Using Gini-based methodology to analyze time series

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Abstract

Most of the literature for analyzing time series measure dispersion using the variance. In this research we use an alternative but parallel framework for analyzing time-series: we use the Gini's Mean Difference (GMD) as an alternative index of variability. The Gini methodology is a rank-based methodology, which takes into account both the variate values and the ranks. It relies only on first order moment assumptions hence it is valid for a wider range of distributions. The GMD shares many properties with the variance, but can be more informative about the properties of distributions that depart from normality. We use one advantage of the Gini: there are two Gini-autocorrelation functions for each pair of variables, which are not necessarily equal. The difference between them, when it exists, can be informative and may assist to identify models with underlying heavy tailed and non-normal innovations. We suggest using Gini-correlograms, a simple graphical tool, to check the symmetry assumption which is natural in the existing methodology.

Key-words: Gini mean difference, Gini correlation, time series, autocorrelation

1. Introduction

Most of the literature dealing with time series measure dispersion and association using the variance and covariance, which are the most popular measures of variability and association. However, these analyses generally rely on assumptions, the validity of which is in question. For example, the autocovariance and autocorrelation, which are based on covariance and correlation of the variable against a time-shifted version of itself, assume symmetry in their variables as a consequence of the definition of covariance.
Recently an alternative approach based on Gini mean difference (GMD) as a measure of spread was introduced in Serfling (2010), Shelef and Schechtman (2011) and Shelef (2013) (for example, Gini-autocovariance, Gini-autocorrelation, to be defined below). These equivalent Gini-based definitions extend the concepts of Gini covariance and Gini correlation introduced by Schechtman and Yitzhaki (1987).

The GMD index shares many properties of the variance, but can be more informative for distributions that depart from normality or symmetry. The most prevalent presentation of the GMD is as the expected absolute difference between two independent and identically distributed (i.i.d) variables X_1 and X_2. Formally, the GMD of *X* is defined as (Gini, 1914)

$$G_X = E\left|X_1 - X_2\right|. \tag{1}$$

Another useful presentation is (Lerman & Yitzhaki, 1984)

$$G_X = 4COV\left(X, F_X(X)\right), \tag{2}$$

where $F_X(X)$ is the cumulative distribution function of *X*. That is, the GMD is (four times) the covariance between a random variable (*X*) and its cumulative distribution function $F_X(X)$.

The GMD takes into account both the values of the random variable and its ranks, and is hence less sensitive to extreme observations than the variance. In addition, because Gini requires only first-order moment assumptions (Stuart & Ord, 1987), the GMD-based method is valid for a wider range of distributions and might be more appropriate for heavy-tailed distributions than variance-based methods.

This paper further contributes by developing Gini-based partial autocorrelations to provide complementary information regarding a time series. Furthermore, we exploit the fact that there are two Gini autocovariances between each pair of variables, and the difference between them, if it exists, can be informative. This property enables a variety of applications that allow checking of some of the hidden assumptions imposed upon when using existing variance-based methods for time series analysis. Harnessing the Gini-based methodology provides an opportunity to identify and deal with cases where a symmetric correlation measure is inappropriate or conventional assumptions about the underlying distribution are not valid.

The structure of the paper is as follows. The next section (2) reviews developments in autoregressive moving average (ARMA) models for non normal distributions and reviews concepts and methods fundamental to the Gini methodology. Section 3 presents the Gini-based autocovariance, autocorrelation and partial autocorrelation. Section 4 illustrates that framework and presents its capabilities via simulations and Section 5 concludes.

2. Background

The standard approach to parameter estimation in AR models is through the Yule–Walker estimates (see, for example, Brockwell and Davis, 1991). Davis and Resnick (1986) establish a weak limit behavior for the sample autocorrelation function (ACF) with heavy-tailed innovations. Andrews, Calder, and Davis (2009) and Trindade, Zhu, and Andrews (2010) consider using maximum likelihood estimation for AR and ARMA processes and Andrews and Davis (2013) deal with AR process with infinite variance. Another method is the least absolute deviation (LAD) estimation, which is widely used for analyzing time series models in a non-Gaussian setting. LAD estimators for ARMA models were developed by Davis, Knight, and Liu (1992), Davis (1996), Davis and Dunsmuir (1997), Calder and Davis (1998), Ling (2005), Pan, Wang, and Yao (2007), Wu and Davis (2010) and others. It should be noted that although LAD requires only a first-order moment assumption, it does not yield closed-form expressions and the solution must be obtained numerically. Some researchers considered related rank-based estimation approaches for AR and ARMA models parameters (see, for example, Koul and Saleh (1993), Koul and Ossiander (1994), Terpstra and Rao (2001), Mukherjee and Bai (2002), Andrews, Davis, and Breidt (2007) and Andrews (2008)). These estimation procedures involve not only the residuals' ranks but also the residuals' values by relying on the *R*-regression suggested by Jaeckel (1972). A weighted quantile regression for AR models with infinite variance errors is suggested in Chen , Li, and Wu (2012). Applying LAD regression, quantile regression and *R*-regression, which are based on minimization of the between-group Gini, ignores some of the variability in the data (as shown in Yitzhaki and Lambert (2012)).

As can be seen from the above discussion, an extensive effort has been invested in developing time series models that are appropriate for distributions that depart from normality, particularly when modeling certain types of financial and engineering data. One of the main purposes of the methodology suggested in this research is to offer simple preliminary tools which can be used to identify the need to employ such models. Furthermore, we offer a parallel framework which enables the user to analyze such series under merely first-order moment assumptions.

