On Quantile Regression for Extremes

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Abstract

Quantile regression has wide applications in many fields. Especially for extreme events, estimating of conditional quantiles at very high or low tails is an interesting and difficult problem. Quantile regression uses an L_1- loss function, and the optimal solution of linear programming for estimating coefficients of regression. This paper proposes a weighted quantile regression method on high quantile regression for certain extreme value sets. The Monte Carlo simulations show good results of the proposed weighted method. Comparisons of the proposed method and existing methods are given. The paper also investigates a real-world example of application on extreme events by using the proposed eighted method.

Keywords: *Bivariate Pareto distribution, conditional quantile, extreme value distribution, generalized Pareto distribution, linear programming, weighted loss function.*

AMS 2010 Subject Classifications: primary: 62G32; secondary: 62J05

1. Motivation and An Example

Extreme value events are present in many areas of real world, such as stock market, extreme weather, gas dispersion and so on, i.e. high winds, heavy rains and large fires cause damages to people or environment. The value, y, of an extreme event or its damage usually is heavy tailed distributed. And also y often related to another variable x. To estimate of high conditional quantiles of y for given x is an important task. In this paper we use quantile regression (Koenker, 2005) to estimate extreme value events. Quantile regression obtains the results which the tradition mean regression method can not provide.

The traditional mean regression is to estimate conditional mean response $\mu_{y|\mathbf{x}}$ for given variables $\mathbf{x} = (1, x_1, x_2, ...x_k)^T$. The model assume that

$$\mu_{y|\mathbf{x}} = E(y|x_1, x_2, ..., x_k) = \mathbf{x}^T\boldsymbol{\beta} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k. \quad (1.1)$$

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We estimate $\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_k)^T$,where $\boldsymbol{\beta} \in R^p$ and $p = k+1$. In general, A least-square (LS) estimator $\widehat{\boldsymbol{\beta}}_{LS}$ is a solution to the following problem

$$\widehat{\boldsymbol{\beta}}_{LS} = \arg \min_{\boldsymbol{\beta} \in R^p} \sum_{i=1}^{n} (y_i - \mathbf{x}_i^T \boldsymbol{\beta})^2. \qquad (1.2)$$

where $\widehat{\boldsymbol{\beta}}_{LS}$ is obtained by minimizing the L_2-squared distance, and $\widehat{\mu}_{LS} = \mathbf{x}^T \widehat{\boldsymbol{\beta}}_{LS}$.

Example. *Rainfall of Alberta (1999 - 2013) (Canada's National Climate Archives)*

Heavy rainfall will cause serious damages such as flash flood, severe erosion or landslide triggering which can intrigue a severe hazard to lives and property. The date were obtained from Canada's National Climate Archives. 5,297 recorded data in millimeter from 1999-2013. (Full data is available at *http://www.climate.weatheroffice.gc.ca/)*. For example, flooding swept over Southern Alberta in July 2013 caused historical water level to reach in Medicine Hat and downtown Calgary was emptied and under water and towns; like High River were completely evacuated for days, and thousands of people were obliged to leave their home. The Alberta flooding is one of the many climate change impacts already being felt around the world, but it also foreshadows the rise in the extreme events that is on the way.

We use 3 millimeter as threshold, since the precipitation under 3 millimeter is unable to cause extreme damage usually. There are $n = 548$ data left.

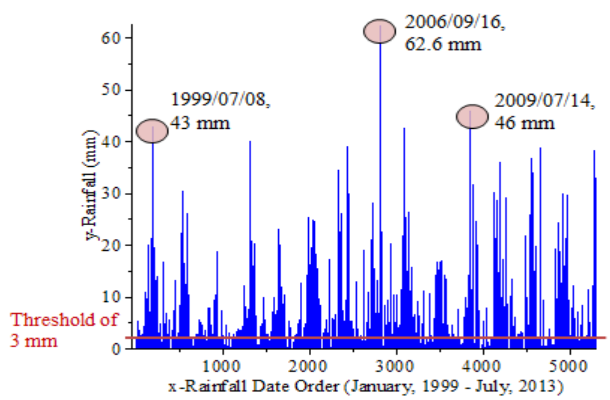

Figure 1. The rainfall (mm) from Alberta between January, 1999 and July, 2013

In Figure 1 x-axis is the precipitation in the order of occurrence, the y-axis is the total precipitation (mm). The average precipitation is $1.26mm$, while the maximum precipitation occurred in September 16^{th}, 2006, over $62.6mm$ raining fell to the ground.

