Example of a Renewal Process with No-mean Inter-arrival Times

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Abstract

We derive the *finite-time* t probability density function (pdf) of the excess, the age and the total life of a renewal process with interarrival times distributed as a no-mean Pareto distribution having shape parameter $\alpha \in (0, 1]$. We compare the time-t pdf's with the corresponding *limiting* pdf's of a renewal process with interarrival times distributed as a finite-mean, *truncated* Pareto distribution having the same shape parameter α. The paper gives an example with fixed values t, α and right truncation point K, such that the limiting pdf's closely approximate the finite-time t pdf's on a subset of support.

Key words: renewal process; no-mean Pareto interarrivals; truncated Pareto interarrivals; finite-time and limiting probability distributions; integral equations; level crossing.

1. Introduction

The probability distributions of the excess, age, and total life in a renewal process at a fixed finite-time $t > 0$, or when $t \to \infty$, have interested statisticians and probabilists for many years. Feller (1966), Smith (1958), Cox (1962), Karlin and Taylor (1975), Ross (1970), and others, have thoroughly considered these pdf's, mostly when the interarrival times have a finite mean. Recent work in statistics and stochastic modelling has generated interest in the finite tme-t distributions of renewal processes when interarrivals have no-mean, heavy-tailed Pareto distributions (e.g., Huang et al., 2013; Harris et al., 2000).

Here, we juxtapose two related renewal processes. Process 1 has interarrivals with a type II *no-mean* Pareto distribution (Kleiber and Kotz, 2003), with a shape parameter $\alpha \in (0, 1]$. The finite-time t pdf's exist, but the limiting pdf's as $t \to \infty$ do not exist because the interarrivls have no mean (see Section 4.1). Process 2 has interarrivals with a *finite-mean* right-truncated Pareto distribution and the same shape parameter α. In Process 2, finite-time pdf's and limiting pdf's, both exist. In general, the time-t pdf's are the solutions of integral equations, which may be tedious to compute. However, the limiting pdf's have well known, simple formulas (e.g., Karlin and Taylor, 1975; Ross,1970). This fact motivates us to determine how to approximate the finite-time t pdf's in Process 1 by using the limiting pdf's in Process 2.

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We denote Process 1 as $\{Z_n\}_{n=1,2,..}$ where $Z_n \underset{dis}{\equiv}$ *no-mean* $Pareto(x,\alpha)$, $x>0$. Then $\{Z_n\}_{n=1,2,...}$ has finite-time t pdf's of excess, age, and total life (Brill, 2014), but *no* limiting pdf's as $t\to\infty$. Let $\{Z_{K,n}\}_{n=1,2,..}$ denote Process 2, where $Z_{K,n} \underset{dis}{\equiv} Pareto(x,\alpha), 0<x<K$, *right- truncated at* $K>0$, which has a *finite mean*. Process $\{Z_{K,n}\}_{n=1,2,...}$ has limiting pdf's of excess, age, and total life as $t\to\infty$, which have very simple expressions (Section 4.1). We will select a pair $\langle t,K\rangle$ such that the *limiting* pdf's in $\{Z_{K,n}\}_{n=1,2,...}$ closely approximate the *finite-timet* pdf's in $\{Z_n\}_{n=1,2,...}$ on subsets of support (Section 5).

Our overall analysis consists of three main stages. The following first two stages result in the finite-time t pdf's of $\{Z_n\}_{n=1,2,...}$ (Brill, 2014). STAGE 1: embed iid (independent and identically distributed) realizations of $\{Z_n\}_{n=1,2,...}$ in cycles of a *regenerative* process $\{X(s)\}_{s\geq 0}$ having a threshold at level t. Then $\{X(s)\}_{s\geq 0}$ has a limiting pdf as $s\to\infty$ (e.g., Smith, 1955; Sigman and Wolff, 1993). STAGE 2: derive the finite-time t pdf's of $\{Z_n\}_{n=1,2,...}$ directly *in terms of* the limiting pdf of $\{X(s)\}_{s\geq 0}$ as $s\to\infty$, using a level crossing procedure (Brill, 2014). The third stage compares the finite-time and corresponding limiting pdf's. STAGE 3: compare the *finite-time* t pdf's of $\{Z_n\}_{n=1,2,...}$ with the corresponding limiting pdf's of $\{Z_{K,n}\}_{n=1,2,...}$, where the truncation point K is chosen so that the limiting pdf's closely approximate the time-t pdf's..