Next, we briefly review part of the Gini-based methodology that is relevant for this paper. The interested reader is referred to Yitzhaki and Schechtman (2013) for a thorough review on the Gini methodology. The GMD forms two asymmetric correlation coefficients between two random variables (Schechtman & Yitzhaki, 1987). Let (X,Y) be two random variables. The two Gini covariances (*G*cov) between them are defined as

$$G\,\mathrm{cov}(Y,X) = COV\big(Y, F_X(X)\big) \text{ and } G\,\mathrm{cov}(X,Y) = COV\big(X, F_Y(Y)\big). \tag{3}$$

The (asymmetric) Gini correlation coefficients are defined as

$$Gcor(Y,X) = \frac{COV\big(Y, F_X(X)\big)}{COV\big(Y, F_Y(Y)\big)} \text{ and}$$

$$Gcor(X,Y) = \frac{COV\big(X, F_Y(Y)\big)}{COV\big(X, F_X(X)\big)}. \tag{4}$$

If (Y,X) has a bivariate normal distribution with (Pearson) correlation ρ, then $Gcor(Y,X) = Gcor(X,Y) = \rho$ (Schechtman & Yitzhaki, 1987). However, in general $Gcor(Y,X)$ and $Gcor(X,Y)$ are not necessarily equal, and even do not necessarily share the same sign. The equality $Gcor(Y,X) = Gcor(X,Y)$ holds if Y,X is exchangeable up to a linear transformation (Schechtman & Yitzhaki, 1987). By "exchangeable up to a linear transformation" it is meant that $(aY+b, cX+d)$ and (X,Y) are equally distributed for some constants *a*,*b*,*c* and *d* with *a* and $c>0$. We note in passing that symmetry is a consequence of the definition of covariance, i.e., $COV(X,Y) = COV(Y,X)$. Hence, the advantage of the Gini is that it enables one to check for symmetry. In time series the autocorrelation is an even function of Y_t and Y_{t-s}, and therefore provides no relevant information with respect to looking forward and backward in time. However, the above-defined Gini correlation is not necessarily symmetric in its arguments and therefore might provide additional information and offer a natural alternative for checking whether looking forward and backward in time makes a difference. For additional properties of the Gini correlation see Schechtman and Yitzhaki (1987, 1999), Yitzhaki (2003), Serfling and Xiao (2007), and Yitzhaki and Schechtman (2013).

The Gini covariances and correlations can be estimated using the sample covariances:

$$\hat{G}\,\mathrm{cov}(Y,X) = \mathrm{cov}\big(Y, R(X)\big), \text{ and}$$

$$\hat{G}cor(Y,X)=\frac{\operatorname{cov}(Y,R(X))}{\operatorname{cov}(Y,R(Y))}=\frac{\sum_{i=1}^{n}(Y_i-\bar{Y})(R(X_i)-\bar{R}(X))\Big/(n-1)}{\sum_{i=1}^{n}(Y_i-\bar{Y})(R(Y_i)-\bar{R}(Y))\Big/(n-1)}, \tag{5}$$

where $R(X_i)$ is the rank of X_i (divided by the sample size), $\bar{R}(X)=\sum_{i=1}^{n}R(X_i)\Big/n$, and $\bar{Y}=\sum_{i=1}^{n}Y_i\Big/n$. The estimator of $Gcor(Y,X)$ is a ratio of two *U*-statistics. Therefore, it is a consistent estimator of $Gcor(Y,X)$ and its distribution converges, for large samples, to the normal distribution (Hoeffding, 1948; Schechtman & Yitzhaki, 1987).

3. Gini-based time series definitions and modeling

3.1. Gini autocovariance

Let Y_t represent a general time series model at discrete time *t* ($t=0,\pm1,\pm2,\ldots$). The autocovariance between Y_t and Y_{t-s} is defined as $\gamma_{t,t-s}=COV(Y_t,Y_{t-s})$ for any lag *s* ($s=0,\pm1,\pm2,\ldots$). Following Serfling (2010), Shelef and Schechtman (2011) and Shelef (2013), define two Gini autocovariances of lag *s* as $\gamma^{G_1}_{(t,t-s)}=COV\left(Y_t,F(Y_{t-s})\right)$ and $\gamma^{G_2}_{(t,t-s)}=COV\left(Y_{t-s},F(Y_t)\right)$, which are parallel to equation (3). We will focus on the fact that the first and second Gini autocovariances can be viewed as the Gini autocovariances looking backward and forward. We assume that Y_t is strictly stationary, so that the joint distributions of $(Y_{t_1},\ldots,Y_{t_k})$ and $(Y_{t_1+s},\ldots,Y_{t_k+s})$ are the same for all positive integers *k* and for all $(t_1,\ldots,t_k)$, $s\in Z$ (Brockwell & Davis, 1991). Hence the conditions below hold for all t,s: $COV\left(Y_t,F(Y_{t-s})\right)=COV\left(Y_{t-j},F(Y_{t-j-s})\right)=\gamma^{G_1}_{(s)}$ and

$COV\left(Y_{t-s},F(Y_t)\right)=COV\left(Y_{t-j-s},F(Y_{t-j})\right)=\gamma^{G_2}_{(s)}$, where all $\gamma^{G_1}_{(s)}$'s and $\gamma^{G_2}_{(s)}$'s are time-independent.

Unlike the existing autocovariance which is symmetric in its variables, the Gini method does not impose that the two Gini autocovariances between Y_t and Y_{t-s} are equal. They will be equal under exchangeability up to a linear transformation (see Section 2 above). For example, consider the AR(1) model, $Y_t=\phi_0+\phi_1Y_{t-1}+\varepsilon_t$. Here,

$$\gamma^{G_1}_{(s)}=COV\left(\phi_0+\phi_1Y_{t-1}+\varepsilon_t,F(Y_{t-s})\right)=COV\left(\phi_1Y_{t-1},F(Y_{t-s})\right)=\phi_1^s\gamma^{G_1}_{(s=0)}, \tag{6}$$

but $\gamma^{G_2}_{(s)}=COV\left(Y_{t-s},F(\phi_0+\phi_1Y_{t-1}+\varepsilon_t)\right)$, which is not necessarily equal to $\gamma^{G_1}_{(s)}$.