The Scatter diagram in Figure 2 seems to be a quandratic tendency of precipitation with the change of maximum temperature and shows that when the temperature is too high or too low, it's unlikely to have heavy rainfall in Alberta. We can use least square method on a polynomial mean regression model

$$E(y|x, x^2) = \beta_0 + \beta_1 x + \beta_2 x^2, \qquad (1.3)$$

where y represents the precipitation (mm), and x represents the maximum temperature (^oC).

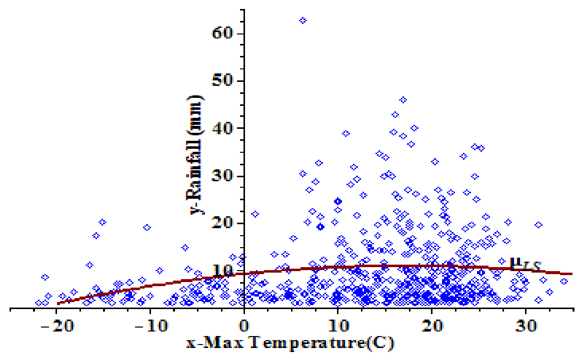

Figure 2. The Scatter diagram and least square mean regression μ_{LS} of Alberta rainfall related to the max temperature between January 1999 and July 2013.

The red curve in Figure 2 is the least square (LS) curve by using *(1.2)* to estimate the mean of the rainfall y for given the maximum temperature x. We have $\widehat{\boldsymbol{\beta}}_{LS}$ by minimizing $(\mathbf{y}-\mathbf{x}^T\widehat{\boldsymbol{\beta}})^T(\mathbf{y}-\mathbf{x}^T\widehat{\boldsymbol{\beta}})$, where $\mathbf{y}=(y_1,y_2,...,y_n)^T$, $\boldsymbol{\beta}=(\beta_0,\beta_1,\beta_2)^T$, $\mathbf{x}_i=(1,x_i,x_i^2)^T$, $i=1,...,n$. But this estimated mean curve only estimate the average rainfalls for given the maximum temperature. We are interested in extreme heavy rainfalls which may do damages. How do we estimate high quantile curve of rainfalls? i.e., estimate 95% conditional quantile curve. The quantile regression method will estimate the high conditional quantiles.

Quantile regression was first introduced by Koenker and Bassett in 1978 (Koenker and Bassett, 1978), but it didn't receive a proper attention until recent years. Quantile regression is an important subject of study and research, it has various applications in different fields (Koenker and Geling, 2001; Wang and Li, 2013). This method is capable of modeling conditional quantiles of a response distribution, which helps to reveal relationships between model variables that are hard to capture by a traditional mean regression. Section 2 introduces extreme value theorem and quantile regression. We proposed a weighted quantile regression method in Section 3. In Section 4, simulations by using both regular and weighted quantile regression methods are performed. In Section 5, we will use three regression methods: mean regression, regular and weighted quantile regression to Example 1.1. Finally, our conclusions are in Section 6.

2. Notation

2.1. Extreme Value Distribution

Let $X_1, X_2, ...$be independent continues random variables with common distribution function (c.d.f.) $F(x)$. We concern the limit behavior of the sample maxima or minima which is $\max(X_1, X_2, ..., X_n)$ or $\min(X_1, X_2, ..., X_n)$ as $n \to \infty$. The interest of extreme value theory (EVT) is in finding possible limit distributions of the sample maxima of independent and identically distributed random variables.

For every continuity point x, let G a nondegenerate distribution function that can occur as a limit are called *extreme value distribution* (Haan and Ferriera, 2006).

