A Comparative Study of TELBS Robust Linear Regression Method

Hong Li* Yongqing Wang †

Abstract
Linear regression is a widely used statistical approach to model the relationship between a dependent variable and one or more independent variables. Regression parameters are often estimated using the method of ordinary least squares (OLS). Unfortunately, OLS estimates are very sensitive to outliers. Tabatabai et. al. (2012) introduced TELBS robust linear regression method. TELBS estimates have high asymptotic efficiency and high breakdown point. In this study we use simulation to assess the performance of TELBS robust technique in comparison with other methods such as M estimate, MM estimate, S estimate, and Least Trimmed Square (LTS) estimate. We examine the presence of outliers in the direction of response variable, covariates direction, and in both the response and covariates direction. In addition, two real data sets are used to illustrate these methods. Some diagnostic measures are introduced and computed to identify the outliers. Results indicate that as the percentage of outliers increases, TELBS method outperforms other methods considered in this study.

Keywords Ordinary Least Squares, TELBS robust estimate, M estimate, MM estimate, Outliers

1. Introduction

Linear regression is one of the most commonly used models for analyzing the effect of explanatory variables on a response variable. It has widespread application in various field of study, including social science, environment, biomedical research. The ordinary least squares (OLS) method has been generally used for regression analysis. However, OLS estimation of parameters is easily affected by the presence of outliers in the data. Outliers (influential points) are observations that are far away from the main pattern of the data. Outliers could be outlying in Y-space, X-space, or both. Usually, outliers outlying in X-space are also referred to as leverage points, such points do not always show up the usual least square residual plots. To remedy this problem, many robust regression methods have been developed that are not easily affected by the outliers, including M estimation, MM estimation, LTS, S estimation, and a newly developed TELBS robust estimation. We give an overview of these methods in Section 2. To illustrate and compare the performance of these methods, we apply them to two real data containing outliers in Section 3. We conduct a simulation study to further compare these methods under various settings in Section 4. Finally, we give a summary and discussion in Section 5.

2. Robust Regression Methods

We consider the standard multiple linear regression model given in the form of

$$y = X\beta + \epsilon$$

where y is n by 1 response vector, $X = (x_{ij})$ is n by p design matrix of predictor variables, β is p by 1 vector of parameters, ϵ is n by 1 vector of random errors. The OLS estimate of parameter vector β is found by minimizing the sum of squared errors.

*Cameron University, 2800 West Gore Blvd., Lawton, OK 73505
†Bowling Green State University, 1600 E Wooster St., Bowling Green, OH 43403

2.1 M Estimation

Huber (1973) introduced the M-estimate that minimize a function ρ of the errors. The objective function is given as

$$min \sum_{i=1}^{n} \rho(\frac{r_i}{\hat{\sigma}}) = min \sum_{i=1}^{n} (\frac{y_i - x_i'\hat{\beta}}{\hat{\sigma}})$$

where $\hat{\sigma}$ is an estimate of scale. A reasonable ρ function should have the following properties:

$\rho(r) \geqslant 0$, $\rho(0) = 0$, $\rho(r) = \rho(-r)$, and $\rho(r_i) \geqslant \rho(r_i')$ for $i = 1, 2, ..., n$. The M estimate of the parameter $\hat{\beta}$ can be obtained by taking partial derivatives with respect to β and setting them equal to 0. The system of normal equations are given by

$$\sum_{i=1}^{n} \psi(\frac{r_i}{\hat{\sigma}}) x_i = 0$$

where ψ is the derivative of ρ and it is a redescending function. Iteratively reweighted least square (IRLS) is one of the commonly used method to solve the nonlinear equations.

In general, M estimate is fairly robust to the outliers in y-direction, however, it is not robust to leverage points (outliers in x-direction).

2.2 LTS Estimation

Least Trimmed Squares (LTS) estimate was proposed by Rousseeuw (1984). Let $r_i = y_i - x_i'\hat{\beta}, i = 1, \ldots, n$, the LTS estimate of parameter is given as

$$\hat{\beta} = \arg min \sum_{i=1}^{h} r_{(i)}^2$$

where $r_1^2 \leq r_2^2 \leq \ldots \leq r_n^2$ are the ordered squared residuals. Usually, h is defined in the range $n/2 + 1 \leq h \leq (3n + p + 1)/4$, with n and p being sample size and number of parameters, respectively. LTS is considered as a high breakdown method with a BP (breakdown point) of 50%.