3.2. Autocorrelation (ACF) and Gini autocorrelation function (GACF)

The autocorrelation function (ACF) between Y_t and Y_{t-s} is defined (under strict stationarity) as

$$ACF(Y_t,Y_{t-s})=\rho_s=\frac{\gamma_s}{\gamma_0}. \tag{7}$$

The most commonly used sample ACF is defined as

$$A\hat{C}F(Y_t, Y_{t-s}) = \hat{\rho}_s = \frac{\sum_{t=1}^{T-s}(Y_{t+s} - \bar{Y})(Y_t - \bar{Y})}{\sum_{t=1}^{T}(Y_t - \bar{Y})^2}, \tag{8}$$

where T is the length of the series and $\bar{Y} = \sum_{t=1}^{T} Y_t / T$ (Box, Jenkins, & Reinsel, 1994). The estimated autocorrelation function is often plotted versus the lag. This plot is called a correlogram and is used as a visual tool to identify a model for a set of data. A different version of ACF was suggested by Davis and Resnick (1985) for the case of heavy tailed data. Feigin and Resnick (1999) discuss the pitfalls of fitting an autoregressive model with heavy-tailed innovations by standard methods. They comment that caution should be taken when fitting models for heavy-tailed data where variances and even means may not exist.

Similar to equation (7) the two Gini autocorrelation functions (GACFs) of order s under strict stationarity are:

$$GACF(Y_t, Y_{t-s}) = \rho_{(s)}^{G_1} = \frac{\gamma_{(s)}^{G_1}}{\gamma_{(s=0)}^{G}} \text{ and } GACF(Y_{t-s}, Y_t) = \rho_{(s)}^{G_2} = \frac{\gamma_{(s)}^{G_2}}{\gamma_{(s=0)}^{G}}. \tag{9}$$

Following equation (6), $\rho_{(s)}^{G_1} = \phi_1^s = \rho_{(s)}$, which indicates that the first GACF is equal to the existing ACF. Therefore, the estimator for the first GACF (to be defined below) can be used to measure the autocorrelation when the second moment does not exist. Furthermore, the two GACFs (and Gini Partial ACFs, to be defined in Section 3.3) can be plotted to result in two Gini-based correlograms. Similarly to the common usage of correlograms to identify the order of ARMA processes, these Gini-based correlograms enable us to graphically check whether the structural symmetry assumption behind the existing ACF, i.e., $ACF(Y_t, Y_{t-s}) = ACF(Y_{t-s}, Y_t)$, is supported by the data or not. A difference between the two GACFs, if it exists, implies that an asymmetric measure such as the GACFs might be more appropriate and will offer more information about the underlying distribution. Note that if ε_t are multivariate normally distributed random variables, then both Y_t and Y_{t-s} are linear combinations of multivariate normal variables. Therefore they are exchangeable up to a linear transformation and hence $GACF(Y_t, Y_{t-s}) = GACF(Y_{t-s}, Y_t)$. On the other hand, if the GACFs differ, it indicates that Y_t and Y_{t-s} are not exchangeable and therefore looking forward and backward at the series is different.

In order to be consistent with the existing sample autocorrelation (equation (8)), we suggest the following GACFs estimates:

$$G\hat{A}CF(Y_t, Y_{t-s}) = \frac{\sum_{t=1}^{T-s}(Y_{t+s} - \bar{Y})(R(Y_t) - \bar{R}(Y_{1:(T-s)}))}{\sum_{t=1}^{T}(Y_t - \bar{Y})(R(Y_t) - \bar{R}(Y_{1:T}))} \tag{10}$$

and

$$G\hat{A}CF(Y_{t-s},Y_t)=\frac{\sum_{t=1}^{T-s}(Y_t-\bar{Y})(R(Y_{t+s})-\bar{R}(Y_{(s+1):T}))}{\sum_{t=1}^{T}(Y_t-\bar{Y})(R(Y_t)-\bar{R}(Y_{1:T}))}. \tag{11}$$

Similar to equation (8), here we use $\bar{Y}$ which is calculated over all observations as the estimator of the process mean. In addition, both denominator and numerator are divided by *T*, so that as s increases the numerator has fewer components and it converges to zero, as expected under strict stationarity. Note that there exist several alternative GACF estimates (see Shelef (2013) for details) and that the differences between the estimates decrease as the length of the sequence increases.

3.3. Gini partial autocorrelation function (Gini-PACF)

Following the Yule-Walker equation in the variance-based method, we have the following system of equations (see, for example, Brockwell and Davis (1991)):

$$\rho_{(j)}=\phi_{s1}\rho_{(j-1)}+...+\phi_{ss}\rho_{(j-s)}\text{, for all } j=1,2,...,s\,. \tag{12}$$

The partial autocorrelation is defined as the last coefficient ϕ_{ss}, which is the autocorrelation between Y_t and Y_{t-s} after adjusting for the effect of the intermediate variables $Y_{t-1},Y_{t-2},\ldots,Y_{t-s+1}$, namely: $\phi_{ss}=cor(Y_t,Y_{t-s}\,|\,Y_{t-1},\ldots,Y_{t-s+1})$.

Similarly, the Gini partial autocorrelation function (Gini-PACF) is $\phi_{ss}^{G_1}=Gcor(Y_t,Y_{t-s}\,|\,Y_{t-1},\ldots,Y_{t-s+1})$. The Gini-PACF is defined as the last coefficient of a partial Gini autoregression equation of order *s*, $Y_t=\phi_{s1}^{G_1}Y_{t-1}+\phi_{s2}^{G_1}Y_{t-2}+\ldots+\phi_{s(s-1)}^{G_1}Y_{t-s+1}+\phi_{ss}^{G_1}Y_{t-s}+\varepsilon_t$ (assuming, without loss of generality, that Y_t is a mean zero process). The Gini covariance between Y_t and Y_{t-j} is $G\operatorname{cov}(Y_t,Y_{t-j})=COV(Y_t,F(Y_{t-j}))=\gamma_{(j)}^{G_1}=\phi_{s1}^{G_1}\gamma_{(j-1)}^{G_1}+...+\phi_{ss}^{G_1}\gamma_{(j-s)}^{G_1}$. Hence,

$$\rho_{(j)}^{G_1}=\phi_{s1}^{G_1}\rho_{(j-1)}^{G_1}+...+\phi_{ss}^{G_1}\rho_{(j-s)}^{G_1}\,. \tag{13}$$