Theorem 1. (Fisher and Tipper,1928, Gnedenko, 1943) The class of extreme value distribution is $G_\gamma(ax+b)$ with $a>0$, b real, where

$$G_\gamma(x) = \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}), & 1+\gamma x > 0 \text{ and } \gamma \neq 0; \\ \exp(-e^{-x}), & \gamma = 0, \end{cases} \quad (2.1)$$

where the parameter γ is called the extreme value index.

The generalize Pareto distribution (GPD) is an example of extreme value distribution.

Definition 2.1. (Haan and Ferriera, 2006).The conditional distribution of $(X-t)/f(t)$, $f(t)$ is a positive nondecreasing function, given $X>t$ has the limit distribution as $t \uparrow x^*$,

$$H_\gamma(x) = \begin{cases} 1-(1+\gamma x)^{-1/\gamma} & \text{if } \gamma \neq 0, \\ 1-e^{-x} & \text{if } \gamma = 0, \end{cases} , where \begin{cases} x \geq 0 & if\ \gamma \geq 0, \\ 0 \leq x \leq -1/\gamma & if\ \gamma < 0. \end{cases} \quad (2.2)$$

This class of distribution function is called the class of *generalized Pareto distribution.*

Heavy-tailed distribution means having a density function that goes to zero less rapidly than an exponential function, that is heavy-tailed distributions whose tails are heavier than the exponential distributions. Extreme value distribution and the GPD n *(2.1)* and *(2.2)* when $\gamma > 0$ are heavy tailed distributions. Extreme value events need to use heavy-tailed distribution to build a good-fit model.

2.2. Quantile Regression

Definition 2.2. The τth quantile of a continuous random variable X with c.d.f. F is defined as

$$Q(\tau) = F^{-1}(\tau) = \inf\{x : F(x) \geq \tau\},\ \ 0<\tau<1, \quad (2.3)$$

where $F(x)$ is right continuous distribution function of variable X.

Definition 2.3. The τth conditional quantile regression of y for given $\mathbf{x} = (1, x_1, x_2, ..., x_k)^T$ is defined as

$$Q_y(\tau|\mathbf{x}) = Q_\tau(y|x_1,x_2,...,x_k) = F^{-1}(\tau|\mathbf{x}) = \mathbf{x}^T\boldsymbol{\beta} = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k,\ 0<\tau<1, \quad (2.4)$$

where $0<\tau<1$, $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, ..., \beta_k)^T$.

Koenker and Bassett (1978) proposed a L_1-loss function to obtain estimator $\hat{\boldsymbol{\beta}}(\tau)$ by solving

$$\hat{\boldsymbol{\beta}}(\tau) = \arg\min_{\boldsymbol{\beta} \in R^p} \sum_{i=1}^{n} \rho_\tau(y_i - \mathbf{x}_i^T \boldsymbol{\beta}(\tau)),\ 0<\tau<1, \quad (2.5)$$

where ρ is a loss function

$$\rho_\tau(u) = u(\tau - I(u<0)) = \begin{cases} u(\tau-1),\ u<0; \\ u\tau,\ u \geq 0. \end{cases} \quad (2.6)$$

Quantile regression problem can be formulated as a linear program problem

$$\min_{(\boldsymbol{\beta},\mathbf{u},\mathbf{v}) \in R^p \times R_+^{2n}} \{\tau \mathbf{1}_n^T \mathbf{u} + (1-\tau)1_n^T \mathbf{v} | \mathbf{1}_n X \boldsymbol{\beta}(\tau) + \mathbf{u} - \mathbf{v} = \mathbf{y}\}, \quad (2.7)$$

where X denotes the $n \times p$ design matrix (Koenker 2005).