2.3 S Estimation

S estimate was proposed by Rousseeuw and Yohai (1984) and defined as

$$\hat{\beta} = \arg minS(r_1(\beta), \ldots, r_n(\beta))$$

where $r_i(\beta)$ is the i^{th} residual, the dispersion $S(\beta)$ is the solution of

$$\frac{1}{n-p} \sum_{i=1}^{n} \rho(\frac{y_i - x_i'\hat{\beta}}{S}) = K$$

where $K = \int \rho(s) d\Phi(s)$ such that $\hat{\beta}$ and $S(\hat{\beta})$ are asymptotically consistent estimate of β and σ for the Gaussian regression model. Rousseeuw and Yohai suggested a redescending influence function as, $\rho(x) = \frac{x^2}{2} - \frac{x^4}{2c^2} + \frac{x^6}{6c^4}, if|x| \leq c$, otherwise, $\rho(x) = \frac{c^2}{6}$.

The turning constant c controls the breakdown value and efficiency of the S estimate. When c=1.548 and K=0.11995, the breakdown value of the S estimate is 50% and the asymptotic efficiency is about 28%. S estimation is usually considered as high breakdown and low efficiency method.

2.4 MM Estimation

MM estimation was introduced by Yohai (1987). It was the first estimate with a high breakdown (50%) and high efficiency under normal distribution assumption. MM estimator has three steps:

1. Compute an initial consistent estimate $\hat{\beta}_0$ with a high BP but possibly low efficiency (LTS estimate and S estimate are two kinds of estimates that can be used as the initial estimate). The commonly adopted influence function for S-estimate is given as

$$\rho(x) = \begin{cases} 3(\frac{x}{c})^2 - 3(\frac{x}{c})^4 + (\frac{x}{c})^6, & \text{if } |x| \leq c \\ 1 & \text{otherwise} \end{cases}$$

2. Calculate the MM estimate of the parameters $\hat{\beta}$ that minimize the expression

$$\sum_{i=1}^{n} \rho(\frac{y_i - x_i'\hat{\beta}}{\hat{\sigma}_0})$$

where $\hat{\sigma}_0$ is the estimate of scale (standard deviation of the residuals) from first step.

3. The final step computes the MM estimate of scale s which is the solution to the equation

$$\frac{1}{n-p} \sum_{i=1}^{n} \rho(\frac{y_i - x_i'\hat{\beta}}{s}) = 0.5$$

2.5 TELBS Robust Regression Method

Tabatabai et. al. (2012) proposed a new robust linear regression method, TELBS in 2012. The TELBS estimate of parameter β is given by

$$\hat{\beta} = \arg min \sum_{i=1}^{n} \frac{\rho_\omega(t_i)}{L_i} \tag{1}$$

where

$$\rho_\omega(x) = 1 - Sech(\omega x)$$

and ω is called turning constant, which is a positive real number. The function $Sech(\cdot)$ is the hyperbolic secant function and t_i is defined by

$$t_i = \frac{(y_i - x_i'\hat{\beta})(1 - h_{ii})}{\sigma} \tag{2}$$

where σ is the error standard deviation, and h_{ii} is the diagonal element of the hat matrix of the form

$$H = X(X'X)^{-1}X',$$

where X is the design matrix. Define $M_j = Median\{|x_{1j}|, |x_{2j}|, \ldots, |x_{nj}|\}$ for $j = 1, ..., p$. Define $L_i = \sum_{j=1}^{p} Max\{M_j, |x_{ij}|\}$. Usually, σ is unknown and we suggested using the estimator proposed by Rousseeuw and Croux (1993), which is given by

$$\hat{\sigma} = 1.1926 Median(Median|r_i - r_j|), 1 \leq i, j \leq n, \tag{3}$$

where r is the residual. Taking the partial derivatives of (1) with respect to parameters and setting them equal to zero results in the following system of equations:

