for measurement error in the mismeasured response, and a Bayesian SLR measurement error model using the mismeasured response. As a reminder, the Bayesian SLR model with no measurement error is very similar to the Bayesian SLR measurement error model given in Section 2.2, with the only difference being the absence of Equation 12.

Three separate simulations were performed for Case 2, each with a different value of the measurement error variance ($\sigma_q^2$) given in Table 1. For each Bayesian model, JAGS was used to obtain samples from the posterior distribution using three independent chains, each with random starting values, and the first 1000 samples were discarded as *burn-in*. Each chain was run for 60,000 iterations and a thinning rate of 12 was applied, resulting in a final chain length of 5,000 iterations. The prior distributions used were $\beta_0 \sim N(0, 100000)$, $\beta_1 \sim N(0, 100000)$, and $\tau_\epsilon \sim gamma(0.5, 2)$.

Distributions of posterior means of $\beta_1$ and $\sigma_\epsilon^2$ are given in Figure 3 for the simulation with $\sigma_q^2 = 4$. The two other simulations showed similar results, therefore the plots are omitted. Panel a) reiterates that the naive estimator of $\beta_1$ is unbiased when measurement error is present in the response variable only and also shows that models using the variable measured with error are slightly more variable than the model using the uncontaminated response variable. In a real world application involving measurement error, you would not have the value of the response variable that is not contaminated by measurement error. This scenario was included in the simulation study to see the effects of measurement error on the width of the distribution of $\beta_1$ as compared to what we would see if measurement error was not present. Panel b) displays the positive bias associated with $\sigma_{\epsilon,naive}^2$ and also shows that the corrected estimate obtained from the measurement error model is approximately unbiased. Panel c) shows the effect of increasing measurement error on the $SE(\hat{\beta}_1)$ based on Equation 18 for the last simulated data set. The $SE(\hat{\beta}_1)$ is approximately $40\%$ larger when $\sigma_q^2 = 4$ as compared to what would have been observed had measurement error not been present (0.066 versus 0.0467).

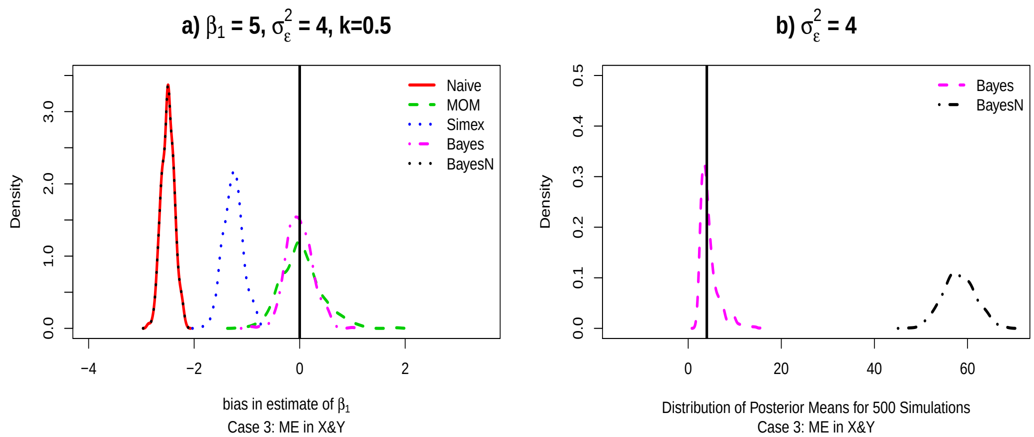


**Figure 4**: Simulation results from one simulation ($N_{sim} = 500$) with data containing measurement error in both the explanatory and response variable. a) Resulting bias distributions of $\beta_1$ for Case 3. A bold vertical line appears at a bias value of 0, indicating an unbiased estimator. b) Resulting distributions of $\sigma^2_\epsilon$ for the Bayesian measurement error correction method (Bayes) and for the naive Bayesian model (BayesN). The true value is indicated with a bold vertical line.

### 3.3 Case 3: Measurement Error in the Response and Explanatory Variable

Three separate simulations were performed for Case 3, each with different values of the measurement error variances ($\sigma^2_u$, $\sigma^2_q$) given in Table 1. Three measurement error correction methods (method of moments, SIMEX with quadratic extrapolation function, and the proposed Bayesian Method) were considered and compared to two naive methods (ordinary least squares method and the naive Bayesian method).

For the Bayesian methods, JAGS was used to obtain samples from the posterior distribution. Three independent chains, each with random starting values, were used and the first 1000 samples were discarded as *burn-in*. Each chain was run for 60,000 iterations and a thinning rate of 12 was applied, resulting in a final chain length of 5,000 iterations. The prior distributions used for the measurement error model were $\beta_0 \sim N(0, 100000)$, $\beta_1 \sim N(0, 100000)$, $\tau_X \sim gamma(0.5, 2)$, and $\tau_\epsilon \sim gamma(0.5, 2)$ while the priors used for the naive model were the same as given for the naive model in Section 3.2.