3. The Proposed Weighted Quantile Regression

3.1. Weighted Quantile Regression

We propose a new weighted quantile regression (WQR) method

$$\hat{\boldsymbol{\beta}}_w(\tau) = \arg\min_{\boldsymbol{\beta}\in R^p} \sum_{i=1}^{n} w_i(x_i,\tau)\rho_\tau(y_i - \mathbf{x}_i^T\boldsymbol{\beta}),\ 0<\tau<1, \qquad (3.1)$$

where $w_i(\mathbf{x}_i,\tau) \in [0,1]$ and $\sum_{i=1}^{n} w_i(\mathbf{x}_i,\tau) = 1,\ i = 1,...,n,$ for $\mathbf{x}_i = (1, x_{i1}, x_{i2}, ..., x_{ik})^T$.

In this paper, we propose the following weights

$$(1)\quad w_i(\mathbf{x}_i,\tau) = \frac{||\mathbf{x}_i||}{\sum_{i=1}^{n} ||\mathbf{x}_i||}, \quad 0<\tau<1; \qquad (3.2)$$

$$(2)\quad w_i(\mathbf{x}_i,\tau) = \frac{1+||\mathbf{x}_i||^{-1}}{\sum_{i=1}^{n} (1+||\mathbf{x}_i||^{-1})}, \quad 0<\tau<1, \qquad (3.3)$$

where $||\mathbf{x}_i|| = \sqrt{x_{i1}^2 + x_{i2}^2 + \cdots + x_{ik}^2}$, k is the number of regressors.

Weighting scheme *(3.3)* is used in simulations in Section 4 to improve accuracy when estimating conditional quantiles of a bivariate Pareto distribution. Weighting scheme *(3.2)* also is used in example in Section 5. Many different weights were tested, and out of all tested weights, the weighting schemes *(3.2)* and *(3.3)* applied into *(3.1)* is the most accurate ones.

3.2. Comparison of quantile regression models

In order to compare the regular and weighted quantile regression models in *(2.5)* and *(3.1)*. We extend the idea of measure goodness of fit by Koenker and Machado (1999), and suggest to use a Relative $R(\tau)$ which is defined as

$$Relative\ R(\tau) = 1 - \frac{V_{weighted}(\tau)}{V_{regular}(\tau)}, \quad -1 \le R(\tau) \le 1, \qquad (3.4)$$

where

$$\begin{aligned} V_{regular}(\tau) &= \sum_{i=1}^{n} d_\tau(y_i,\hat{y}_i) \\ &= \sum_{y_i \ge \mathbf{x}_i^T\hat{\boldsymbol{\beta}}^{(\tau)}} \frac{\tau}{n}\left|y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}}(\tau)\right| + \sum_{y_i < \mathbf{x}_i^T\hat{\boldsymbol{\beta}}^{(\tau)}} \frac{(1-\tau)}{n}\left|y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}}(\tau)\right|, \qquad (3.5) \end{aligned}$$

where $\hat{\boldsymbol\beta}(\tau)$ is obtained by *(2.5)*.

$$\begin{aligned} V_{weighted}(\tau) &= \sum_{i=1}^{n} w_i d_\tau(y_i, \hat{y}_i) \\ &= \sum_{y_i \geq \mathbf{x}_i^T \hat{\boldsymbol\beta}^{(\tau)}} w_i \tau \left| y_i - \mathbf{x}_i^T \hat{\boldsymbol\beta}_w(\tau) \right| + \sum_{y_i < \mathbf{x}_i^T \hat{\boldsymbol\beta}^{(\tau)}} w_i (1-\tau) \left| y_i - \mathbf{x}_i^T \hat{\boldsymbol\beta}_w(\tau) \right|, \quad (3.6) \end{aligned}$$

where w_i can be given in *(3.2)*, $0 < w_i \leq 1$, $\sum_{i=1}^n w_i = 1$, $\hat{\boldsymbol\beta}_w(\tau)$ is obtained by *(3.1)*.

