A Test for Comparing the Location of Two Quadratic Growth Curves

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Abstract
The difference in the location for two quadratic growth curves is compared in this paper. For a 2^{nd} degree polynomial, the vertex gives the location of the curve in the XY plain. We present an approximate confidence region for the difference of vertices of two quadratic growth curves using both the gradient and delta methods. To test directly on the vertices, we derive a quadratic-form statistic under the null hypothesis that there is no shift in the location of the vertices in two mixed linear models. The statistic has an approximate chi-squared distribution. We compare the test statistic with an F statistic, which is derived for indirect test on the difference in the location of the vertices based on the intercept and slope parameters. We also present results for a simulation study conducted to assess the influence of sample size, measurement time points and nature of the random effects. Simulation results show that the test statistic performs well in terms of Type I error rate and power. The test statistic is applied to the Tell Efficacy Longitudinal Study, in which sound identification scores for children are modeled as quadratic growth curves for two independent groups, control and treatment. The interpretations of shift in the location of the vertices are also presented.

Key Words: Random Effect, Mixed Model, Quadratic Growth Curve, Vertex, Confidence Region, Power Function

1. Introduction

Many longitudinal studies are designed to investigate a characteristic of an individual, where the characteristic is measured repeatedly over time for each study participant. Often the individuals are considerably correlated across measurement observations. A multivariate model with general unrestricted covariance structure may be used to analyze these correlated data, but the growth curve model is usually applied. The analyses of growth curves focus on the explanation of within-individual variation by the aging process or natural development. The relation between time t and response y cannot be adequately described by a linear trend model in some longitudinal studies. Adding a square term of the fixed effect time t to the model gives a quadratic growth curve model, which can often describes the true unknown model very well. The coefficient parameters of fixed effect are necessary to determine the growth curve. The vertex of a quadratic curve gives the location of such a curve, which is useful in order to solve an optimization problem. By all means reasonable, it is important to derive the confidence region of the parabola's vertex as well as the confidence interval of t-coordinate and y-coordinate. For two independent groups, such as control and treatment, the confidence region as well as the confidence interval for the difference of vertices of two quadratic growth curves are interesting. Both the t-coordinate and y-coordinate of vertex are given by a non-linear combination of the model fixed regression coefficients, not simply only one of them. However, common statistical computer packages usually display confidence intervals for the fixed regression coefficient, but not for any of their functions.

In Section 2, two models, mixed model and growth curve model, and three methods, the gradient method, the delta method and mean response method are reviewed and the

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confidence region as well as power function are derived. To show the validity of the test statistics, simulations using parameters and sample sizes and power analysis for testing the difference between the vertices of two groups are given in Section 3. An application of analysis of Tell Efficacy Study using these statistics is presented in Section 4. Conclusion and discussion are drawn in Section 5.

2. Models and Methods for Confidence Set

The confidence region for the difference of vertices for two growth curves is investigated. Two growth curve models are considered; one is second-order mixed model with random intercept and the other is second-order mixed model with random intercept and random slope. For two independent samples, they are defined as follows,
Second-order mixed model with only random intercept,

$$y_{ij} = \beta_0^{(mid)} + \beta_0^{(eff)} D_i + \beta_1^{(mid)} t_{ij} + \beta_1^{(eff)} D_i t_{ij} + \beta_2^{(mid)} t_{ij}^2 + \beta_2^{(eff)} D_i t_{ij}^2 + \alpha_{0i} + \varepsilon_{ij} \quad (1)$$

where

$$D_i = \begin{cases} -1 & if\ y_{ij}\ comes\ from\ control\ group\ C, \\ +1 & if\ y_{ij}\ comes\ from\ treatment\ group\ T. \end{cases}$$

is a dummy variable to indicate the group,
n is the number of time points, N is the number of individuals,
β's are regression coefficients of fixed effect,
α_{0i} is random effect, $\alpha_{0i} \sim N(0, \sigma^2_{\alpha_0})$,
ε_{ij} is the random error term for the i^{th} individual at the j^{th} occasion, $\varepsilon_{ij} \sim N(0, \sigma_e^2)$,
α_{0i} and ε_{ij} are independent, $Cov(\alpha_{0i}, \varepsilon_{ij}) = 0$ for all i,
y_{ij} denotes response variable for the i^{th} individual at j^{th} occasion, $\boldsymbol{E}(y_{ij}) = \beta_0 + \beta_1 t_{ij} + \beta_2 t_{ij}^2$ and $\Sigma_{\boldsymbol{y}} = \boldsymbol{ZGZ'} + \boldsymbol{R}$, where $\boldsymbol{G} = \sigma^2_{\alpha_0}$ and $\boldsymbol{R} = \sigma_e^2 \boldsymbol{I}$.
From model (1), the individual models for control and treatment groups respectively are,

$$y_{ij} = \beta_0^{(C)} + \beta_1^{(C)} t_{ij} + \beta_2^{(C)} t_{ij}^2 + \alpha_{0i} + \varepsilon_{ij} \qquad for\ group\ C,$$

$$y_{ij} = \beta_0^{(T)} + \beta_1^{(T)} t_{ij} + \beta_2^{(T)} t_{ij}^2 + \alpha_{0i} + \varepsilon_{ij} \qquad for\ group\ T,$$

Second-order mixed model with random intercept and random slope,

$$y_{ij} = \beta_0^{(mid)} + \beta_0^{(eff)} D_i + \beta_1^{(mid)} t_{ij} + \beta_1^{(eff)} D_i t_{ij} + \beta_2^{(mid)} t_{ij}^2 + \beta_2^{(eff)} D_i t_{ij}^2 + \alpha_{0i} + \alpha_{1i} t_{ij} + \varepsilon_{ij} \quad (2)$$

where

$$D_i = \begin{cases} -1 & if\ y_{ij}\ comes\ from\ control\ group\ C, \\ +1 & if\ y_{ij}\ comes\ from\ treatment\ group\ T. \end{cases}$$

is a dummy variable to indicate the group,
n is the number of time points, N is the number of individuals,
β's are regression coefficients of fixed effect,
α_{0i} and α_{1i} are random effects, $\alpha_{0i} \sim N(0, \sigma^2_{\alpha_0})$, $\alpha_{1i} \sim N(0, \sigma^2_{\alpha_1})$,
ε_{ij} are random error terms, $\varepsilon_{ij} \sim N(0, \sigma_e^2)$,
α_{0i}, α_{1i} and ε_{ij} are mutually independent, $Cov(\alpha_{0i}, \varepsilon_{ij}) = 0$ and $Cov(\alpha_{1i}, \varepsilon_{ij}) = 0$,
y_{ij} denotes response variable for the i^{th} individual at j^{th} occasion, $\boldsymbol{E}(y_{ij}) = \beta_0 + \beta_1 t_{ij} + \beta_2 t_{ij}^2$

and $\Sigma_{\boldsymbol{y}} = \boldsymbol{ZGZ'} + \boldsymbol{R}$, where $\boldsymbol{G} = \begin{pmatrix} \sigma^2_{\alpha_0} & \sigma_{\alpha_0\alpha_1} \\ \sigma_{\alpha_0\alpha_1} & \sigma_{\alpha_1} \end{pmatrix}$, and $\boldsymbol{R} = \sigma^2_e \boldsymbol{I}$.
From model (2), the distinct models for control and treatment group are,

$$y_{ij} = \beta_0^{(C)} + \beta_1^{(C)} t_{ij} + \beta_2^{(C)} t_{ij}^2 + \alpha_{0i} + \alpha_{1i} t_{ij} + \varepsilon_{ij} \qquad for\ group\ C,$$

$$y_{ij} = \beta_0^{(T)} + \beta_1^{(T)} t_{ij} + \beta_2^{(T)} t_{ij}^2 + \alpha_{0i} + \alpha_{1i} t_{ij} + \varepsilon_{ij} \qquad for\ group\ T,$$