Relevant level crossing methods are discussed in Brill (2008), and references therein. Regenerative processes are discussed in Smith (1955), Sigman and Wolff (1993), among others.

Section 2 details the renewal process $\{Z_n\}_{n=1,2,...}$. Section 3 defines the regenerative process $\{X(s)\}_{s\geq 0}$, gives its limiting pdf, and also the finite-time t pdf's of excess, age and total life in $\{Z_n\}_{n=1,2,...}$ *in terms of* the limiting pdf of $\{X(s)\}_{s\geq 0}$. Section 4 defines the renewal process $\{Z_{K,n}\}_{n=1,2,...}$, and gives its limiting pdf's of excess, age and total life. Section 5 compares the finite-time t pdf's of $\{Z_n\}_{n=1,2,...}$ with the limiting pdf's of $\{Z_{K,n}\}_{n=1,2,...}$. Note that all figures are at the end of the paper.

2. Renewal process $\{Z_n\}_{n=1,2,...}$ with no-mean Pareto interarrivals

Let the interarrivals be $Z_n \underset{dis}{\equiv} Z \underset{dis}{=} Pareto(x,\alpha),\ x>0, \alpha\in(0,1]$,with

$$\left.\begin{array}{ll}\text{cdf} & B(x) = 1-(1+x)^{-\alpha},\ x\in[0,\infty),\\ \text{ccdf} & \bar{B}(x) = 1-B(x) = (1+x)^{-\alpha},\ x\in[0,\infty),\\ \text{pdf} & b(x) = \frac{d}{dx}B(x) = \alpha\,(1+x)^{-\alpha-1},\ x\in(0,\infty),\end{array}\right\} \quad (1)$$

where $B(x)=P(Z\leq x)$. If $\alpha>0$ then Z is heavy tailed (Sigman, 1999); if $\alpha\in(0,1]$ then Z has no mean. The functions in (1) are plotted in Fig. 1, when $\alpha=0.5$. The limiting pdf's of excess, age, and total life do not exist when $\alpha=0.5$.

A 1-dimensional diagram of $\{Z_n\}_{n=1,2,...}$ is given in Fig. 2, where time $t>0$ is fixed, $N(t)=$ *number of renewals in* $(0,t)$, N_t $=$ *number of renewals required to first exceed* t = $\min\{n|\sum_{i=1}^{n} Z_i > t\}$ – a stopping time $=N(t)+1$. Note that $E(N_t)=E(N(t))+1\equiv m(t)+1$, where $m(t)$ is the renewal function (Karlin and Taylor, 1975).

3. Regenerative process $\{X(s)\}_{s\geq 0}$ and its limiting pdf

We construct a regenerative process $\{X(s)\}_{s\geq 0}$ having state space $[0,\infty)$ and a threshold at level $t>0$, from the renewal process $\{Z_n\}_{n=1,2,...}$ (Brill, 2014; Fig. 3). Some properties of $\{X(s)\}_{s\geq 0}$ are: $X(0)=0$; sample paths make upward jumps $\underset{dis}{=} Z$ at an arbitrary Poisson rate $\lambda=1$; whenever a jump upcrosses level t, $\{X(s)\}_{s\geq 0}$ jumps *downward* immediately into level 0, completing a cycle, *and immediately starts a new cycle*. $X(s), s>0$, is a non-decreasing step function on each cycle (Fig. 3). This construction rotates $\{Z_n\}_{n=1,2,...}$ 90^o counterclockwise, and creates an infinite number of iid realizations in the cycles. (More details are given in Brill, 2014.)