As can be seen from the equation above, each ACF ($\rho_{(s)}$) (in equation (12)) is replaced by the relevant first GACF ($\rho_{(s)}^{G_1}$). A natural alternative is to plug-in the second GACF ($\rho_{(s)}^{G_2}$) instead of the first GACF ($\rho_{(s)}^{G_1}$). As a result, an additional version of the Gini-PACF is formed, which we call the second Gini-PACF. The PACF can be estimated by approximating the Yule-Walker estimates of the successive AR processes using $\hat{\rho}_s$ as estimates of the theoretical autocorrelations (see Wei, 1993; Box et al., 1994). Accordingly, we suggest estimating the two Gini-PACFs using the following two systems of equations:

$$\hat{\rho}_{(j)}^{G_1}=\hat{\phi}_{s1}^{G_1}\hat{\rho}_{(j-1)}^{G_1}+\hat{\phi}_{s2}^{G_1}\hat{\rho}_{(j-2)}^{G_1}+...+\hat{\phi}_{s(s-1)}^{G_1}\hat{\rho}_{(j-s+1)}^{G_1}+\hat{\phi}_{ss}^{G_1}\hat{\rho}_{(j-s)}^{G_1} \tag{14}$$

and

$$\hat{\rho}_{(j)}^{G_2}=\hat{\phi}_{s1}^{G_2}\hat{\rho}_{(j-1)}^{G_2}+\hat{\phi}_{s2}^{G_2}\hat{\rho}_{(j-2)}^{G_2}+...+\hat{\phi}_{s(s-1)}^{G_2}\hat{\rho}_{(j-s+1)}^{G_2}+\hat{\phi}_{ss}^{G_2}\hat{\rho}_{(j-s)}^{G_2}, \tag{15}$$

which are to be solved for the two last coefficients $\hat{\phi}_{ss}^{G_1}$ and $\hat{\phi}_{ss}^{G_2}$ for $s=1,2,\ldots$.

4. Simulation results

This section illustrates the method by graphically examining Gini-correlograms. First, we use the AR(1) model, $Y_t = \phi_0 + \phi_1 Y_{t-1} + \varepsilon_t$, with independent and normally distributed innovations as a benchmark. Theoretically, the two GACFs and the existing ACF are equal. Therefore, we expect that in the simulation (due to random errors) their values will be close to one another. In addition, the two Gini regression coefficients should be close to the OLS regression coefficient. The parameters in our simulation are: $\phi_0 = 0$; $y_0 = 0$; $\phi_1 = 0.5$, 0.7 or 0.9 and T=100 and 200. For each set of parameters we generated R=10,000 original series and calculated the means and standard errors of the ACF, PACF, GACFs and Gini-PACFs for each lag. It is expected that because the model is AR(1), the ACF will decay to 0 and the PACF will cut-off after the first lag. Figure 1 presents the correlograms and the Gini-correlograms for the means of ACF, PACF, GACFs and Gini-PACFs for each lag, for $\phi_1 = 0.7$ and T=200. Detailed results for different values of ϕ_1 and T are given in Table 1 in the Appendix. From Figure 1 it can be seen that because the model is AR(1), the ACF decays to 0 and the PACF cuts-off after the first lag. Furthermore, as expected, the two GACFs are close to one another and to the existing ACF. The standard errors of the differences between the estimators are at most 0.002. (The standard errors are naturally very small because the number of replications is large (10,000)).

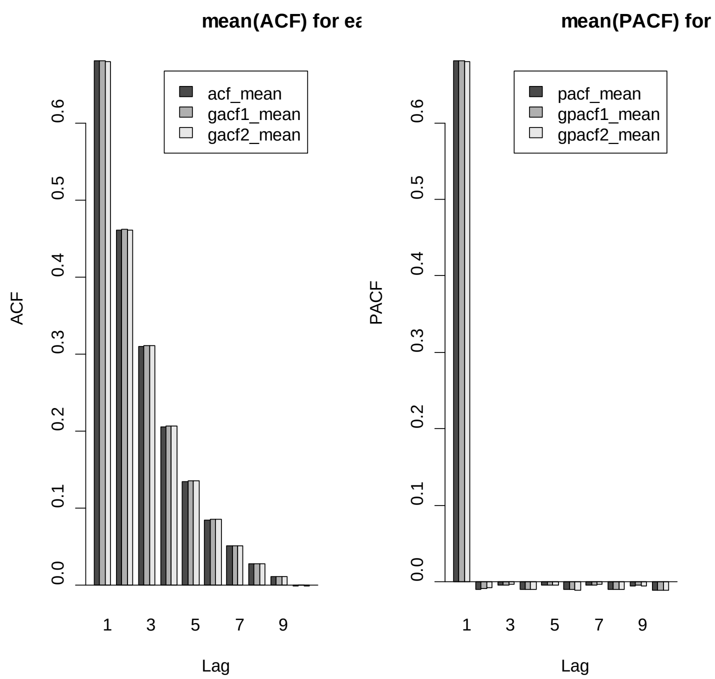

Figure 1. AR(1) model with normally distributed innovations, T=200, $\phi_1 = 0.7$ - correlograms and Gini-correlograms

Next, we concentrate on two non-normal distributions for the innovations: Pareto, which is asymmetric, and t(2) which is symmetric and heavy-tailed. Note that for these distributions only the first-order moment is finite. Hence, theoretically, OLS cannot be used but Gini regression can. In this simulation T=200, 500 and 1000; ε_t are i.i.d. innovations drawn from a centered Pareto distribution. The Pareto distribution used here is with a shape parameter of 1.5 and a scale of 1 (the resulting cdf is $F(x)=1-x^{-1.5}$). In order to get the centered distribution, we subtract the mean 3. Figure 2 presents the correlograms and the Gini-correlograms for the means of ACF, PACF, GACF and Gini-PACF for each lag, for T=500 and $\phi_1=0.7$. Detailed results for different values of ϕ_1 are given in Table 2 in the Appendix. Similar patterns occur for the different sample sizes, not presented here.