Bias distributions for $\beta_1$ resulting from the simulation with $(\sigma^2_u, \sigma^2_q) = (4, 4)$ are shown in Panel a) of Figure 4. Similar results were obtained for the other two cases, therefore the plots are omitted. Measurement error in the response variable does not affect the bias associated with $\beta_1$, therefore this plot looks nearly the same as the plot in Panel b) of Figure 2 and the results presented in Section 3.1 hold. Panel b) contains the distribution of posterior means for $\sigma^2_\epsilon$ from the two Bayesian models. Since measurement error is present in both the explanatory and response variables, a much larger positive bias is associated with $\sigma^2_{\epsilon,naive}$ as compared to Case 2. The Bayesian measurement error model produces an estimator of $\sigma^2_\epsilon$ that is much closer to the true value than the naive model.

## 4. Discussion

Measurement error in the explanatory variable is often the focus of measurement error correction methods, as the naive estimator of $\beta_1$ is biased towards 0. Simulation studies were used to compare three measurement error correction methods in this scenario. The magnitude of the measurement error variance ($\sigma^2_u$) relative to the variance of the unobserved explanatory variable ($\sigma_X^2$) can have a big impact on the performance of the methods, as seen in Section 3.1. All methods performed relatively well when the measurement error variances were small, however the SIMEX method and the method of moments correction had noticeable performance issues when measurement error variances increased. The SIMEX method consistently underestimated the value of $\beta_1$. Although the method of moments correction provided an estimator that appeared unbiased in all cases, it is highly variable when measurement error variances are large. The Bayesian measurement error model provided an approximately unbiased estimator of $\beta_1$ in all cases considered in the simulation study and results were as variable or less variable than the method of moments estimator.

Although the naive estimator of $\beta_1$ is unbiased when measurement error is present in the response variable only, there are still reasons to fit a measurement error model to account for the extra variability present. The measurement error model can help guide researchers in future studies with deciding where to focus resource allocation. If measurement error is present in the response variable only, and if reducing or eliminating it is possible in future studies, a $\sqrt{\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_q^2}} * 100\%$ reduction in $SE(\hat{\beta}_1)$ is possible, resulting in an increase in power. Also, models are sometimes built with the goal of prediction. In order to obtain a prediction interval, the estimated value of $\sigma_\epsilon^2$ is used. When prediction intervals are of interest, the posterior predictive intervals computed using the Bayesian measurement error model will be more precise than those computed from the naive Bayesian model.

When uncorrelated measurement error is present in both the explanatory and response variable, the naive estimator of $\beta_1$ is biased towards 0 and the naive estimator of $\sigma_\epsilon^2$ is too large. A measurement error model in this case allows researchers to obtain better (less biased) estimates of $\beta_1$ and can also help researchers with making decisions regarding future studies, as it may be possible to obtain more precise estimates of $\beta_1$ if reducing the measurement error in the response variable is possible.

Measurement errors should be taken into account when they are present in order to obtain the most accurate and precise results. The Bayesian measurement error model provides approximately unbiased results and easily extends to cases where non-constant measurement error is present and also to other models (multiple linear regression, nonlinear regression, etc.), suggesting a variety of different areas of application for these methods. The impacts and adjustments for correlated measurement errors in the response and explanatory variables are a subject for future work.

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## REFERENCES

Berkson, J. (1950), "Are There Two Regressions?", *Journal of the American Statistical Association*, 45, 164–180.

Buonaccorsi, J.P. (2010), *Measurement Error Models, Methods, and Applications*, London: Chapman & Hall.

Carroll, R.J., Ruppert, D., Stefanski, L.A., and Crainiceanu, C.M., (2006), *Measurement Error in Nonlinear Models. A Modern Perspective* (2nd ed.), London: Chapman & Hall.

Cook, J.R. and Stefanski, L.A. (1994), "Simulation-extrapolation estimation in parametric measurement error models," *Journal of the American Statistical Association*, 89, 1314–1328.

Fuller, W.A. (1987), *Measurement Error Models*, New York: John Wiley & Sons, Inc.

Geman, S. and Geman, D. (1984), "Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images," *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 6, 721–741

Gilks, W.R., Richardson, S., and Spiegelhalter, D.J. (1996), *M*arkov Chain Monte Carlo in Practice, New York: Chapman and Hall.

Plummer, M. (2003), "JAGS: A program for analysis of Bayesian graphical models using Gibbs sampling," *Proceedings of the 3rd International Workshop on Distributed Statistical Computing (DSC 2003). March*, 1–10.

R Core Team (2014). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria.

Ronning, G., and Rosemann, G. (2008), "SIMEX estimation in case of correlated measurement errors," *Advances in Statistical Analysis*, 92, 391–404.

Stefanski, L.A. (2000), "Measurement Error Models," *American Statistical Association*, 95, 1353–1358.

Su, Y.S. and Yajima, M. (2014). R2jags: A Package for Running jags from R. R package version 0.04-03.