$\{X(s)\}_{s\geq 0}$ has a limiting pdf with support $[0,t)$, because it is a regenerative process (e.g., Sigman and Wolff, 1993; Smith, 1955). Moreover, the limiting pdf at jump instants is the same as the time-average limiting pdf, due to Poisson occurrences (Wolff, 1982). We denote the mixed limiting pdf of $\{X(s)\}_{s\geq 0}$ as $s\to\infty$ by $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$, where $\pi_0^{(t)}=\lim_{s\to\infty}P(X(s)=0)$. The limiting cdf, and normalizing condition, are respectively

$$F^{(t)}(x)=\pi_0^{(t)}+\int_{y=0}^{x} f^{(t)}(y)dy,\ x\in[0,t);\ F^{(t)}(t)=1. \tag{2}$$

The pdf $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$ is the key for obtaining the finie-time t pdf's of $\{Z_n\}_{n=1,2,...}$.

3.1. Integral equation for $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$

Applying a level crossing technique (Brill, 2014), we obtain the Volterra integral equation

$$f^{(t)}(x)=\pi_0^{(t)}\alpha(1+x)^{-\alpha-1}+\alpha\int_{y=0}^{x}(1+x-y)^{-\alpha-1}f^{(t)}(y)dy,\ 0<x<t. \tag{3}$$

We use a computational procedure (Section 3.1.1) to solve (3) for $f^{(t)}(x), 0<x<t$, and apply the normalizing condition in (2) to compute $\pi_0^{(t)}$.

3.1.1. Computational procedure for $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}, x\in(0,t)$

The computation for solving (3) is based on the Riemann-sum definition of an integral on a finite interval. The resulting numerical solution is a step function on a partition of $(0,t)$ with a "small" norm $h>0$. We choose h such that $t=Nh$ where N is a positive integer. In our example, $t=400$, $h=0.1$, $N=4000$.

Outline of computational procedure We first choose the values of t, h, and N. Let $f_*^{(t)}(x)=\frac{f^{(t)}(x)}{\pi_0^{(t)}},\ x\in(0,t)$, which transforms (3) into the following integral equation for $f_*^{(t)}(x), 0<x<t$,

$$f_*^{(t)}(x)=\alpha(1+x)^{-\alpha-1}+\alpha\int_{y=0}^{x}(1+x-y)^{-\alpha-1}f_*^{(t)}(y)dy,\ 0<x<t. \tag{4}$$

The computation consists of the following five steps.

Step 1. Compute and store $b(ih) = \alpha(1+ih)^{-\alpha-1}, i = 0, ..., N$. (This is the first term on the right hand side of (4) when $x = ih$.)

Step 2. From (4), we start the computation with $f_*^{(t)}(0^+) = f_*^{(t)}(0 \cdot h) = \alpha$, and compute for $i = 1, ..., N$,

$$\begin{aligned} f_*^{(t)}(ih) &= \alpha(1+ih)^{-\alpha-1} + \int\limits_{y=0}^{ih} \alpha(1+ih-y)^{-\alpha-1} f_*^{(t)}(y)dy \\ &= \alpha(1+ih)^{-\alpha-1} + \sum_{j=1}^{i} \int\limits_{y=(j-1)h}^{jh} \alpha(1+ih-y)^{-\alpha-1} f_*^{(t)}(y)dy, \end{aligned}$$

(Step 2 gives $f_*^{(t)}(ih), i = 1, ..., N$).

Step 3. From Step 2, compute a step function

$$f_*^{(t)}(x) = f_*^{(t)}((i-1)h),\ x \in ((i-1)\,h, ih)\,,\ i = 1, ..., N.$$

Step 4. Compute $\int_{x=0}^{t} f_*^{(t)}(x)dx = h\sum_{i=1}^{N} f_*^{(t)}(ih)$.