For both employed models, the relationship between the regression coefficients are,

$$\beta_k^{(C)} = \beta_k^{(mid)} - \beta_k^{(eff)} \qquad for\ k = 0,1,2,$$

$$\beta_k^{(T)} = \beta_k^{(mid)} + \beta_k^{(eff)} \qquad for\ k = 0,1,2,$$

2.1 Delta Method for Difference of Vertices

The second-order no-intercept mixed model with random intercept for control and treatment groups is,

$$y_{ij} = \beta_0 D_i + \beta_1 D_i t_{ij} + \beta_2 D_i t_{ij}^2 + \alpha_{0i} + \varepsilon_{ij}$$

The equivalent model is,

$$y_{ij} = \beta_0^{(C)} + \beta_0^{(T)} + \beta_1^{(C)} t_{ij} + \beta_1^{(T)} t_{ij} + \beta_2^{(C)} t_{ij}^2 + \beta_2^{(T)} t_{ij}^2 + \alpha_{0i} + \varepsilon_{ij} \tag{3}$$

The second-order no-intercept mixed model with random intercept and slope for control and treatment groups is,

$$y_{ij} = \beta_0 D_i + \beta_1 D_i t_{ij} + \beta_2 D_i t_{ij}^2 + \alpha_{0i} + \alpha_{1i} t_{ij} + \varepsilon_{ij}$$

The equivalent model is,

$$y_{ij} = \beta_0^{(C)} + \beta_0^{(T)} + \beta_1^{(C)} t_{ij} + \beta_1^{(T)} t_{ij} + \beta_2^{(C)} t_{ij}^2 + \beta_2^{(T)} t_{ij}^2 + \alpha_{0i} + \alpha_{1i} t_{ij} + \varepsilon_{ij} \tag{4}$$

For model (3) and (4), let $\boldsymbol{b}' = (b_0^{(C)}, b_0^{(T)}, b_1^{(C)}, b_1^{(T)}, b_2^{(C)}, b_2^{(T)})$ be the maximum likelihood estimator (MSE) of the regression coefficients $\boldsymbol{\beta}' = (\beta_0^{(C)}, \beta_0^{(T)}, \beta_1^{(T)}, \beta_1^{(T)}, \beta_2^{(C)}, \beta_2^{(C)})$. Provided that the covariance parameters of random effects are unknown, $\boldsymbol{b}$ is approximately normally distributed in large sample with mean $\boldsymbol{\beta}$ and covariance $\Sigma_{\boldsymbol{b}}$, i.e. $\boldsymbol{b} \overset{a}{\sim} (\boldsymbol{\beta}, \Sigma_{\boldsymbol{b}})$. Where,

$$\Sigma_{\boldsymbol{b}} = \begin{pmatrix} \sigma^2_{b_0^{(C)}} & 0 & \sigma_{b_0^{(C)} b_1^{(C)}} & 0 & \sigma_{b_0^{(C)} b_2^{(C)}} & 0 \\ 0 & \sigma^2_{b_0^{(T)}} & 0 & \sigma_{b_0^{(T)} b_1^{(T)}} & 0 & \sigma_{b_0^{(T)} b_2^{(T)}} \\ \sigma_{b_0^{(C)} b_1^{(C)}} & 0 & \sigma^2_{b_1^{(C)}} & 0 & \sigma_{b_1^{(C)} b_2^{(C)}} & 0 \\ 0 & \sigma_{b_0^{(T)} b_1^{(T)}} & 0 & \sigma^2_{b_1^{(T)}} & 0 & \sigma_{b_1^{(T)} b_2^{(T)}} \\ \sigma_{b_0^{(C)} b_2^{(C)}} & 0 & \sigma_{b_1^{(C)} b_2^{(C)}} & 0 & \sigma^2_{b_2^{(C)}} & 0 \\ 0 & \sigma_{b_0^{(T)} b_2^{(T)}} & 0 & \sigma_{b_1^{(T)} b_2^{(T)}} & 0 & \sigma^2_{b_2^{(T)}} \end{pmatrix} = (\boldsymbol{X}'\Sigma_{\boldsymbol{y}}^{-1}\boldsymbol{X})^{-1}.$$

The distinct estimated covariances for control and treatment groups are,

$$\hat{\Sigma}_{\boldsymbol{b}^{(T)}} = \begin{pmatrix} \hat{\sigma}^2_{b_0^{(T)}} & \hat{\sigma}_{b_0^{(T)} b_1^{(T)}} & \hat{\sigma}_{b_0^{(T)} b_2^{(T)}} \\ \hat{\sigma}_{b_0^{(T)} b_1^{(T)}} & \hat{\sigma}^2_{b_1^{(T)}} & \hat{\sigma}_{b_1^{(T)} b_2^{(T)}} \\ \hat{\sigma}_{b_0^{(T)} b_2^{(T)}} & \hat{\sigma}_{b_1^{(T)} b_2^{(T)}} & \hat{\sigma}^2_{b_2^{(T)}} \end{pmatrix}, \qquad \hat{\Sigma}_{\boldsymbol{b}^{(C)}} = \begin{pmatrix} \hat{\sigma}^2_{b_0^{(C)}} & \hat{\sigma}_{b_0^{(C)} b_1^{(C)}} & \hat{\sigma}_{b_0^{(C)} b_2^{(C)}} \\ \hat{\sigma}_{b_0^{(C)} b_1^{(T)}} & \hat{\sigma}^2_{b_1^{(C)}} & \hat{\sigma}_{b_1^{(C)} b_2^{(C)}} \\ \hat{\sigma}_{b_0^{(C)} b_2^{(T)}} & \hat{\sigma}_{b_1^{(C)} b_2^{(C)}} & \hat{\sigma}^2_{b_2^{(C)}} \end{pmatrix}.$$