4. Simulations

In this Section, Monte Carlo simulations are performed. We generate m random samples with size n for each from the bivariate Pareto distribution in Figure 3 with c.d.f.

$$F(x,y) = 1 - x^{-\alpha} - y^{-\alpha} - (x+y+1)^{-\alpha},\ x > 1,\ y > 1,\ \alpha > 0. \quad (4.1)$$

and the τth conditional quantile function of y for given x is

$$Q_y(\tau|x) = 1 - x\left(1 - \frac{1}{(1-\tau)^{-1/(\alpha+1)}}\right), \quad 0 < \tau < 1. \quad (4.2)$$

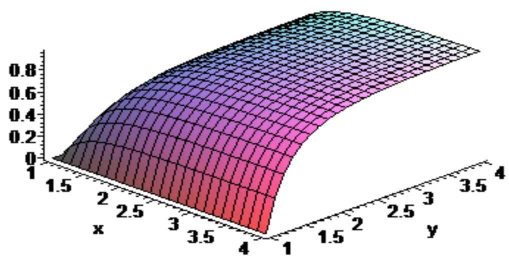

Figure 3. Bivariate Pareto distribution function with $\alpha = 3$.

Assume that the true conditional quantile is $Q_y(\tau|x) = \beta_0 + \beta_1 x$. We use two quantile regression methods:

1. The regular quantile regression (QR) Q_R estimation based on *(2.5)*,

$$Q_R(\tau|x) = \hat{\beta}_0(\tau) + \hat{\beta}_1(\tau)x \quad (4.3)$$

2. The weighted quantile regression (WQR) Q_W estimation based on *(3.1)*

$$Q_W(\tau|x) = \hat{\beta}_{w0}(\tau) + \hat{\beta}_{w1}(\tau)x \quad (4.4)$$

with weight and the weight $w_i = \frac{1+||x_i||^{-1}}{\sum_i (1+||x_i||^{-1})}$ in *(3.3)*.

For each method, we generate size $n = 200$, $m = 1,000$ samples. which can be estimated for each simulated sample. For ith sample, we have $Q_{R,i}(\tau|x)$ and $Q_{W,i}(\tau|x)$, $i = 1, ...m$.

We use $\alpha = 3$, we compare the two estimates: $Q_R(\tau|x)$ in *(4.3)* and $Q_W(\tau|x)$ in *(4.4)*, with the true quantile function in *(4.2)*. Figure 4 shows that $Q_W(\tau|x)$ is closer to the true conditional distribution function when $\tau = 0.95$, let's say Q_W behaves more efficient than Q_R.

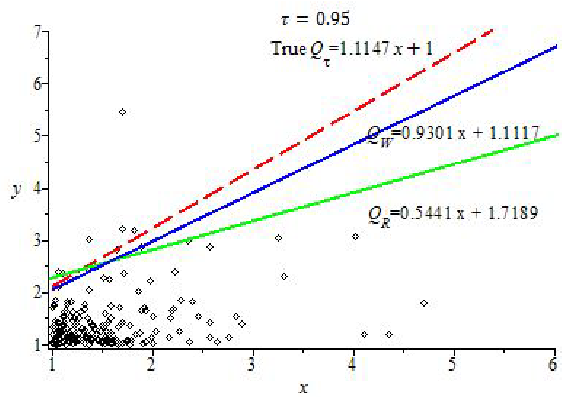

Figure 4. Simulation for $\tau = 0.95$. Q_R−green solid; Q_W−blue solid; ture Q_τ−red dash.

We design the simulation MSE (SMSE) of the estimators by *(4.3)* and *(4.4)*:

$$SMSE(Q_R(\tau|x)) = \frac{1}{m}\sum_{i=1}^{m}\int_{1}^{N}\left(Q_{R,i}(\tau|x) - Q_y(\tau|x)\right)^2 dx; \qquad (4.5)$$

$$SMSE(Q_W(\tau|x)) = \frac{1}{m}\sum_{i=1}^{m}\int_{1}^{N}\left(Q_{W,i}(\tau|x) - Q_y(\tau|x)\right)^2 dx, \qquad (4.6)$$

where the true τth conditional quantile of bivaraite Pareto distribution $Q_y(\tau|x)$ is in *(4.2)*, N is a finite x value such that the c.d.f in *(4.1)* $F(N, N) \approx 1$. The simulation efficiencies (SEFF) are given by

$$SEFF(Q_W(\tau|x)) = \frac{SMSE(Q_R(\tau|x))}{SMSE(Q_W(\tau|x))}. \qquad (4.7)$$

where $SMSE(Q_R(\tau|x))$ and $SMSE(Q_W(\tau|x))$ are given in *(4.5)* and *(4.6)*.