Step 5. Denote the approximation of $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{x\in(0,t)}$ as $\left\{\hat{\pi}_0^{(t)}, \hat{f}^{(t)}(x)\right\}_{x\in(0,t)}$. Then

$$\begin{aligned} \hat{f}^{(t)}(x) &= \hat{\pi}_0^{(t)} f_*^{(t)}(x),\ x \in (0,t), \\ \hat{\pi}_0^{(t)} + \int_0^t \hat{f}^{(t)}(x)dx &= \hat{\pi}_0^{(t)} + \hat{\pi}_0^{(t)} \int_0^t f_*^{(t)}(x)dx = 1, \end{aligned}$$

which we solve for $\hat{\pi}_0^{(t)}$.

The pdf $\left\{\hat{\pi}_0^{(t)}, \hat{f}^{(t)}(x)\right\}_{0<x<t}$ is plotted in Fig. 4, when $\alpha = 0.5$, $t = 400$, $h = 0.1$.

Section 3.2 uses $\left\{\hat{\pi}_0^{(t)}, \hat{f}^{(t)}(x)\right\}_{0<x<t}$ to approximate the time-t pdf's of excess, age and total life of $\{Z_n\}_{n=1,2...}$. Section 5 (using Section 4.1) compares the time-t pdf's of $\{Z_n\}_{n=1,2...}$ with the corresponding *limiting* pdf's of renewal process $\{Z_{K,n}\}_{n=1,2...}$ (Section 4).

3.2. The finite-time t pdf's of the renewal process $\{Z_n\}_{n=1,2,...}$

For the renewal process $\{Z_n\}_{n=1,2,...}$ we denote the *finite-time* t excess, age and total life as γ_t, δ_t and β_t respectively. The corresponding pdf's are denoted by: $f_{\gamma_t}(x)$, $0 < x < \infty$; $\{\pi_{\delta_t}, f_{\delta_t}(x)\}_{0<x<t}$ where $\pi_{\delta_t} = P(\delta_t = t)$; $f_{\beta_t}(x)$, $0 < x < \infty$. From Brill (2014), the pdf's are given in terms of $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$ as

$$f_{\gamma_t}(x) = b(t+x) + \int_{y=0}^{t} b(t+x-y)\frac{f^t(y)}{\pi_0^{(t)}}dy,\ 0 < x < \infty, \tag{5}$$

$$\begin{aligned}
f_{\delta_t}(x) &= \bar{B}(x)\frac{f^{(t)}(t-x)}{\pi_0^{(t)}},\ 0<x<t; \qquad \pi_{\delta_t}=\bar{B}(t), &(6)\\
f_{\beta_t}(x) &= b(x)\int_{y=t-x}^{t}\frac{f^{(t)}(y)}{\pi_0^{(t)}}dy,\ 0<x<t, &(7)\\
f_{\beta_t}(x) &= b(x)\left(1+\int_{y=t-x}^{t}\frac{f^{(t)}(y)}{\pi_0^{(t)}}dy\right),\ t<x<\infty. &(8)
\end{aligned}$$

Remark. From (7) and (8), $f_{\beta_t}(x)$ has a discontinuity at $x=t$ of size

$$f_{\beta_t}(t^+)-f_{\beta_t}(t^-)=b(t)=\alpha(1+t)^{-\alpha-1}.$$

In order to approximate the time-t pdf's in (5)-(8), we will substitute

$$\left\{\hat{\pi}_0^{(t)},\hat{f}^{(t)}(x)\right\}_{0<x<t}\ \text{ for }\ \left\{\pi_0^{(t)},f^{(t)}(x)\right\}_{0<x<t}.$$

The approximate time-t pdf's of $\{Z_n\}_{n=1,2,...}$, i.e., $\hat{f}_{\gamma_t}(x),\ x>0$; $\left\{\hat{\pi}_{\delta_t},\hat{f}_{\delta_t}(x)\right\}_{0<x<t}$; $\hat{f}_{\beta_t}(x),\ x>0$, are specified in terms of the basic approximation $\left\{\hat{\pi}_0^{(t)},\hat{f}^{(t)}(x)\right\}_{0<x<t}$ (Section 3.1.1). Examples of these time-t pdf's are plotted in Figs. 5, 6 and 7 respectively.