Let $\boldsymbol{V}^{(C)'} = (V_x^{(C)}, V_y^{(C)})$ and $\boldsymbol{V}^{(T)'} = (V_x^{(T)}, V_y^{(T)})$ denote the vertices of the control and treatment groups respectively, then $\boldsymbol{V}^{(C)}, \boldsymbol{V}^{(T)}$ and their estimates $\hat{\boldsymbol{V}}^{(C)}, \hat{\boldsymbol{V}}^{(T)}$ follows,

$$V_x^{(C)} = \frac{-\beta_1^{(C)}}{2\beta_2^{(C)}}, \qquad V_y^{(C)} = \beta_0^{(C)} - \frac{\beta_1^{(C)2}}{4\beta_2^{(C)}}, \qquad V_x^{(T)} = \frac{-\beta_1^{(T)}}{2\beta_2^{(T)}}, \qquad V_y^{(T)} = \beta_0^{(T)} - \frac{\beta_1^{(T)2}}{4\beta_2^{(T)}}.$$

$$\hat{V}_x^{(C)} = \frac{-b_1^{(C)}}{2b_2^{(C)}}, \qquad \hat{V}_y^{(C)} = b_0^{(C)} - \frac{b_1^{(C)2}}{4b_2^{(C)}}, \qquad \hat{V}_x^{(T)} = \frac{-b_1^{(T)}}{2b_2^{(T)}}, \qquad \hat{V}_y^{(T)} = b_0^{(T)} - \frac{b_1^{(T)2}}{4b_2^{(T)}}.$$

For treatment and control group, the first-order partial derivative of $\hat{\boldsymbol{V}}^{(T)}$ with respect to $\boldsymbol{\beta}^{(T)}$ evaluated at $\boldsymbol{\beta}^{(T)} = \boldsymbol{b}^{(T)}$ is,

$$\frac{\partial \boldsymbol{V}^{(T)}}{\partial \boldsymbol{\beta}^{(T)}}\Big|_{\boldsymbol{\beta}^{(T)}=\boldsymbol{b}^{(T)}} = \hat{D}^{(T)} = \begin{pmatrix} 0 & -\frac{1}{2}b_2^{(T)-1} & \frac{1}{2}b_1^{(T)}b_2^{(T)-2} \\ 1 & -\frac{1}{2}b_1^{(T)}b_2^{(T)-1} & \frac{1}{4}b_1^{(T)2}b_2^{(T)-2} \end{pmatrix}.$$

$$\frac{\partial \boldsymbol{V}^{(C)}}{\partial \boldsymbol{\beta}^{(C)}}\Big|_{\boldsymbol{\beta}^{(C)}=\boldsymbol{b}^{(C)}} = \hat{D}^{(C)} = \begin{pmatrix} 0 & -\frac{1}{2}b_2^{(C)-1} & \frac{1}{2}b_1^{(C)}b_2^{(C)-2} \\ 1 & -\frac{1}{2}b_1^{(C)}b_2^{(C)-1} & \frac{1}{4}b_1^{(C)2}b_2^{(C)-2} \end{pmatrix}.$$

When the sample size tends to be large, based on the multivariate delta method, $\hat{\boldsymbol{V}}^{(T)}$, the estimate of $\boldsymbol{V}^{(T)}$ for treatment group, is approximately multivariate normally distributed with mean $\boldsymbol{V}^{(T)}$ and covariance $\Sigma_{\hat{\boldsymbol{V}}^{(T)}}$, i.e., $\hat{\boldsymbol{V}}^{(T)} \overset{a}{\sim} MVN(\boldsymbol{V}^{(T)}, \Sigma_{\hat{\boldsymbol{V}}^{(T)}})$. Using the estimated covariance $\hat{\Sigma}_{\hat{\boldsymbol{V}}^{(T)}}$, where

$$\hat{\Sigma}_{\hat{\boldsymbol{V}}^{(T)}} = D^{(T)}\hat{\Sigma}_{\boldsymbol{b}^{(T)}}D^{(T)'} = \begin{pmatrix} \hat{\sigma}^2_{\hat{V}_x^{(T)}} & \hat{\sigma}_{\hat{V}_x^{(T)}\hat{V}_y^{(T)}} \\ \hat{\sigma}_{\hat{V}_x^{(T)}\hat{V}_y^{(T)}} & \hat{\sigma}^2_{\hat{V}_y^{(T)}} \end{pmatrix}$$

$\hat{V}_x^{(T)} \overset{a}{\sim} N(V_x^{(T)}, \sigma^2_{\hat{V}_x^{(T)}})$ and $\hat{V}_y^{(T)} \overset{a}{\sim} N(V_y^{(T)}, \sigma^2_{\hat{V}_y^{(T)}})$. Similarly, the estimated vertex for control group $\hat{\boldsymbol{V}}^{(C)} \overset{a}{\sim} MVN(\boldsymbol{V}^{(C)}, \Sigma_{\hat{\boldsymbol{V}}^{(C)}})$ and $\hat{V}_x^{(C)} \overset{a}{\sim} N(V_x^{(C)}, \sigma^2_{\hat{V}_x^{(C)}})$, $\hat{V}_y^{(C)} \overset{a}{\sim} N(V_y^{(C)}, \sigma^2_{\hat{V}_y^{(C)}})$, where

$$\hat{\Sigma}_{\hat{\boldsymbol{V}}^{(C)}} = D^{(C)}\hat{\Sigma}_{\boldsymbol{b}^{(C)}}D^{(C)'} = \begin{pmatrix} \hat{\sigma}^2_{\hat{V}_x^{(C)}} & \hat{\sigma}_{\hat{V}_x^{(C)}\hat{V}_y^{(C)}} \\ \hat{\sigma}_{\hat{V}_x^{(C)}\hat{V}_y^{(C)}} & \hat{\sigma}^2_{\hat{V}_y^{(C)}} \end{pmatrix}$$

The summation of two independent normal distribution as described by Casella and Berger (2002), is normal with the summation of mean and variance. Define the difference between the two vertices of control and treatment group, $\boldsymbol{V}^{(diff)'} = \boldsymbol{V}^{(T)'} - \boldsymbol{V}^{(C)'} = (V_x^{(diff)}, V_y^{(diff)})$. Suppose that control group and treatment group are independent, the covariance of $\boldsymbol{V}^{(diff)}$ is $\Sigma_{\boldsymbol{V}}^{(diff)} = \Sigma_{\boldsymbol{V}^{(C)}} + \Sigma_{\boldsymbol{V}^{(T)}}$. The distribution for the difference of $x-$coordinates, $V_x^{(diff)} = V_x^{(T)} - V_x^{(C)}$, and the difference of $y-$coordinates, $V_y^{(diff)} = V_y^{(T)} - V_y^{(C)}$, are $\hat{V}_x^{(diff)} \overset{a}{\sim} N\left((V_x^{(T)} - V_x^{(C)}), (\sigma^2_{\hat{V}_x^{(T)}} + \sigma^2_{\hat{V}_x^{(C)}})\right)$ and $\hat{V}_y^{(diff)} \overset{a}{\sim} N\left((V_y^{(T)} - V_y^{(C)}), (\sigma^2_{\hat{V}_y^{(T)}} + \sigma^2_{\hat{V}_y^{(C)}})\right)$. Therefore, the approximate $(1-\alpha)\%$ confidence interval of $\hat{V}_x^{(diff)}$ is

$$(\hat{V}_x^{(diff)} - Z_{1-\alpha/2}\hat{\sigma}_{\hat{V}_x^{(diff)}},\ \hat{V}_x^{(diff)} + Z_{1-\alpha/2}\hat{\sigma}_{\hat{V}_x^{(diff)}}).$$