$$\sum_{i=1}^{n} \frac{\psi_\omega(t_i)}{L_i} \frac{\partial t_i}{\partial \beta_i} = 0 \tag{4}$$

where $\psi_\omega = \omega Sech(\omega x) Tanh(\omega x)$, which is the derivative of ρ_ω. The weight w_i is defined as

$$w_i = \frac{\psi(t_i)(1 - h_{ii})}{\sigma(y_i - x_i'\hat{\beta})L_i} \tag{5}$$

Then the equation (4) can be written as

$$\sum_{i=1}^{n} w_i(y_i - x_i'\hat{\beta})x_i = 0$$

Denote the weight matrix by W, it is a diagonal matrix. The elements on the main diagonal are $w_1, w_2, \ldots, w_n$. Therefore, the estimate of the parameter β is given by

$$\hat{\beta} = (X'WX)^{-1}X'Wy \tag{6}$$

The following procedures are used to estimate the parameter.

Step 1. Set $\hat{\sigma}^0 = 1$, calculate an initial estimate of vector β by minimizing the function given in (1).

Step 2. Calculate $\hat{\sigma}$ and weights w_i by using equation (3) and (5), then obtain the weight matrix W.

Step 3. Calculate $\hat{\beta}$ using equation (6).

Repeat step 2 to 3 until convergence occurs.

TELBS estimates of linear regression parameters has influence functions bounded in both the explanatory and the response variable direction. It has high breakdown point and high asymptotic efficiency. In all examples and simulations considered in this study, TELBS method is evaluated under an efficiency level of 85% (ω=0.628).

3. Applications

To study the performance of TELBS and compare it with other robust methods, we apply each method, LS, S, LTS, M, MM, and TELBS to two data sets. All results are obtained by using R 2.12. We first consider a brain and body weight data. This example was taken from a larger data set in Weisberg (1980) and Jerison (1973). It gives the brain weight and body weight of 65 animals. We used a logarithmic transformation (common log) for both variables, and a scatter diagram of the transformed data is given in Figure 1 (left). We can see there are 3 outlying points. Tabaitabai (2012) proposed a new diagnostic measure called S_h. We examine outliers by using three diagnostic measures: Cook's distance (CD), robust Studentized residual (SR), and S_h. We give a brief introduction of these measures here.

Cook's distance is widely used for identifying outliers. In Tabaitabai (2012), we suggested a robust Cook's distance using TELBS estimates of parameters, which is given by

$$CD_i = \frac{h_{ii}t_i^2}{p(1 - h_{ii})^4}, \quad i = 1, 2, ..., n$$

where p is the number of parameters, t_i and h_{ii} are defined in Section 2.5.

Table 1: Summary of diagnostic measures for brain weight data

Observation	CD	SR	S_h
1	0.0069	3.6013	-0.8091
2	0.0249	5.5704	0.1022
3	0.0011	-1.3180	-0.5288
4	0.0163	-3.3414	2.1828
⋮	⋮	⋮	⋮
62	0.0061	3.3907	-0.8374
63	2.2524	-24.5325	8.2490
64	1.7425	-22.2431	7.7345
65	4.2448	-25.6678	13.6624

The Studentized residual using TELBS estimates of parameters has the form

$$SR_i = \frac{t_i}{\hat{\sigma}\sqrt{1-h_{ii}}}, \quad i = 1, 2, ..., n$$

where $\hat{\sigma}$ is defined by (3).

In addition to considering the elements of the main diagonal of the hat matrix h_{ii}, we recommended a new influence measure in Tabaitabai (2012), which is defined as

$$S_h(i) = \frac{h_{ii} - Median(h_{ii})}{\hat{\sigma}_h}, \quad i = 1, 2, ..., n$$

where $\hat{\sigma}_h = 1.1926 Median(Median|h_{ii} - h_{jj}|), 1 \leq i, j \leq n$. Large value of $|S_h(i)|$ indicates the presence of an influential observation. This measure seems to be very good for identifying the leverage points based on the results in Tabaitabai (2012) and this study.