4. Renewal process $\{Z_{K,n}\}_{n=1,2,...}$ with right-truncated Pareto interarrivals

Let Z_K denote a right-truncated Pareto random variable, truncated at $K>0$, with the same shape parameter α as Z (formla (1)). We now consider the renewal process $\{Z_{K,n}\}_{n=1,2,...}$ with iid interarrivals $Z_{K,n}\underset{dis}{\equiv}Z_K$. Substituting from (1) we obtain the cdf, ccdf and pdf of Z_K, respectively, as

$$\left.\begin{aligned}
&B_K(x)=\tfrac{B(x)}{B(K)}=\tfrac{1-(1+x)^{-\alpha}}{1-(1+K)^{-\alpha}},\ x\in(0,K),\\
&\bar{B}_K(x)\ =1-B_K(x)=1-\tfrac{1-(1+x)^{-\alpha}}{1-(1+K)^{-\alpha}},\ x\in(0,K),\\
&b_K(x)=\tfrac{d}{dx}B_K(x)=\tfrac{\alpha(1+x)^{-\alpha-1}}{1-(1+K)^{-\alpha}},\ x\in(0,K).
\end{aligned}\right\}\qquad(9)$$

From $\bar{B}_K(x)$ in (9), for all $\alpha>0$, Z_K has a finite mean given by

$$E(Z_K)=\int_{x=0}^{K}\bar{B}_K(x)dx=\int_{x=0}^{K}\left(1-\frac{1-(1+x)^{-\alpha}}{1-(1+K)^{-\alpha}}\right)dx,$$

which evaluates to

$$E(Z_K)=\begin{cases}K-\frac{(-\alpha+1)K-(1+K)^{-\alpha+1}+1}{(-\alpha+1)\left(1-(1+K)^{-\alpha}\right)},\ \alpha>0,\alpha\neq1,\\(1+\frac{1}{K})\ln(1+K)-1,\ \alpha=1.\end{cases}\qquad(10)$$

Since $E(Z_K)<\infty$, the limiting pdf's of excess, age and total life exist in $\{Z_{K,n}\}_{n=1,2,...}$ as $t\to\infty$ (see Section 4.1).

4.1. Limiting pdf's of excess, age and total life of renewal process $\{Z_{K,n}\}_{n=1,2,...}$

Denote the *limiting* excess, age and total life by γ_K, δ_K and β_K respectively; with corresponding pdf's $f_{K,\gamma}(x)$, $0 < x < K$; $f_{K,\delta}(x)$, $0 < x < K$; $f_{K,\beta}(x)$, $0 < x < K$. Using the well known formulas for the limiting pdf's (e.g., Karlin and Taylor, 1975; Ross, 1970; Brill, 2008 - chapter 10.2), and substituting from (9), we obtain

$$f_{K,\gamma}(x) = \frac{1}{E(Z_K)}\bar{B}_K(x) = \frac{1}{E(Z_K)}\left(1 - \frac{1-(1+x)^{-\alpha}}{1-(1+K)^{-\alpha}}\right),\ 0 < x < K\ , \quad (11)$$

$$f_{K,\delta}(x) = \frac{1}{E(Z_K)}\bar{B}_K(x) = \frac{1}{E(Z_K)}\left(1 - \frac{1-(1+x)^{-\alpha}}{1-(1+K)^{-\alpha}}\right),\ 0 < x < K, \quad (12)$$

$$f_{K,\beta}(x) = \frac{1}{E(Z_K)}xb_K(x) = \frac{1}{E(Z_K)}x\left(\frac{\alpha\,(1+x)^{-\alpha-1}}{1-(1+K)^{-\alpha}}\right),\ 0 < x < K, \quad (13)$$

where $E(Z_K)$ is given in (10).