Similarly, the approximate $(1-\alpha)\%$ confidence interval of $\hat{V}_y^{(diff)}$ is

$$(\hat{V}_y^{(diff)} - Z_{1-\alpha/2}\hat{\sigma}_{\hat{V}_y^{(diff)}},\ \hat{V}_y^{(diff)} + Z_{1-\alpha/2}\hat{\sigma}_{\hat{V}_y^{(diff)}}).$$

2.2 Gradient Method for Difference of X- Coordinates for Same Quadratic Term

When assuming the quadratic terms of two growth curves are the same, $\beta_2^{(C)} = \beta_2^{(T)} = \beta_2$, the confidence interval for the difference of x-coordinates is illustrated (Martin Bachmaier, Test and confidence set for the difference of the x-coordinates of the vertices of two quadratic regression models, Stat Papers, 51:285-296, 2010.). For model (1) and (2), the $x-$coordinates of vertices for control and treatment groups are,

$$V_x^{(C)} = \frac{-\beta_1^{(C)}}{2\beta_2^{(C)}} = \frac{-(\beta_1^{(mid)} - \beta_1^{(eff)})}{2\beta_2^{(mid)}}, \qquad V_x^{(T)} = \frac{-\beta_1^{(T)}}{2\beta_2^{(T)}} = \frac{-(\beta_1^{(mid)} + \beta_1^{(eff)})}{2\beta_2^{(mid)}},$$

$$\hat{V}_x^{(C)} = \frac{-b_1^{(C)}}{2b_2^{(C)}} = \frac{-(b_1^{(mid)} - b_1^{(eff)})}{2b_2^{(mid)}}, \qquad \hat{V}_x^{(T)} = \frac{-b_1^{(T)}}{2b_2^{(T)}} = \frac{-(b_1^{(mid)} + b_1^{(eff)})}{2b_2^{(mid)}},$$

From large sample theory, if the distribution of one parameter is unknown, normal distribution could be applied to estimate the distribution of this parameter. Hence for large sample the distribution for the estimate of difference of the two vertices, $\hat{V}^{(diff)}$ is approximately normal. Because of the large degrees of freedom, z score can be used as an approximation to t score,

$$\begin{aligned}
& V_x^{(diff)} \in C(V_x^{(T)} - V_x^{(C)}) \\
\Leftrightarrow\ & \frac{(b_1^{(eff)} + b_2 V_x^{(diff)})^2}{\hat{Var}(b_1^{(eff)}) + 2V_x^{(diff)}\hat{Cov}(b_1^{(eff)}, b_2) + [V_x^{(diff)}]^2 \hat{Var}(b_2)} \leqslant Z_{1-\alpha/2}^2 \\
\Leftrightarrow\ & (b_1^{(eff)} + b_2 V_x^{(diff)})^2 \leqslant \Big(\hat{Var}(b_1^{(eff)}) + 2V_x^{(diff)}\hat{Cov}(b_1^{(eff)}, b_2) + [V_x^{(diff)}]^2 \hat{Var}(b_2)\Big) \cdot Z_{1-\alpha/2}^2 \\
\Leftrightarrow\ & A \cdot [V_x^{(diff)}]^2 + B \cdot V_x^{(diff)} + C \leqslant 0, \\
& where, A = b_2^2 - \hat{Var}(b_2) \cdot Z_{1-\alpha/2}^2 \\
& B = 2b_1^{(eff)} b_2 - 2\hat{Cov}(b_1^{(eff)}, b_2) \cdot Z_{1-\alpha/2}^2 \\
& C = [b_1^{(eff)}]^2 - \hat{Var}(b_1^{(eff)}) \cdot Z_{1-\alpha/2}^2 .
\end{aligned} \tag{5}$$

To solve the inequality, if $A \neq 0$, then $A \cdot x_0^2 + B \cdot x_0 + C$ is a parabola. It has two nulls if the discriminant $D = B^2 - 4AC$ is positive. With regard to the numerical stability concerning small values of $4AC$, we compute either zero in two different ways:

$$x_{01} = \begin{cases} \frac{-2C}{B - \sqrt{B^2 - 4AC}} & when\ B < 0, \\ \frac{-B - \sqrt{B^2 - 4AC}}{2A} & when\ B \geqslant 0. \end{cases} \qquad x_{02} = \begin{cases} \frac{-B + \sqrt{B^2 - 4AC}}{2A} & when\ B \leqslant 0, \\ \frac{-2C}{B + \sqrt{B^2 - 4AC}} & when\ B > 0. \end{cases}$$

Therefore when $A > 0$ and $D > 0$, this leads to a two-sided confidence interval $[x_{01}, x_{02}]$. When $A < 0$ and $D > 0$, the confidence interval goes to $(-\infty, x_{02}] \bigcup [x_{01}, +\infty)$. In this project, only the first situation is considered, i.e. the confidence interval for the difference of x-coordinates for vertices $\hat{V}_x^{(diff)}$ is $[x_{01}, x_{02}]$.

2.3 Mean Response Method for Difference of Y-Coordinates

Given the $x-$coordinates of two vertices for control and treatment groups $\hat{V}_x^{(C)}, \hat{V}_x^{(T)}$, the difference of $y-$coordinate of vertex $\hat{V}_y^{(diff)}$ can be calculated as,

$$\hat{V}_y^{(diff)} = \hat{V}_y^{(T)} - \hat{V}_y^{(C)} = (b_0^{(T)} + b_1^{(T)} \cdot \hat{V}_x^{(T)} + b_2^{(T)} \cdot \hat{V}_x^{(T)2}) - (b_0^{(C)} + b_1^{(C)} \cdot \hat{V}_x^{(C)} + b_2^{(C)} \cdot \hat{V}_x^{(C)2})$$

where $\hat{V}_y^{(diff)}$ is treated as a difference of the mean responses $\hat{V}_y^{(C)}$ and $\hat{V}_y^{(T)}$. The normal distribution can be applied instead of the t distribution when sample size goes large, then the difference of $y-$coordinate of vertex $\hat{V}_y^{(diff)}$ distributes approximately normally,

$$\frac{\hat{V}_y^{(diff)} - V_y^{(diff)}}{\hat{\sigma}^2_{\hat{V}_y^{(diff)}}} \overset{a}{\sim} N(0,1) \ ,$$

Therefore the $(1-\alpha)\%$ confidence interval of $V_y^{(diff)}$ is

$$(\hat{V}_y^{(diff)} - Z_{1-\alpha/2}\hat{\sigma}_{\hat{V}_y^{(diff)}}, \ \hat{V}_y^{(diff)} + Z_{1-\alpha/2}\hat{\sigma}_{\hat{V}_y^{(diff)}}).$$

The estimated variance of $V_y^{(diff)}$ for the mean response method is equivalent to the estimated variance for the delta method. Hence the conclusion is drawn that the two methods provide identical confidence interval for $y-$coordinate.