Table 1 gives the values of the three measures for some of the observations using TELBS as a robust estimator of regression parameters. Observations 63, 64, 65 are identified as outliers. To investigate weather a larger brain is required to govern a heavier body, a linear regression model is used to fit the data with brain weight (y) and body weight (x). Each method was fitted to the data and the fitted lines for LS and TELBS are given in Figure 1 (right). The result of estimates for each method is given in Table 2. Based on the result, body weight has a significant effect on brain weight. Among five robust regression methods, MM and TELBS provide the closest estimates to LS with the removal of three outliers.

Another example we consider is a breast cancer data. Lea (1965) discussed the relationship between mean annual temperature and the mortality rate for a type of breast cancer in women. The subjects were residents of certain regions of Great Britain, Norway, and Sweden. A scatter diagram of temperature (x) vs. mortality index (y) (Figure 2 (left)) shows a strong positive relationship between the two variables. We compute the values of the three measures for the breast cancer data using TELBS as a robust estimator of regression parameters, the result is given in Table 3. The data set contains a single outlier, observation 15, which has the largest values for each measure.

A simple linear regression model is used to fit the data. The fitted line for LS, TELBS, and LS (with removal of an outlier) are given in Figure 2 (right). The estimates of parameters for each method are given in Table 4. M, MM give similar estimates as LS. LTS, S, and TELBS give better fit in comparison to LS with the removal of an outlier. Among the five robust methods, only TELBS can identify the significant effect of both the constant term and temperature. In addition, to examine the goodness of fit for each model, a robust

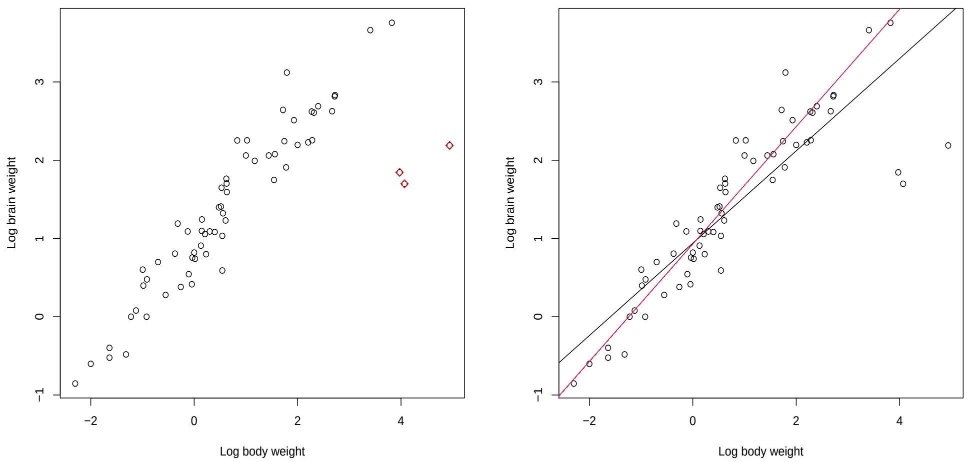

Figure 1: Left: Scatter diagram for brain weight data. Right: Scatter diagram with LS fit (black line), TELBS fit (red line), and LS fit with removal of 3 outliers (dashed line).

Table 2: Summary of estimates for brain weight data for six comparison models

	Parameter	Estimate	Standard errors	P-value
LS	Constant	0.9432	0.0704	< 0.0001
	Body weight	0.5915	0.0412	< 0.0001
LS (3 outliers	Constant	0.9271	0.0799	< 0.0001
removed)	Body weight	0.7517	0.0464	< 0.0001
S	Constant	0.8650	-	-
	Body weight	0.7470	-	-
LTS	Constant	0.8475	-	-
	Body weight	0.7713	-	-
M	Constant	0.9242	0.0462	< 0.0001
	Body weight	0.6985	0.0270	< 0.0001
MM	Constant	0.9196	0.0426	< 0.0001
	Body weight	0.7460	0.0249	< 0.0001
TELBS	Constant	0.9289	0.0435	< 0.0001
	Body weight	0.7499	0.0255	< 0.0001