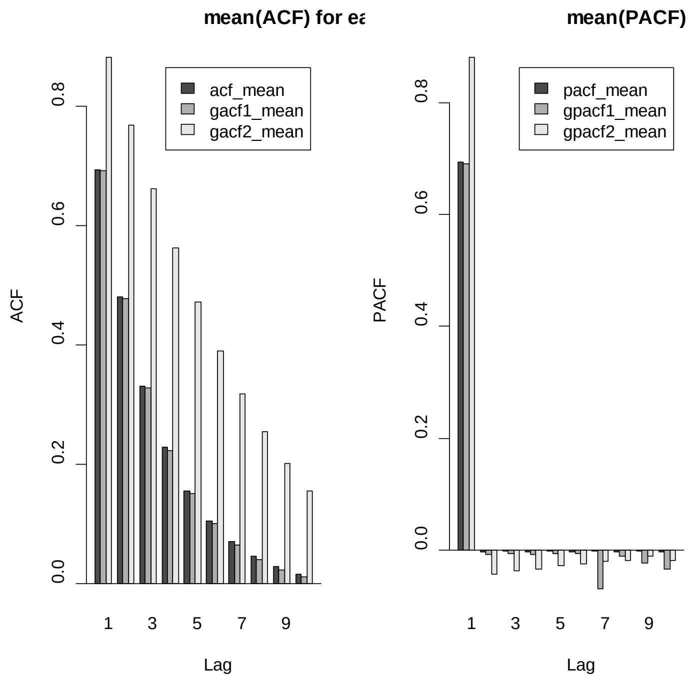

Figure 2. **AR(1) model with Pareto innovations, T=500, $\phi_1=0.7$ - correlograms and Gini-correlograms.**

Because the model is AR(1), the ACF should decay to 0 and the PACF should cut-off after the first lag. From Figure 2, we see that the ACF and PACF as well as both GACF and both Gini-PACF correlograms follow these patterns. However, the correlogram for $mean(\hat{\rho}^{G_2}_{(s)})$ decays slower than that of $mean(\hat{\rho}^{G_1}_{(s)})$ (even at lag=10, where the ACF and the first GACF are close to zero ($mean(\hat{\rho}^{G_1}_{(s=10)})=0.011$), the second GACF is still high ($mean(\hat{\rho}^{G_2}_{(s=10)})=0.156$)). The standard errors of the differences between the estimators are at most 0.002. The two GACFs are different for each lag, indicating the asymmetric nature of the autocorrelation. Similar patterns occur for different sample sizes (not presented here). The variance-based method imposes a symmetric covariance structure as a consequence of the definition of covariance. Therefore, the method assumes one ACF and one PACF for each lag. However, such a symmetric measure seems inadequate for the data set simulated here. Furthermore, the importance of this finding is that the difference between the two GACFs indicates the non-normality of the innovations (as opposed to the case of normally distributed innovations, where the GACFs were equal).

Simulation results for the AR(1) model with t(2), for T=500 and $\phi_1=0.7$ are given in Table 3 in the Appendix. As in the case of Pareto innovations, there appears to be a

difference between the Gini-correlograms which indicates the non-normality of the innovations. Similar patterns were observed with innovations from a Log-normal (0,1) distribution (not shown here).
The use of the two GACFs and the two Gini-PACFs is applicable also in MA(1) models, where $Y_t = \theta_0 + \varepsilon_t + \theta_1\varepsilon_{t-1}$, in that the Gini-correlograms can assist to identify that the MA(1) model has non-normally distributed innovations. In the case of MA(1) model, it is expected that the PACF will decay and the ACF will cut-off (see, for example, Brockwell and Davis (1991)). As an example, Figure 3 presents the correlograms and the Gini-correlograms for the means of ACF, PACF, GACFs and Gini-PACFs, for *T*=500 for Pareto innovations (detailed results are reported in Table 4 in the Appendix).

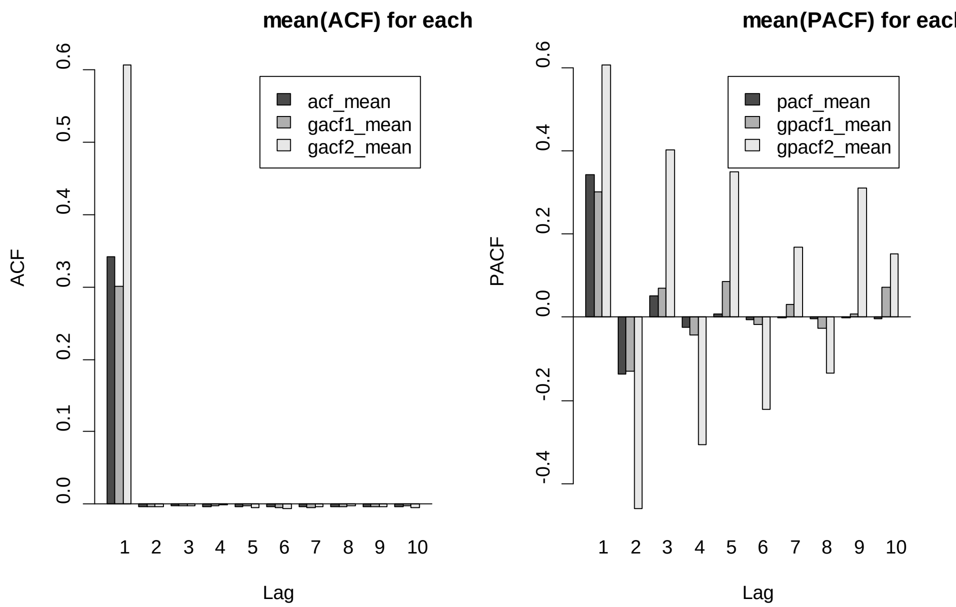

Figure 3. MA(1) model with Pareto innovations, *T*=500, $\theta_1 = 0.4$ - correlograms and Gini-correlograms.