2.4 Confidence Region for the Difference of Vertices

In order to compute a confidence region for the difference of vertices, the large sample chi-square distribution for a quadratic form is applied. The chi-square distribution with k degrees of freedom is the distribution of a sum of the squares of k independent standard normal random variables. As proven, the estimated difference of vertex follows an approximate multivariate normal distribution,

$$\hat{V}^{(diff)} \overset{a}{\sim} MVN\left(\boldsymbol{V}^{(diff)}, \Sigma_{\hat{\boldsymbol{V}}^{(diff)}}\right)$$

where $\Sigma_{\hat{\boldsymbol{V}}^{(diff)}} = \Sigma_{\hat{\boldsymbol{V}}^{(T)}} + \Sigma_{\hat{\boldsymbol{V}}^{(C)}}$. For the bivariate standard normal distribution in vector form, the sum of the squares of two independent standard normal variables is chi-square distribution with two degrees of freedom:

$$\begin{pmatrix} \hat{V}_x^{(diff)} - V_x^{(diff)} \\ \hat{V}_y^{(diff)} - V_y^{(diff)} \end{pmatrix}' \hat{\Sigma}^{-1}_{\hat{\boldsymbol{V}}^{(diff)}} \begin{pmatrix} \hat{V}_x^{(diff)} - V_x^{(diff)} \\ \hat{V}_y^{(diff)} - V_y^{(diff)} \end{pmatrix} \sim \chi^2_{(2)} \ .$$

Therefore the approximate $(1-\alpha)\%$ confidence region for the difference of the vertices for two groups is

$$\begin{pmatrix} \hat{V}_x^{(diff)} - V_x^{(diff)} \\ \hat{V}_y^{(diff)} - V_y^{(diff)} \end{pmatrix}' \hat{\Sigma}^{-1}_{\hat{\boldsymbol{V}}^{(diff)}} \begin{pmatrix} \hat{V}_x^{(diff)} - V_x^{(diff)} \\ \hat{V}_y^{(diff)} - V_y^{(diff)} \end{pmatrix} \leqslant \chi^2_{1-\alpha,2} \ ,$$

The confidence region is an ellipse from this equation.

3. Power Analysis

Power plays an important role to reject the null hypothesis of same vertex for two groups given that the vertices of two groups are actually different. The null hypothesis,

$$H_0 : \boldsymbol{V}^{(C)} = \boldsymbol{V}^{(T)}$$

where $\boldsymbol{V}^{(C)}$ and $\boldsymbol{V}^{(T)}$ are distinct vertices of control and treatment group may be tested indirectly with an $F-$test or directly by a Chi-square test depends on the quadratic term.

3.1 Power Function for F Test

Repeated measurements on two samples, control and treatment, from a population can be presented by a split plot design model,

$$y_{ijk} = \mu_{...} + \rho_{i(k)} + \tau_j + \gamma_k + (\tau\gamma)_{jk} + \varepsilon_{ijk}$$

where,
y_{ijk} is the response for ith subject at jth occasion for group k,
$\mu_{...}$ is a constant for grand mean,
$\rho_{i(k)}$ is the random effect for subject i nested within group k, and $\rho_{i(k)}$ are independent $N(0, \sigma_\rho^2)$,
τ_j is the fixed time effect and τ_j are constants subject to the restriction $\sum \tau_j = 0$,
γ_k is the fixed group effect and γ_k are constants subject to the restriction $\sum \gamma_k = 0$,
ε_{ij} are independent $N(0, \sigma_\varepsilon^2)$, and independent of the $\rho_{i(k)}$,
$i = 1, 2, ..., N; j = 1, 2, ...r; k = 1, 2$. N is the sample size and r is the number of occasions.
The 2^{nd} order linear mixed regression model with covariance structure compound symmetry in longitudinal study is,

$$y_{ijk} = \beta_0 X_{ijk} + \beta_1 X_{ijk} t_{ij} + \beta_2 X_{ijk} t_{ij}^2 + \alpha_{0i} + \varepsilon_{ijk},$$

which is equivalent to model (3). In order to test the difference between two vertices of control and treatment group $H_0 : \boldsymbol{V}^{(C)} = \boldsymbol{V}^{(T)}$, F test statistic can be used to test the null hypothesis $H_0 : \boldsymbol{\beta}^{(C)} = \boldsymbol{\beta}^{(T)}$, if the quadratic terms of two groups are equal, i.e. $\beta_2^{(C)} = \beta_2^{(T)}$. Given that the different quadratic terms $\beta_2^{(C)} \neq \beta_2^{(T)}$, the F test is an indirect test. The equivalent null hypothesis for the F test is $H_0 : \boldsymbol{C\beta} = \mathbf{0}$, where

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_0^{(C)} \\ \beta_1^{(C)} \\ \beta_2^{(C)} \\ \beta_0^{(T)} \\ \beta_1^{(T)} \\ \beta_2^{(T)} \end{pmatrix}$$

The F test statistic is,

$$F = \frac{\boldsymbol{b}'\boldsymbol{C}[\boldsymbol{C}'(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{C}]^{-1}\boldsymbol{C}'\boldsymbol{b}}{rank(\boldsymbol{C})}, \tag{6}$$

with the non-centrality parameter

$$\lambda_1 = \boldsymbol{\beta}'\boldsymbol{C}[\boldsymbol{C}'(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}C]^{-1}\boldsymbol{C}'\boldsymbol{\beta},$$

where $\boldsymbol{V} = \boldsymbol{ZGZ}' + \boldsymbol{R}$. The numerator degrees of freedom is $rank(C)$, and the denominator degrees of freedom $2N(r-1) - rank(C)$ by Between Within method, and the exact power function is,

$$Power = Prob\left\{F_{\lambda_1} > F_{1-\alpha}\right\},$$

where $F_{1-\alpha}$ is the critical value for the central F distribution with Type I error rate α.