Table 3: Summary of diagnostic measures for breast cancer data

Observation	CD	SR	S_h
1	0.0042	0.0287	2.1400
2	0.1058	0.1751	1.0225
3	0.0159	0.0671	1.0937
4	0.0004	0.0115	0.5614
⋮	⋮	⋮	⋮
14	0.0081	-0.0542	0.4343
15	6.6114	0.4135	10.0281
16	0.0016	-0.0097	6.6091

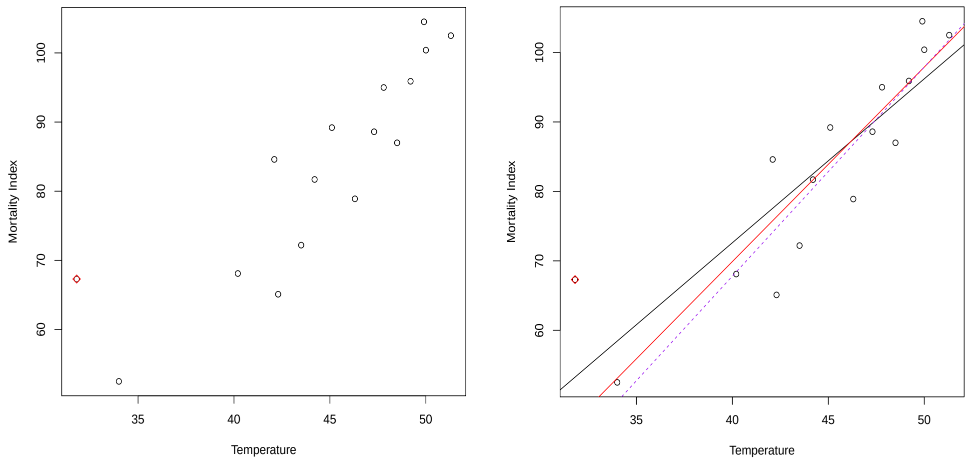

Figure 2: Left: Scatter diagram for breast cancer data. Right: Scatter diagram with LS fit (black line), TELBS fit (red line), and LS fit with removal of an outlier (dashed line).

version of the coefficient of determination R^2 is evaluated by using the formula given on pp.69 in Tabaitabai (2012). The result is included in Table 4. For LS method, the R^2 increases from 0.76 to 0.85 after removing the outlier. All robust methods give R^2 0.86 or higher.

4. A Simulation Study

To further evaluate the performance of the TELBS estimates in comparison with M, MM, S, and LTS estimates, we conduct a simulation study under a small sample size (n=15) and a relative large sample size (n=30). We consider different contamination levels under various direction of contamination such as x-direction, y-direction, and both x and y direction. The simulation study is performed with R 2.12 and based on 5000 simulations. We consider a linear regression models with two covariates (x_1 and x_2) and generate both x_1 and x_2 and the random errors from a standard normal distribution with parameters 1, 3, and 3 for intercept and two covariates respectively. To evaluate the robustness of these estimates, we randomly choose $10\%, 20\%, 40\%$ of the data and contaminate them by magnifying their size by a factor of 100, first in the direction of response variable (y), explanatory variables (both x1 and x2), then both the response and explanatory variables (y, x1, and x2). The bias is estimated by the equation $Bias = |\frac{\sum_{i=1}^{m}(\hat{\beta}_i)}{m} - \beta|$, where m is the number of simulations. The mean square error is estimated by $MSE = \frac{\sum_{i=1}^{m}(\hat{\beta}_i - \beta)^2}{m}$.

Table 5 and 6 give the results of Bias and MSE for each method for sample size of 15 and 30 respectively when the contamination is in the x_1, x_2 direction. Table 7 and 8 give the results of Bias and MSE for each method for sample size of 15 and 30 respectively when the contamination is in the y-direction. Table 9 and 10 give the results of Bias and MSE for each method for sample size of 15 and 30 respectively when the contamination is in both x_1, x_2, and y directions. By examining the simulation results, we see that M estimation underperforms in all cases especially in the x_1, x_2 direction. It fails to give a close estimate of the parameters when the contamination level increases to 20% or higher.