As the figure shows, the simulation results verify that for MA(1) models with Pareto innovations the PACF (and two Gini-PACFs) decay and the ACF (and two GACFs) cut-off. However, the second Gini-PACF decays slower than the first one, indicating that the innovations are non-normally distributed. Similar patterns of large differences between the two GACFs and between the two Gini-PACFs when looking backward and forward were also observed in AR(2), MA(2) and ARMA(1,1) models (results are not shown here).

5. Discussion and conclusions

The proposed Gini-based framework makes several contributions to the field of time series analysis. First, the Gini-based measures are valid under only first-order moment assumptions. Therefore, the suggested framework establishes an approach which is valid regardless of the underlying distribution, whether the distribution is heavy-tailed or not, as well as if it has infinite variance or not, as long as it has a first-order moment. Second, the GMD takes into account both the values and the ranks of the variables, thereby reflecting sensitivity to the values themselves and, hence, no information is lost as occurs with other estimators which are based only on ranks. The GMD is a variability measure

which is between the variance (which is sensitive to extreme values) and measures that are based only on ranks (which ignore the variable values). Third, the Gini-based methodology provides a more informative approach because the two bi-directional (forward and backward) Gini autocorrelations between each pair of variables are not necessarily equal, as is the case by definition in the variance-based method. As a result, it offers a built-in capability to discriminate forward and backward directions.
Another contribution is that the suggested Gini-based framework facilitates Gini-based correlograms for examining assumptions hidden behind existing methodology. The common usage of correlograms and partial correlograms in time series analysis is to identify a model for the data. This paper further contributes to this field by offering a process of plotting and comparing two Gini-based correlograms and two Gini-based partial correlograms. The comparison equips the user with a simple graphical tool which enables checking whether the use of a symmetric correlation measure is adequate for the data set. Simulations were used to illustrate the added value which can be gained by applying the suggested Gini-based measures and Gini-correlograms for identifying departures from normality. The simulations show that both the two Gini autocorrelations and the two Gini partial autocorrelations are close to one another for AR(1) models with normally distributed innovations. When using AR(1) and MA(1) models with non-normally distributed innovations (Pareto, *t*(2) or Log-normal), however, the two Gini autocorrelations and the two Gini partial autocorrelations depart from one another. Such differences imply that the variance-based symmetric measures might be inadequate under non-normality of the innovations. Similar results were found also for ARMA processes of different orders.

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Appendix: Simulation results for quantitative comparison

Table 1 presents the means of $\hat{\rho}_{(s)}, \hat{\phi}_{ss}, \hat{\rho}_{G_1(s)}, \hat{\rho}_{G_2(s)}, \hat{\phi}_{ss}^{G_1}, \hat{\phi}_{ss}^{G_2}$, for AR(1) models with Normal innovations, where *R*=10,000, *T*=100 and 200; $\phi_1 = 0.5$, 0.7 and 0.9.

Table 1. Means of $\hat{\rho}_{(s)}, \hat{\phi}_{ss}, \hat{\rho}_{(s)}^{G_1}, \hat{\rho}_{(s)}^{G_2}, \hat{\phi}_{ss}^{G_1}$ and $\hat{\phi}_{ss}^{G_2}$ in AR(1) models with Normal innovations, *T*=100 and 200; $\phi_1 = 0.5$, 0.7 and 0.9.

T=100

$lag(s)$	$mean(\hat{\rho}_{(s)})$	$mean(\hat{\rho}_{(s)}^{G_1})$	$mean(\hat{\rho}_{(s)}^{G_2})$	$mean(\hat{\phi}_{ss})$	$mean(\hat{\phi}_{ss}^{G_1})$	$mean(\hat{\phi}_{ss}^{G_2})$
$\phi_1 = 0.5$						
1	0.4697	0.4710	0.4703	0.4697	0.4710	0.4703
2	0.2132	0.2144	0.2143	-0.0196	-0.0202	-0.0196
3	0.0875	0.0887	0.0884	-0.0120	-0.0111	-0.0116
4	0.0271	0.0282	0.0277	-0.0204	-0.0210	-0.0212
5	-0.0005	0.0002	<0.0001	-0.0097	-0.0092	-0.0091
6	-0.0136	-0.0131	-0.0131	-0.0197	-0.0204	-0.0204
7	-0.0216	-0.0214	-0.0211	-0.0118	-0.0116	-0.0112
8	-0.0249	-0.0249	-0.0247	-0.0199	-0.0206	-0.0208
9	-0.0265	-0.0265	-0.0257	-0.0114	-0.0109	-0.0100
10	-0.0277	-0.0277	-0.0271	-0.0210	-0.0216	-0.0220
$\phi_1 = 0.7$						
1	0.6608	0.6614	0.6599	0.6608	0.6614	0.6599
2	0.4311	0.4328	0.4314	-0.0203	-0.0193	-0.0183
3	0.2746	0.2770	0.2759	-0.0126	-0.0110	-0.0108
4	0.1691	0.1717	0.1707	-0.0204	-0.0209	-0.0209
5	0.0986	0.1010	0.1004	-0.0101	-0.0096	-0.0095
6	0.0508	0.0527	0.0525	-0.0206	-0.0219	-0.0214