5. Comparison of time-t pdf's of $\{Z_n\}_{n=1,2,...}$ with limiting pdf's of $\{Z_{K,n}\}_{n=1,2,...}$

The limiting pdf's of $\{Z_{K,n}\}_{n=1,2,...}$ are given in (11), (12), and (13). The time-t pdf's of $\{Z_n\}_{n=1,2,...}$ are relatively tedious to compute, requiring the numerical solutions of Volterra integral equations (Section 3.2). Let ξ represent γ, δ, or β. Approximating $f_{\xi_t}(x)$, would be more efficient if we could determine a truncation point K_ξ of $b(x),\ x > 0$, such that the limiting pdf $f_{K,\xi}(x)$ closely approximates the pdf of $f_{\xi_t}(x)$ over a range of support of interest to the analyst,

Conjecture. If the time point of interest t of $\{Z_n\}_{n=1,2,...}$ is large (see Fig. 2), then there exists some truncation point $K_\xi \geq t$ of $b(x), x > 0$, such that the *limiting* pdf $f_{K,\xi}(x)$ in (11), (12) or (13) closely approximates $f_{\xi_t}(x)$ over subsets of support of $f_{\xi_t}(x)$.

For *any* $t > 0$, we compute K_ξ^* as the solution of K in the equation $f_{K,\xi}(0^+) = f_{\xi_t}(0^+)$ iff $1/E(Z_K) = f_{\xi_t}(0^+)$, for $\xi = \gamma$, δ (formulas (10) and (5)-(6). In the example of this paper, the shape parameter of $Pareto(x, \alpha)$ is $\alpha = 0.5$. For the total life pdf, we arbitrarily take K_β^* = 0.95 *quantile of* $b(x), x > 0$. For *large* $t > 0$, each limiting pdf $f_{K,\xi}(x)$, $\xi = \gamma$, δ, or β, closely approximates the corresponding time-t pdf over subsets of the positive real numbers. (Here "closely approximates" means the approximations are close in Figs. 8. 9, and 10). (Future work will consider metrics that quantify the qualitative notion "closely approximates").

In this paper, we utilize the following policy (Policy 1) for obtaining the truncation point K_ξ, $\xi = \gamma$, δ, β.

Policy 1

Choose K_ξ such that $f_{\xi_t}(0^+) = f_{K,\xi}(0^+)$, $\xi = \gamma$, δ.

Excess

From (5), $f_{\gamma_t}(0^+) = \alpha(1+t)^{-\alpha-1} + \alpha \int\limits_{y=0}^{t} (1+t-y)^{-\alpha-1}\frac{f^{(t)}(y)}{\pi_0^{(t)}}dy$. From (11), solve $\left(f_{\gamma_t}(0^+) = \frac{1}{E(Z_K)}, K\right)$, which gives $K_\gamma = K$. Compare $f_{\gamma_t}(x)$ with $f_{K,\gamma}(x)$, when t = 400, $\alpha = .5$, K_γ = 3877.5672, $E(Z_K)$ = 61.2781.

Age

From (6), $f_{\delta_t}(0^+) = \frac{f^{(t)}(t)}{\pi_0^{(t)}}$; $\pi_{\delta_t} = (1+t)^{-\alpha}$. From (12), solve $\left(f_{\delta_t}(0^+) = \frac{1}{E(Z_K)}, K\right)$, which gives $K_\delta = K$. Compare $f_{\delta_t}(x)$ with $f_{K,\delta}(x)$ when t = 400, $\alpha = .5$, K_δ = 3877.5672, $E(Z_{K_\delta})$ = 61.2781.

Total life

From (7) and (13), $f_{\beta_t}(0^+) = f_{K,\beta}(0^+) = 0$, which does not provide an equation for K_β. Therefore, we choose K_β = .95 quantile of $Pareto(x, \alpha)$ (.05 VaR), arbitrarily. When $\alpha = .5$, the .05 VaR of $Pareto(x, \alpha)$ is $x = 399$ (≈ 400). Compare $f_{\beta_t}(x)$ with $f_{K,\beta}(x)$, when t = 400, $\alpha = .5$, K_β = 3877.5672; $E(Z_{K_\beta})$ = 61.2781 (in this case $K_\beta = K_\delta = K_\gamma$). Compare $f_{\gamma_t}(x)$ with $f_{K,\gamma}(x)$ when t = 400; $\alpha = .5$, K_γ = 3877.5672, $E(Z_{K_\gamma})$ = 61.2781.