Table 4: Summary of estimates for breast cancer data for six comparison models

	Parameter	Estimate	Standard errors	P-value
LS	Constant	-21.7947	0.0704	< 0.0001
(R^2=0.76)	Temperature	2.3577	0.0412	< 0.0001
LS (1 outlier	Constant	-52.6181	15.8239	0.0055
removed)	Temperature	3.0152	0.3466	< 0.0001
(R^2=0.85)				
S	Constant	-47.7873	-	-
(R^2=0.89)	Temperature	2.9296	-	-
LTS	Constant	-48.5596	-	-
(R^2=0.91)	Temperature	2.9600	-	-
M	Constant	-32.0048	14.8267	0.9756
(R^2=0.87)	Temperature	2.5792	0.3301	< 0.0001
MM	Constant	-29.0534	16.6295	0.9487
(R^2=0.86)	Temperature	2.5161	0.3702	< 0.0001
TELBS	Constant	-42.1086	14.2273	0.0031
(R^2=0.87)	Temperature	2.7956	0.3167	< 0.0001

LTS, S, and MM estimation perform well in most cases except for the x_1, x_2 direction and a contamination level of 40% for both sample sizes, they provide a relative large bias and MSE. In addition, MM and S method fail to give a good estimate for the y-direction with a contamination level of 40% when the sample size is small. TELBS outperforms all other methods in all cases considered, it provides similar or smaller bias and MSE in comparison with other methods under each case.

Table 5: Bias and MSE with contamination in x_1, x_2 direction (n=15)

	Par	LTS	S	M	MM	TELBS
10%	β_0	0.0063	0.0143	0.0433	0.0036	0.0133
Bias	β_1	0.0308	0.0251	2.8605	0.0308	0.0293
	β_2	0.0453	0.0024	2.8787	0.0201	0.0041
MSE	β_0	0.3578	0.2009	1.4704	0.1013	0.1104
	β_1	0.5463	0.3102	8.3910	0.1811	0.1208
	β_2	0.4892	0.3025	8.4568	0.1765	0.1248
20%	β_0	0.0013	0.0111	0.0615	0.0086	0.0004
Bias	β_1	0.1122	0.0478	2.9603	0.0954	0.0243
	β_2	0.0333	0.0438	2.9595	0.0656	0.0004
MSE	β_0	0.3453	0.2407	1.4053	0.1277	0.1123
	β_1	0.6102	0.4066	8.7679	0.3716	0.1471
	β_2	0.5491	0.3729	8.7689	0.3598	0.1578
40%	β_0	0.0184	0.0249	0.0116	0.0098	0.0074
Bias	β_1	1.1986	2.0264	2.9694	0.5988	0.0945
	β_2	1.2098	2.0440	2.9698	0.4667	0.0846
MSE	β_0	0.7477	0.8525	0.8245	0.4445	0.1517
	β_1	3.8151	6.1377	8.8176	1.9756	0.2917
	β_2	3.7654	6.1337	8.8202	1.6455	0.3258

Par represents parameter

Table 6: Bias and MSE with contamination in x_1, x_2 direction (n=30)

	Par	LTS	S	M	MM	TELBS
10%	β_0	0.0006	0.0135	0.0106	0.0096	0.0003
Bias	β_1	0.0175	0.0024	2.9442	0.0100	0.0048
	β_2	0.0193	0.0029	2.9463	0.0072	0.0068
MSE	β_0	0.1618	0.1091	0.6456	0.0430	0.0472
	β_1	0.1775	0.1213	8.6988	0.0646	0.0520
	β_2	0.1829	0.1289	8.7063	0.0622	0.0524
20%	β_0	0.0001	0.0092	0.0449	0.0041	0.0097
Bias	β_1	0.0058	0.0245	2.9683	0.0093	0.00003
	β_2	0.0007	0.0154	2.9684	0.0025	0.0001
MSE	β_0	0.1471	0.0939	0.5305	0.0508	0.0523
	β_1	0.1752	0.1216	8.8106	0.0744	0.0574
	β_2	0.1790	0.1297	8.8114	0.0729	0.0542
40%	β_0	0.0146	0.0063	0.0305	0.0054	0.0036
Bias	β_1	1.0612	1.7339	2.9695	1.7953	0.0105
	β_2	1.0295	1.7392	2.9697	1.7851	0.0114
MSE	β_0	0.1774	0.2492	0.3478	0.2372	0.0602
	β_1	3.2150	5.2153	8.8182	5.3454	0.0703
	β_2	3.2074	5.2251	8.8194	5.3484	0.0656