When the linear mixed model contains both random intercept and random slope terms, the equivalent 2^{nd} order model is model (4). To test the null hypothesis $H_0 : \boldsymbol{\beta}^{(C)} = \boldsymbol{\beta}^{(T)}$, an approximate F test statistic (6) is employed, with the approximate degrees of freedom which can be computed by either Satterwaite method or Konward and Ronger (1997) method. The approximate power function is,

$$Power = Prob\left\{F_{\lambda_1} > F_{1-\alpha}\right\}.$$

3.2 Power Function for Chi-Square Test

The non-central chi-square distribution can be applied as a direct test to compute power for the null hypothesis $H_0 : \boldsymbol{V}^{(C)} = \boldsymbol{V}^{(T)}$. As proven, $\hat{V}^{(diff)} \overset{a}{\sim} MVN\left(\boldsymbol{V}^{(diff)}, \Sigma_{\hat{\boldsymbol{V}}^{(diff)}}\right)$, then $\hat{V}^{(diff)\prime}\Sigma^{-1}_{\hat{\boldsymbol{V}}^{(diff)}}\hat{V}^{(diff)}$ distributes approximately as a non-central chi-square with 2 degrees of freedom and the non-centrality parameter

$$\begin{aligned}
\lambda_2 &= V^{(diff)\prime}\Sigma^{-1}_{\hat{\boldsymbol{V}}^{(diff)}}V^{(diff)} \\
&= \begin{pmatrix} \frac{-\beta_1^{(T)}}{2\beta_2^{(T)}} - \frac{-\beta_1^{(C)}}{2\beta_2^{(C)}} \\ \beta_0^{(T)} - \frac{\beta_1^{(T)2}}{4\beta_2^{(T)}} - \beta_0^{(C)} + \frac{\beta_1^{(C)2}}{4\beta_2^{(C)}} \end{pmatrix}' \Sigma_{\hat{\boldsymbol{V}}^{(diff)}} \begin{pmatrix} \frac{-\beta_1^{(T)}}{2\beta_2^{(T)}} - \frac{-\beta_1^{(C)}}{2\beta_2^{(C)}} \\ \beta_0^{(T)} - \frac{\beta_1^{(T)2}}{4\beta_2^{(T)}} - \beta_0^{(C)} + \frac{\beta_1^{(C)2}}{4\beta_2^{(C)}} \end{pmatrix}.
\end{aligned}$$

That is, $\hat{V}^{(diff)\prime}\Sigma^{-1}_{\hat{\boldsymbol{V}}^{(diff)}}\hat{V}^{(diff)} \overset{a}{\sim} \chi^2_{2,\lambda_2}$. Under the null hypothesis, the non-centrality parameter $\lambda_2 = 0$. The power function is,

$$Power = Prob\left\{\chi^2_{2,\lambda_2} > \chi^2_{1-\alpha,2}\right\},$$

where $\chi^2_{1-\alpha,2}$ is the critical value given test size level α. Using the estimated covariance $\hat{\Sigma}_{\hat{\boldsymbol{V}}^{(diff)}}$, the consistent statistic for $\Sigma_{\hat{\boldsymbol{V}}^{(diff)}}$, the decision rule is, reject the null hypothesis if

$$\begin{pmatrix} \boldsymbol{V}_x^{(T)} - \boldsymbol{V}_x^{(C)} \\ \boldsymbol{V}_y^{(T)} - \boldsymbol{V}_y^{(C)} \end{pmatrix}' \hat{\Sigma}_{\hat{\boldsymbol{V}}^{(diff)}} \begin{pmatrix} \boldsymbol{V}_x^{(T)} - \boldsymbol{V}_x^{(C)} \\ \boldsymbol{V}_y^{(T)} - \boldsymbol{V}_y^{(C)} \end{pmatrix} > \chi^2_{1-\alpha,2},$$

otherwise do not reject the null hypothesis.

4. Analysis of Simulation Results

4.1 Two Quadratic Growth Curves With Same Quadratic Term

For mixed model with only random intercept (1), 1000 data sets are generated respectively for control and treatment group with the regression coefficient parameters $\beta_0^{(C)}$, $\beta_0^{(T)}$, $\beta_1^{(C)}$, $\beta_1^{(T)}$ and same quadratic term β_2 equal to 2, 2, 8, 8.1 and -1, and covariance coefficients $\sigma^2_{\alpha_0}$ equals 1 for sample size 20 and 100. The true distinct model for control group is,

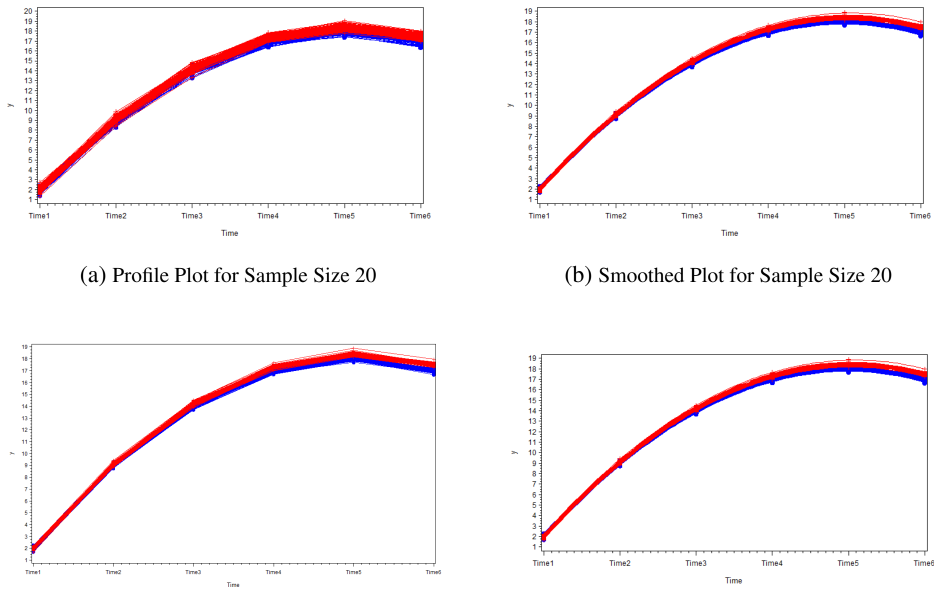
$$y_{ij} = 2 + 8x_{ij} - x_{ij}^2 + \alpha_{0i} + \varepsilon_{ij}, \qquad i = 1,2,...,N \quad j = 0,1,...,5.$$

The true model for treatment group is,

$$y_{ij} = 2 + 8.1x_{ij} - x_{ij}^2 + \alpha_{0i} + \varepsilon_{ij}, \qquad i = 1,2,...,N \quad j = 0,1,...,5.$$

With the vertices for control and treatment group $\boldsymbol{V}' = (4, 18)$ and $\boldsymbol{V}' = (4.05, 18.4025)$, the difference between them is $\boldsymbol{V}^{(diff)\prime} = (0.05, 0.4025)$. The profile plots and smoothed profile plots are shown in Figure 1. For a better display, only 100 datasets are randomly selected from each group; red represents treatment group and blue is for control group. Quadratic trend is intuitively suggested from the figure. The red curves are above the blue curves which indicates the *y*-coordinate of vertex for treatment group is larger than that for control group.

The results of simulation for confidence intervals of difference of *x*-coordinates are shown in Table 1. In this table, symbol I represents delta method and symbol II represents gradient method. The results include the empirical coverage as well as lower bound and upper bound for the empirical coverage. From the columns of the empirical coverage, two



(a) Profile Plot for Sample Size 20

(b) Smoothed Plot for Sample Size 20

(c) Profile Plot for Sample Size 100

(d) Smoothed Plot for Sample Size 100

Figure 1: Profile and Smoothed Plots for Mixed Model with Only Random Intercept

out of the 12 conditions had the nominal coverage outside the bounds; they are Type I error rate 0.1 sample size 20 for both methods. The reason for the bias should be the small sample size, since for sample size 100, both methods performs well for all the different error rates. The conclusion is that both delta method and gradient method are applicable for the confidence interval of the difference for x-coordinates.