Future work will consider additional policies for obtaining the truncation points K_ξ, ξ = γ, δ, β .

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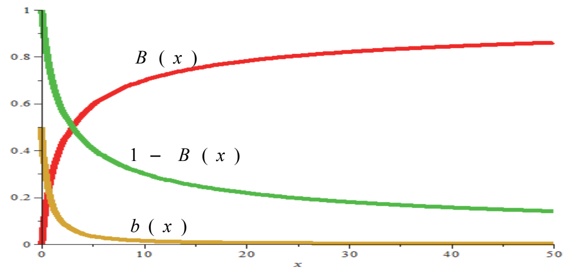

Figure 1: Pdf , cdf, ccdf of $Z \underset{dis}{=} Pareto(x, \alpha)$ when $\alpha = 0.5$

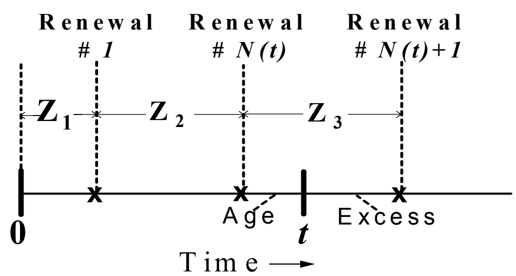

Figure 2: Renewal process $\{Z_n\}_{n=1,2,...}$. Total life = age + excess.

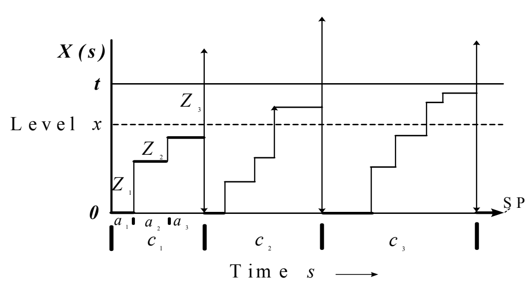

Figure 3: Sample path of regenrative process $\{X(s)\}_{s \geq 0}$, $t \in (0, \infty)$. $a_i \underset{dis}{=}$ exponential with mean 1. c_j = cycle of $\{X(s)\}_{s \geq 0}$. SP = system point (leading point of sample path).

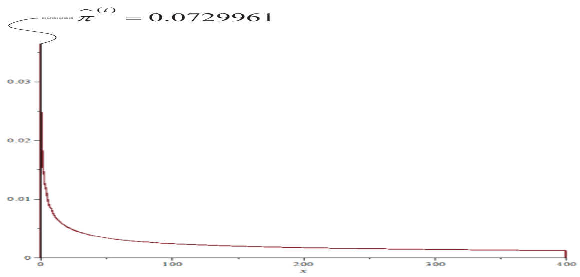

Figure 4: Approximation $\{\hat{\pi}_0^{(t)}, \hat{f}^{(t)}(x)\}_{x\in(0,t)}$ of solution to equation (3) for the regenrative process $\{X(s)\}_{s\geq 0}$, $\alpha = 0.5$, $t = 400$; $\hat{\pi}_0^{(t)} = 0.072996$

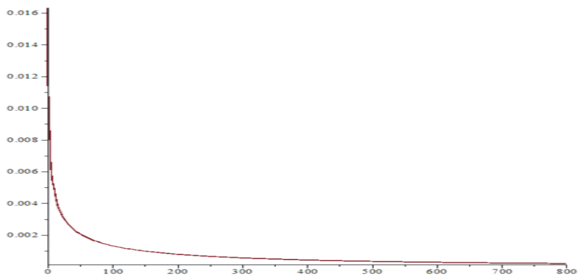

Figure 5: $\widehat{f}_{\gamma_t}(x)$, $x \in (0, 800)$, $\alpha = 0.5$, $t = 400$, $h = 0.1$, $N = 4000$.