Table 7: Bias and MSE with contamination in y-direction (n=15)

	Par	LTS	S	M	MM	TELBS
10%	β_0	0.0143	0.0078	0.0204	0.0097	0.0002
Bias	β_1	0.0415	0.0184	1.2148	0.0156	0.0034
	β_2	0.0247	0.0164	1.0158	0.0187	0.0149
MSE	β_0	0.2929	0.1992	22.7041	0.1042	0.1227
	β_1	0.3675	0.2245	144.0982	0.1153	0.1614
	β_2	0.4045	0.2210	145.7049	0.1239	0.1526
20%	β_0	0.0143	0.0201	2.4941	0.0049	0.0137
Bias	β_1	0.0118	0.0115	9.6629	0.0167	0.0037
	β_2	0.0146	0.0133	8.0754	0.0082	0.0043
MSE	β_0	0.2673	0.1701	401.0656	0.1025	0.1324
	β_1	0.3623	0.1972	1484.186	0.1278	0.1651
	β_2	0.3676	0.2014	1105.711	0.1332	0.1687
40%	β_0	0.0097	1.4106	32.4270	4.2684	0.0013
Bias	β_1	0.0018	4.4432	98.1562	18.6975	0.0071
	β_2	0.0203	4.6559	100.8857	19.6652	0.0214
MSE	β_0	0.1644	370.1686	4344.272	1001.424	0.1673
	β_1	0.2174	1181.942	16414.03	3050.492	0.2545
	β_2	0.2200	1293.672	17105.19	3449.291	0.2808

Table 8: Bias and MSE with contamination in y-direction (n=30)

	Par	LTS	S	M	MM	TELBS
10%	β_0	0.0183	0.0047	0.0355	0.0075	0.0156
Bias	β_1	0.0088	0.0179	0.1092	0.0068	0.0061
	β_2	0.0207	0.0111	0.1175	0.0009	0.0024
MSE	β_0	0.1582	0.1002	0.0635	0.0435	0.0507
	β_1	0.1904	0.1140	0.0713	0.0465	0.0552
	β_2	0.1922	0.1103	0.0742	0.0431	0.0581
20%	β_0	0.0077	0.0175	0.0038	0.0022	0.0077
Bias	β_1	0.0023	0.0062	0.9126	0.0045	0.0055
	β_2	0.0026	0.0109	1.1691	0.0001	0.0076
MSE	β_0	0.1404	0.0896	24.1282	0.0475	0.0503
	β_1	0.1586	0.0979	47.0104	0.0491	0.0603
	β_2	0.1701	0.0921	108.3892	0.0498	0.0584
40%	β_0	0.0096	0.0023	27.9896	0.0087	0.0061
Bias	β_1	0.0047	0.0071	86.2142	0.0134	0.0074
	β_2	0.0010	0.0107	87.1350	0.0014	0.0054
MSE	β_0	0.0882	0.0811	2259.844	0.0727	0.0643
	β_1	0.0865	0.0823	11145.62	0.0834	0.0751
	β_2	0.1040	0.0912	11531.11	0.0809	0.0737

Table 9: Bias and MSE with contamination in both x_1, x_2, and y-direction (n=15)