$lag(s)$	$mean(\hat{\rho}_{(s)})$	$mean(\hat{\rho}_{(s)}^{G_1})$	$mean(\hat{\rho}_{(s)}^{G_2})$	$mean(\hat{\phi}_{ss})$	$mean(\hat{\phi}_{ss}^{G_1})$	$mean(\hat{\phi}_{ss}^{G_2})$
7	0.0173	0.0186	0.0187	-0.0125	-0.0120	-0.0118
8	-0.0051	-0.0041	-0.0040	-0.0203	-0.0211	-0.0213
9	-0.0202	-0.0195	-0.0189	-0.0121	-0.0115	-0.0108
10	-0.0306	-0.0300	-0.0291	-0.0214	-0.0221	-0.0222
$\phi_1 = 0.9$						
1	0.8477	0.8471	0.8436	0.8477	0.8471	0.8436
2	0.7158	0.7166	0.7112	-0.0226	-0.0154	-0.0132
3	0.6010	0.6035	0.5972	-0.0149	-0.0105	-0.0094
4	0.5018	0.5058	0.4992	-0.0222	-0.0207	-0.0199
5	0.4162	0.4214	0.4149	-0.0127	-0.0112	-0.0105
6	0.3419	0.3477	0.3418	-0.0231	-0.0235	-0.0227
7	0.2769	0.2831	0.2778	-0.0148	-0.0130	-0.0137
8	0.2208	0.2271	0.2226	-0.0218	-0.0226	-0.0217
9	0.1721	0.1786	0.1749	-0.0143	-0.0126	-0.0128
10	0.1300	0.1365	0.1335	-0.0225	-0.0237	-0.0236
T=200						
$\phi_1 = 0.5$						
1	0.4840	0.4847	0.4845	0.4840	0.4847	0.4845
2	0.2313	0.2320	0.2318	-0.0090	-0.0092	-0.0093
3	0.1062	0.1068	0.1067	-0.0058	-0.0056	-0.0055
4	0.0450	0.0453	0.0453	-0.0097	-0.0102	-0.0100
5	0.0151	0.0153	0.0154	-0.0047	-0.0043	-0.0042
6	0.0003	0.0003	0.0004	-0.0099	-0.0104	-0.0105
7	-0.0071	-0.0069	-0.0070	-0.0051	-0.0047	-0.0049
8	-0.0104	-0.0101	-0.0101	-0.0098	-0.0098	-0.0098
9	-0.0124	-0.0123	-0.0121	-0.0054	-0.0055	-0.0052
10	-0.0146	-0.0145	-0.0141	-0.0116	-0.0120	-0.0117
$\phi_1 = 0.7$						
1	0.6811	0.6812	0.6802	0.6811	0.6812	0.6802
2	0.4613	0.4621	0.4613	-0.0099	-0.0093	-0.0083
3	0.3100	0.3110	0.3106	-0.0048	-0.0044	-0.0038
4	0.2060	0.2070	0.2069	-0.0097	-0.0099	-0.0098
5	0.1343	0.1353	0.1354	-0.0049	-0.0046	-0.0045
6	0.0846	0.0856	0.0855	-0.0101	-0.0104	-0.0108
7	0.0506	0.0514	0.0514	-0.0043	-0.0041	-0.0038
8	0.0274	0.0280	0.0280	-0.0096	-0.0099	-0.0104
9	0.0109	0.0116	0.0114	-0.0056	-0.0049	-0.0053
10	-0.0009	-0.0003	-0.0006	-0.0108	-0.0114	-0.0111
$\phi_1 = 0.9$						
1	0.8749	0.8742	0.8722	0.8749	0.8742	0.8722
2	0.7642	0.7641	0.7610	-0.0104	-0.0061	-0.0046
3	0.6665	0.6672	0.6635	-0.0058	-0.0033	-0.0020
4	0.5801	0.5816	0.5777	-0.0108	-0.0101	-0.0095

$lag(s)$	$mean(\hat{\rho}_{(s)})$	$mean(\hat{\rho}^{G_1}_{(s)})$	$mean(\hat{\rho}^{G_2}_{(s)})$	$mean(\hat{\phi}_{ss})$	$mean(\hat{\phi}^{G_1}_{ss})$	$mean(\hat{\phi}^{G_2}_{ss})$
5	0.5038	0.5060	0.5021	-0.0052	-0.0040	-0.0037
6	0.4362	0.4391	0.4353	-0.0111	-0.0112	-0.0107
7	0.3766	0.3798	0.3763	-0.0058	-0.0055	-0.0051
8	0.3241	0.3277	0.3243	-0.0098	-0.0095	-0.0098
9	0.2777	0.2814	0.2783	-0.0052	-0.0054	-0.0050
10	0.2366	0.2404	0.2375	-0.0109	-0.0111	-0.0113

Table 2 presents the means of $\hat{\rho}_{(s)}, \hat{\phi}_{ss}, \hat{\rho}_{G_1(s)}, \hat{\rho}_{G_2(s)}, \hat{\phi}^{G_1}_{ss}, \hat{\phi}^{G_2}_{ss}$ for AR(1) models with Pareto innovations, *R*=10,000, *T*=500, $\phi_1 = 0.5$, 0.7 and 0.9.

Table 2. Means of $\hat{\rho}_{(s)}, \hat{\phi}_{ss}, \hat{\rho}^{G_1}_{(s)}, \hat{\rho}^{G_2}_{(s)}, \hat{\phi}^{G_1}_{ss}, \hat{\phi}^{G_2}_{ss}$ in AR(1) models with Pareto innovations, *T*=500, $\phi_1 = 0.5$, 0.7 and 0.9.