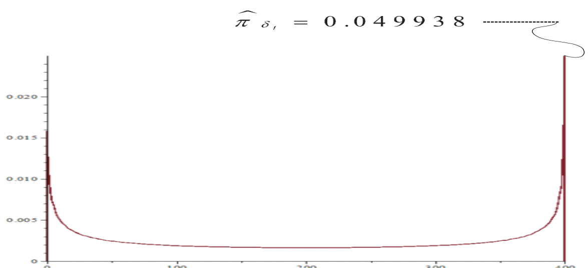

Figure 6: $\left\{\hat{\pi}_{\delta_t}, \widehat{f}_{\delta_t}(x)\right\}_{0<x<t}$, $\alpha = 0.5$, $t = 400$, $h = 0.1$, $N = 4000$. $\hat{\pi}_{\delta_t} = 0.049938$

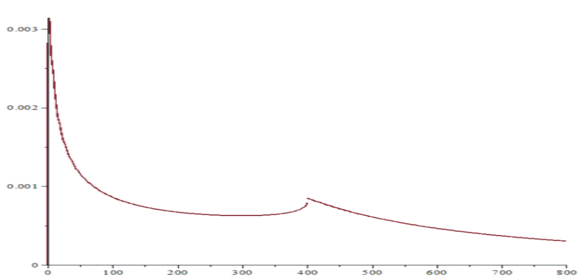

Figure 7: $\widehat{f}_{\beta_t}(x)$, $x \in (0, 800)$, $\alpha = 0.5$, $t = 400$, $h = 0.1$, $N = 4000$. Discontinuity at $t = +b(t) = \alpha(1+t)^{-\alpha-1}$

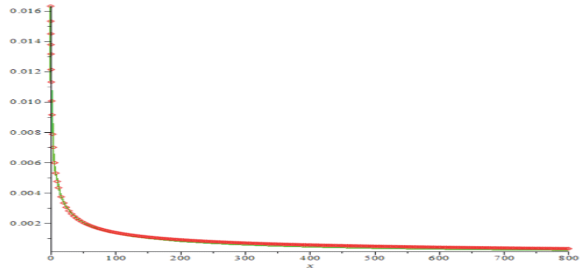

Figure 8: $f_{\gamma_t}(x)$ versus $f_{K,\gamma}(x), 0 < x < 800$. $\alpha = 0.5$, $t = 400. f_{\gamma_t}$: green line; $f_{K,\gamma}$: red dots

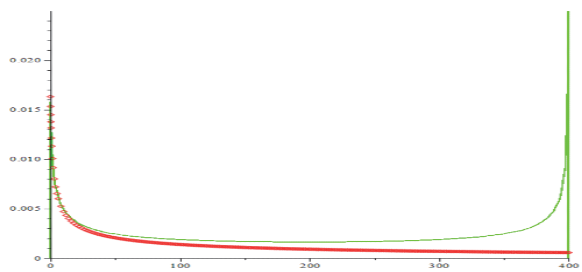

Figure 9: $f_{\delta_t}(x)$ versus $f_{K,\delta}(x), 0 < x < t$. $\alpha = 0.5$, $t = 400. f_{\delta_t}$: green line; $f_{K,\delta}$: red dots

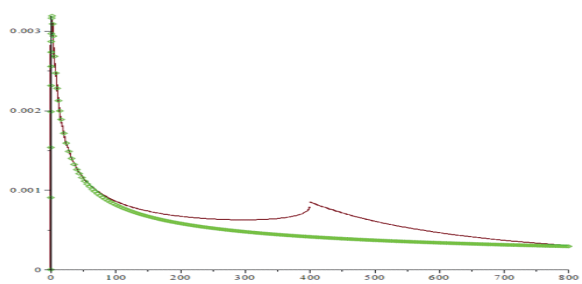

Figure 10: $f_{\beta_t}(x)$ versus $f_{K,\beta}(x)$, $0 < x < 800$. $\alpha = 0.5$, $t = 400. f_{\beta_t}$: black line; $f_{K,\beta}$: green line