Table 1: Confidence Interval for Difference of x-Coordinates

Type I Error	Sample Size	Empirical Coverage I	lower bound	upper bound	Empirical Coverage II	lower bound	upper bound
0.01	100	0.993	0.98621	0.99979	0.994	0.98771	1.00029
0.01	20	0.991	0.98331	0.99869	0.991	0.98331	0.99869
0.05	100	0.955	0.94215	0.96785	0.955	0.94215	0.96785
0.05	20	0.959	0.94671	0.97129	0.961	0.94900	0.97300
0.1	100	0.91	0.89511	0.92489	0.91	0.89511	0.92489
0.1	20	0.931	0.91782	0.94418	0.931	0.91782	0.94418

The results of simulation for confidence intervals for the difference of y-coordinates are displayed in Table 2; the results for the delta method and mean response method are identical. The table contains the empirical coverage as well as lower bound and upper bound for the empirical coverage. From the table, all the 18 conditions had the nominal coverage within the bounds. Therefore, both delta method and mean response method are appropriate to compare the difference of y-coordinates.

Table 3 shows the simulation results of the confidence region for the difference of vertices. The table includes the empirical coverage as well as lower bound and upper bound for the empirical coverage. From the table, none of the 6 conditions had the nominal coverage outside the bounds. Hence, the approximate chi-square distribution with two degrees of freedom applied to obtain the confidence region for the difference of vertices are practicable.

Table 2: Confidence Interval for Difference of y-Coordinates

Type I Error	Sample Size	Empirical Coverage	lower bound	upper bound
0.01	100	0.987	0.97778	0.99622
0.01	20	0.986	0.97643	0.99557
0.05	100	0.94	0.92528	0.95472
0.05	20	0.954	0.94102	0.96698
0.1	100	0.893	0.87692	0.90908
0.1	20	0.906	0.89082	0.92118

Table 3: Confidence Region of Difference of two vertices

Type I Error	Sample Size	Empirical Coverage	lower bound	upper bound
0.01	100	0.992	0.98475	0.99925
0.01	20	0.989	0.98051	0.99749
0.05	100	0.956	0.94329	0.96871
0.05	20	0.95	0.93649	0.96351
0.1	100	0.897	0.88119	0.91281
0.1	20	0.916	0.90157	0.93043

4.2 Direct Chi-square Test vs Indirect F Test

For linear mixed model with only random intercept, twelve datasets are generated with difference regression coefficients, different variances of random effect, different sample sizes but same time measurements. There are six occasions for $j = 1, 2.25, 3.5, 5.25, 6.5, 7.75$; and sample sizes are selected to be 20 and 100. Two variances chosen for the random effect are 1.44 and 78 with apparent difference between them. The regression coefficients for control and treatment groups with vertex are listed in Table 4.As mentioned, both direct F test and indirect chi-square test can be applied to test the difference between the vertices from two groups. For the F test, the null hypothesis is $H_0 : \boldsymbol{\beta}^{(C)} = \boldsymbol{\beta}^{(T)}$; and for the chi-square test, the null hypothesis is $H_0 : \boldsymbol{V}^{(diff)'} = \boldsymbol{0}$. Provided that the quadratic term of two groups are equal, i.e. $\beta_2^{(C)} = \beta_2^{(T)}$, F test and chi-square test are equivalent.

Table 4: Parameters for Power Analysis

		β_0	β_1	β_2	Vertex
Parameter I	Control	3.73	1.41	-0.062	(11.37, 11.75)
	Treatment	2.98	2.29	-0.062	(18.47, 24.13)
Parameter II	Control	4.0552	1.5184	-0.08737	(8.69, 10.65)
	Treatment	4.3705	1.9574	-0.08737	(11.20, 15.33)
Parameter III	Control	9.92	1.149	-0.5818	(9.87, 15.59)
	Treatment	9.4378	1.4489	-0.5818	(12.45, 18.46)

The results of power analysis for the twelve combinations are displayed in Table 5. For each combination, 1000 datasets are generated to test whether the difference of the two vertices is significant via F test and Chi-square test. From the table, for the smaller random effect variance, no matter which set of regression coefficient is applied, both the two tests always reject the null hypothesis. The only exception is sample size 20 with third set of coefficients; the F test rejects two hundred more times than the chi-square test which indicates the higher power of F Test. When the variance of random effect becomes large, it is more obvious that F test rejects more than the Chi-square test for almost every combination which leads to the higher power of the F test. The conclusion is that given the same quadratic term for two groups, F test is more powerful than chi-square test.

Table 5: Chi-square Test vs F-Test with Denominator Degrees of Freedom Between-Within

			Parameters I		Parameters II		Parameters III	
			$N=20$	$N=100$	$N=20$	$N=100$	$N=20$	$N=100$
$\sigma^2_{\alpha_o}$ = 1.44	F Test	Reject	1000	1000	1000	1000	977	1000
		DNR	0	0	0	0	23	0
	χ^2 Test	Reject	990	1000	1000	1000	745	1000
		DNJ	10	0	0	0	255	0
$\sigma^2_{\alpha_o}$ = 78	F Test	Reject	1000	1000	733	1000	422	972
		DNJ	0	0	267	0	578	28
	χ^2 Test	Reject	306	951	238	997	63	647
		DNJ	694	49	762	3	937	353

* Mixed model with only random intercept

5. Application

We apply the proposed test statistics on the study of growth of language and early literacy skills in preschoolers who have developmental speech and language impairment. The confidence intervals and confidence region for the difference of vertices from control and treatment groups are performed.

5.1 Description of Study

U.S. Department of Education data for the Individuals with Disabilities Education Act (IDEA) demonstrate that 13% of four-year olds and five-year olds are receiving special education services in preschool and that 82% of these children indicate developmental speech and language impairment (DSLI) as a primary diagnosis. Young children with DSLI often fail to develop crucial pre-literacy skills, which will place those children at high risk for later reading failure and literacy difficulties.

In a recent study, examining the efficacy of ”Teaching Early Literacy and Language” (TELL) curriculum in promoting the early literacy and oral language growth trajectories of preschoolers with DSLI is performed. The variables in TELL curriculum (Wilcox and Gray, 2011) including a series of instructions, scripted teaching activities, materials for implementation of oral language and early literacy activities, and professional development for teachers. They target one specific skill (e.g., vocabulary, identification of beginning sounds in a word) or small set of skills (e.g., inferential language, print concepts, letter sounds and identification) over a relatively short period of time (e.g., weeks). The TELL curriculum has shown positive results in oral language and early literacy activities in an earlier small randomized controlled trial. In contrast, researchers compare those trajectories of children who received the TELL curriculum with those who were randomly assigned to control classes.