$lag(s)$	$mean(\hat{\rho}_{(s)})$	$mean(\hat{\rho}^{G_1}_{(s)})$	$mean(\hat{\rho}^{G_2}_{(s)})$	$mean(\hat{\phi}_{ss})$	$mean(\hat{\phi}^{G_1}_{ss})$	$mean(\hat{\phi}^{G_2}_{ss})$
$\phi_1 = 0.5$						
1	0.495	0.493	0.811	0.495	0.493	0.811
2	0.243	0.242	0.631	-0.003	-0.012	-0.084
3	0.118	0.116	0.469	-0.002	-0.012	-0.063
4	0.056	0.053	0.332	-0.003	-0.020	-0.044
5	0.024	0.022	0.223	-0.002	-0.103	-0.030
6	0.010	0.006	0.142	-0.002	-0.007	-0.013
7	0.002	-0.002	0.085	-0.002	-0.106	-0.009
8	-0.002	-0.006	0.047	-0.003	-0.219	-0.011
9	-0.004	-0.008	0.023	-0.002	0.016	0.021
10	-0.004	-0.009	0.010	-0.003	-0.005	0.014
$\phi_1 = 0.7$						
1	0.693	0.692	0.882	0.693	0.692	0.882
2	0.480	0.477	0.768	-0.004	-0.009	-0.044
3	0.331	0.328	0.661	-0.001	-0.006	-0.037
4	0.228	0.224	0.562	-0.003	-0.008	-0.035
5	0.156	0.151	0.471	-0.002	-0.006	-0.028
6	0.105	0.100	0.390	-0.003	-0.006	-0.025
7	0.070	0.065	0.318	-0.002	-0.070	-0.021
8	0.046	0.041	0.255	-0.003	-0.012	-0.018
9	0.029	0.023	0.201	-0.002	-0.024	-0.012
10	0.016	0.011	0.156	-0.004	-0.034	-0.019
$\phi_1 = 0.9$						
1	0.891	0.890	0.947	0.891	0.890	0.947
2	0.794	0.791	0.896	-0.004	-0.006	-0.013
3	0.708	0.703	0.847	-0.002	-0.002	-0.013
4	0.630	0.624	0.799	-0.004	-0.006	-0.013
5	0.560	0.553	0.753	-0.003	0.001	-0.012

$lag(s)$	$mean(\hat{\rho}_{(s)})$	$mean(\hat{\rho}_{(s)}^{G_1})$	$mean(\hat{\rho}_{(s)}^{G_2})$	$mean(\hat{\phi}_{ss})$	$mean(\hat{\phi}_{ss}^{G_1})$	$mean(\hat{\phi}_{ss}^{G_2})$
6	0.498	0.490	0.709	-0.003	-0.006	-0.011
7	0.442	0.433	0.666	-0.002	-0.001	-0.011
8	0.392	0.383	0.625	-0.004	-0.015	-0.012
9	0.347	0.337	0.585	-0.002	0.019	-0.011
10	0.307	0.297	0.547	-0.004	-0.040	-0.011

Table 3 presents the means of $\hat{\rho}_{(s)}, \hat{\phi}_{ss}, \hat{\rho}_{G_1(s)}, \hat{\rho}_{G_2(s)}, \hat{\phi}_{ss}^{G_1}, \hat{\phi}_{ss}^{G_2}$ for AR(1) models with *t*(2) innovations, *R*=10,000, *T*=500 and $\phi_1 = 0.7$.

Table 3. Means of $\hat{\rho}_{(s)}, \hat{\phi}_{ss}, \hat{\rho}_{(s)}^{G_1}, \hat{\rho}_{(s)}^{G_2}, \hat{\phi}_{ss}^{G_1}, \hat{\phi}_{ss}^{G_2}$ in AR(1) models with *t*(2) innovations, *T*=500 and $\phi_1 = 0.7$.

$lag(s)$	$mean(\hat{\rho}_{(s)})$	$mean(\hat{\rho}_{(s)}^{G_1})$	$mean(\hat{\rho}_{(s)}^{G_2})$	$mean(\hat{\phi}_{ss})$	$mean(\hat{\phi}_{ss}^{G_1})$	$mean(\hat{\phi}_{ss}^{G_2})$
1	0.692	0.692	0.788	0.692	0.692	0.788
2	0.478	0.478	0.612	-0.004	-0.006	-0.021
3	0.329	0.329	0.470	-0.002	-0.001	-0.015
4	0.226	0.226	0.355	-0.003	-0.007	-0.013
5	0.154	0.153	0.265	-0.002	-0.001	-0.009
6	0.104	0.103	0.195	-0.004	-0.008	-0.009
7	0.069	0.068	0.141	-0.002	-0.002	-0.005
8	0.044	0.043	0.100	-0.004	-0.012	-0.007
9	0.027	0.025	0.069	-0.002	-0.004	-0.003
10	0.015	0.013	0.046	-0.004	-0.006	-0.005

Table 4 presents the means of $\hat{\rho}_{(s)}, \hat{\phi}_{ss}, \hat{\rho}_{G_1(s)}, \hat{\rho}_{G_2(s)}, \hat{\phi}_{ss}^{G_1}, \hat{\phi}_{ss}^{G_2}$ for MA(1) models with Pareto innovations, *R*=10,000, *T*=500, $\theta_0 = 0$ and $\theta_1 = 0.4$.

Table 4. Means of $\hat{\rho}_{(s)}, \hat{\phi}_{ss}, \hat{\rho}_{(s)}^{G_1}, \hat{\rho}_{(s)}^{G_2}, \hat{\phi}_{ss}^{G_1}, \hat{\phi}_{ss}^{G_2}$ in MA(1) models with Pareto innovations, *T*=500, $\theta_0 = 0$ and $\theta_1 = 0.4$.

$lag(s)$	$mean(\hat{\rho}_{(s)})$	$mean(\hat{\rho}_{(s)}^{G_1})$	$mean(\hat{\rho}_{(s)}^{G_2})$	$mean(\hat{\phi}_{ss})$	$mean(\hat{\phi}_{ss}^{G_1})$	$mean(\hat{\phi}_{ss}^{G_2})$
1	0.342	0.300	0.607	0.342	0.300	0.607
2	-0.003	-0.004	-0.003	-0.137	-0.130	-0.461
3	-0.003	-0.002	-0.002	0.052	0.070	0.402
4	-0.003	-0.002	-0.001	-0.025	-0.043	-0.306
5	-0.003	-0.002	-0.005	0.007	0.086	0.350
6	-0.003	-0.005	-0.006	-0.007	-0.017	-0.222
7	-0.003	-0.004	-0.004	-0.001	0.030	0.169
8	-0.003	-0.004	-0.003	-0.004	-0.026	-0.134
9	-0.003	-0.004	-0.004	-0.002	0.009	0.312
10	-0.003	-0.002	-0.004	-0.003	0.071	0.152