	Par	LTS	S	M	MM	TELBS
10%	β_0	0.0122	0.0097	0.2276	0.0073	0.0065
Bias	β_1	0.0088	0.0026	0.0878	0.0162	0.0078
	β_2	0.0091	0.0101	0.1919	0.0182	0.0161
MSE	β_0	0.3143	0.2312	4.3738	0.1174	0.1113
	β_1	0.4150	0.3584	21.6543	0.2469	0.1338
	β_2	0.4478	0.3159	22.2062	0.2289	0.1433
20%	β_0	0.0257	0.0002	0.3366	0.0014	0.0106
Bias	β_1	0.0003	0.0152	0.0514	0.0123	0.0212
	β_2	0.0248	0.0181	0.0562	0.0093	0.0283
MSE	β_0	0.3565	0.2052	2.0022	0.1325	0.1222
	β_1	0.4818	0.3436	5.2538	0.2807	0.1283
	β_2	0.5007	0.3403	6.5749	0.2593	0.1392
40%	β_0	0.0167	0.0061	6.0666	0.0169	0.0265
Bias	β_1	0.0532	0.0231	0.0448	0.0175	0.1012
	β_2	0.0228	0.0179	0.0371	0.0210	0.0276
MSE	β_0	0.3509	0.2635	160.056	0.2073	0.1496
	β_1	0.4819	0.3831	0.7726	0.3756	0.2257
	β_2	0.4859	0.3997	0.9163	0.3631	0.1975

Table 10: Bias and MSE with contamination in both x_1, x_2, and y-direction (n=30)

	Par	LTS	S	M	MM	TELBS
10%	β_0	0.0134	0.0047	0.1341	0.0006	0.0021
Bias	β_1	0.0176	0.0065	0.0248	0.0116	0.0091
	β_2	0.0147	0.0154	0.0240	0.0092	0.0012
MSE	β_0	0.1674	0.1172	0.3124	0.0478	0.0474
	β_1	0.1827	0.1388	2.8842	0.0973	0.0581
	β_2	0.1759	0.1408	3.2812	0.0964	0.0565
20%	β_0	0.0046	0.0107	0.3152	0.0011	0.0015
Bias	β_1	0.0040	0.0165	0.0214	0.0030	0.0069
	β_2	0.0041	0.0054	0.0356	0.0029	0.0034
MSE	β_0	0.1579	0.1116	0.2796	0.0527	0.0491
	β_1	0.2466	0.1444	0.8944	0.1323	0.0548
	β_2	0.1935	0.1449	0.8763	0.1269	0.0563
40%	β_0	0.0023	0.0005	6.2696	0.0075	0.0067
Bias	β_1	0.0029	0.0078	0.0227	0.0012	0.0142
	β_2	0.0338	0.0073	0.0080	0.0221	0.0077
MSE	β_0	0.1286	0.0936	120.129	0.0802	0.0601
	β_1	0.1945	0.1595	0.3154	0.1551	0.0708
	β_2	0.1758	0.1606	0.3416	0.1545	0.0686

5. Discussion

We introduce some commonly used robust linear regression methods and a newly developed TELBS method in this article. We study the performance of these methods by using two examples and a simulation study. Results indicate that M estimation fails to provide good estimates in some cases especially when the sample size is small and the outliers are in the x-direction. LTS, S, MM perform well in most cases except when the contamination level is high (40%). TELBS robust method performs well in all cases considered and outperforms other methods considered in this study as the percentage of outliers increases. It provides a flexible and powerful alternative to the practitioners in the field of robust linear regression.

REFERENCES

Alema, O.G. (2011). Comparison of Robust Regression Methods in Linear Regression. Int.J. Contemp. Math. Sciences, 6:9, 409-421.

Maronna, R. A., Martin, R. D., and Yohai, V. J. (2006). *Robust Statistics: Theory and Methods*. New York: Wiley.

Rousseeuw, P. J., Leroy, A. M. (1987). *Robust Regression and Outlier Detection*. New York: Wiley.

Rousseeuw, P. J., Croux, C. (1993). Alternative to the Median Absolute Deviation. Journal of American Statistical Association. 88, 1273-1283.

SAS Institute Inc., Robust Regression and Outlier Detection with the Robustreg Procedure, USA, 2006.

Schumacker, E.R., Monahan P.M., Mount, E. R. (2002). A Comparison of OLS and Robust Regression Using S-PLUS. Multiple Linear Regression Viewpoints. 28, 10-13.

Tabatabai, M., Eby, W., Li, H, *et al.* (2012). TELBS Robust Linear Regression Method. Open Access Medical Statistics. 2, 65-84.

Yohai, V.J. (1987). High Breakdown-point and High Efficiency Robust Estimates for Regression. The Annals of Statistics, 15, 642-656.