In our study, we focus on one specific item from TELL curriculum, Curriculum Based Measurement (CBM) Letter Sound Identification (SoundID). Fifty-seven children with DSLI are randomly assigned to offer the TELL curriculum or accept those with business as usual (BAU). The efficacy variable, SoundID test score were obtained by six follow-up time measurements. The mean and standard deviation of SoundID scores for both children with DSLI from TELL and BAU curriculum are displayed in Table 6. On average, compared to the children received BAU, children who accepted TELL curriculum have higher

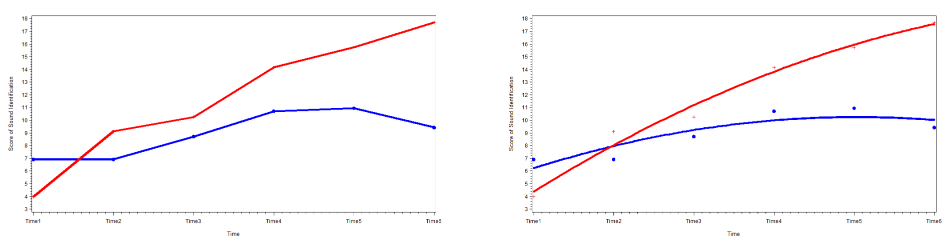
SoundID scores from the second time point.

Table 6: Sound Identification Score by Group (TELL Curriculum vs. Control): Mean, Standard Deviations at Each Occasion

Variables	TELL ($n = 32$)	Control ($n = 25$)
	Mean (SD)	Mean (SD)
SoundID (T1) Scores	3.970 (4.730)	6.917 (9.180)
SoundID (T2) Scores	9.120 (7.310)	6.920 (8.524)
SoundID (T3) Scores	10.260 (8.080)	8.720 (9.095)
SoundID (T4) Scores	14.148 (8.023)	10.714 (9.670)
SoundID (T5) Scores	15.741 (7.744)	10.955 (8.666)
SoundID (T6) Scores	17.692 (8.480)	9.429 (8.818)

5.2 Comparison of Vertices for TELL Curriculum and Control Group

The profile plot and smoothed profile plot for children with DSLI received TELL curriculum and BAU are shown in Figure 2, which indicate the quadratic curve for model. The model for children accepted TELL curriculum was conducted first; two models are compared based on the three criteria AIC, AICc and BIC displayed in Table 7, they are linear mixed model with only random intercept and compound symmetry structure and linear mixed model with random intercept and slope and unstructured covariance structure. The three criteria all suggest the later model by the smaller values of the criteria.



(a) Profile Plot for TELL Efficacy Example (b) Smoothed Plot for TELL Efficacy Example

Figure 2: Profile and Smoothed Plots for TELL Efficacy Example

Table 7: Model Selection for Children Received TELL Curriculum

Information Criteria	Random Intercept Model	Random intercept and Slope Model
AIC	986.8	957.1
AICc	986.9	957.4
BIC	989.7	963.0

The proposed methods for the confidence interval of vertex are applied, the estimated vertex, lower and upper limits are displayed in Table 8. The delta method for the confidence interval of x-coordinate is (3.885, 15.958), while the gradient method obtain the confidence interval (6.686, 76.715) which is too wide to be usable. The delta method and mean response method for the confidence intervals of y-coordinate are identical (11.682, 29.536).

Table 8: Confidence Intervals of Vertex for Children with DSLI Who Received TELL Curriculum

Method	Vertex	Lower Limit	Upper Limit
Delta for x-coordinate	9.921	3.885	15.958
Gradient for x-coordinate	9.921	6.686	76.715
Delta for y-coordinate	20.609	11.682	29.536
Mean Response for y-coordinate	20.609	11.682	29.536

Similar model analysis are also used to children with DSLI who accepted BAU, the results for model selection and confidence interval are displayed in Table 9 and Table 10. For x-coordinate of the estimated vertex, confidence interval from the delta method is (4.104, 7.699) while that from gradient method is (4.746, 11.422). For y-coordinate of the estimated vertex, confidence interval from the delta method and mean response method are both (6.909, 14.528).

Table 9: Model Selection for Children Received BAU

Information Criteria	Random Intercept Model	Random intercept and Slope Model
AIC	696.9	687.4
AICc	697.0	687.7
BIC	699.3	692.3

Table 10: Confidence Intervals of Vertex for Children with DSLI Who Received BAU

Method	Vertex	Lower Limit	Upper Limit
Delta for x-coordinate	5.886	4.104	7.669
Gradient for x-coordinate	5.886	4.746	11.422
Delta for y-coordinate	10.718	6.909	14.528
Mean Response for y-coordinate	10.718	6.909	14.528

For the Letter Sound Identification of children with DSLI who received the TELL curriculum, the estimated vertex is 20.61 letters at 10th scheduled visit, while for children with DSLI in BAU, the estimated vertex is 10.72 letters at 6th scheduled visit. The TELL curriculum treatment produced a shrift up to 9.89 letters and a shift to the right of 4 time measurements. The vertex of treatment is outside the scope of the occasion, and results can be interpreted that children from BAU class have reached a plateau at 6th scheduled visit but that children accepted TELL curriculum would continue to increase proficiency after the 6th visit. In order to test the difference between the two groups, methods for confidence set for difference of vertices in the second proposed topic are applied; the results are displayed in Table 11. For the difference of x-coordinates of vertices, the gradient method is not applicable since the quadratic term for control children and TELL children are not equal, which is against the assumption. The results illustrate that the time for children reach the plateau is not significantly different while the sound identification of letters are significantly different, which indicates the advantages of the TELL curriculum.

6. Conclusion and Discussion

Several methods for confidence interval and confidence region for the difference of vertices from two independent groups with quadratic growth curve model were discussed in

Table 11: Confidence Intervals for difference of Vertices for Control and TELL Children

Method	Difference of Vertices	Lower Limit	Upper Limit
Delta for x-coordinates	4.035	-2.259	10.329
Delta for y-coordinates	9.891	0.184	19.329
Mean Response for y-coordinates	9.891	0.184	19.329

this report. Initially, the delta method and gradient method were performed for confidence interval of the difference of x-coordinates for the vertices, while the delta method and mean response method for the difference of$y$-coordinates. The approximate chi-square distribution with two degrees of freedom were derived in the confidence region analysis and power analysis. Furthermore, in the simulation study, three different sample sizes were chosen in order to examine the influence of size for all the methods. Three different Type I error rates were chosen as well for the purpose of making the methods more convincible. Depending on all the simulation results, a conclusion could be drawn that all methods described in this study for confidence region of the difference of vertices for quadratic growth curves of 2^{nd} degree polynomial are applicable for different sample sizes, different Type I error rates and different models. For the power analysis, the indirect F test and direct chi-square test are compared, and F test has larger power than the chi-square test. An interesting topic for further research can be dealing with vertices of quadratic growth curves with heterogeneity in the random-effects population.

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