Table 1. shows the simulation SMSE-Q_R, SMSE-Q_W and SEFF for different τ values.

Table 1. Simulation Mean Squared Errors (SMSE) of Estimating $Q_y(\tau|\mathbf{x})$, $n = 200,\ m = 1000$

τ	0.95	0.96	0.97	0.98	0.99
SMSE-Q_R	22.64	27.37	60.05	104.06	228.00
SMSE-Q_W	20.76	26.80	42.87	99.39	234.74
SEFF-Q_W	1.09	1.02	1.40	1.05	0.97

Figure 5 shows most of the EFFs are larger than 1, so Q_W is more efficient than Q_R when $\tau = 0.95, 0.96$ and up to 0.99.

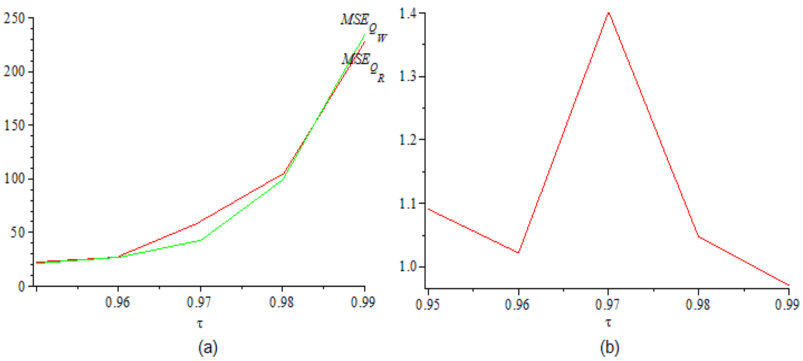

Figure 5. (a) SMSE of Q_R−red solid and Q_W−green solid; (b) Simulation efficiency of the Q_W relative to the Q_R.

From over all results of simulations, we conclude that:

(1) Figures 5(a) and 5(b) show that for almost every value of τ when using Q_W, with the proposed weight *(3.2)* are more efficient relative to the classical quantile estimator Q_R.

(2) Figure 4 presents the weighted 0.95th quantile regression Q_W is closer to the true 0.95th quantile Q, than the classical 0.95th quantile regression line Q_R, for $x > 2.5$.

5. Example of Applications

Quantile regression is used in many application areas: medical diagnosis, survival analysis, social science, environmental studies, hydrology, market return, forecasting. In this sction, we apply the following three regression models to Example 1.1 the Rainfall Example in Section 1:

1. The traditional mean linear regression (LS) $\widehat{\boldsymbol{\beta}}_{LS}$in *(1.2);*
2. The regular quantile regression Q_R estimator $\widehat{\boldsymbol{\beta}}(\tau)$ in *(2.5)*;
3. The proposed weighted quantile regression Q_W estimator $\widehat{\boldsymbol{\beta}}_w(\tau)$ in *(3.1)* with weight *(3.2).*

5.1. Goodness-of-Fit Tests

To fit the data to GPD in *(2.2)*, we transform the data $y^* = (y-\mu)/\sigma$, where $\mu = 3$, and obtained maximum likelihood estimator (MLE) $\widehat{\sigma}_{MLE} = 5.8061$ and $\widehat{\gamma}_{MLE} = 0.1781$. Figure 6 is a log-log plot. We can see that the data fit the GPD in *(2.2)* well.

We also perform three goodness-of-fit test including Kolmogorov test $(K-S)$ (Kolmogorov, 1933), Anderson-Darling test $(A-D)$ and Cramer-von-Mises test $(C-v-M)$ (see Anderson and Darling, 1952). The Hypothesis for all the three tests is

$$H_0 : F(x) = F^*(x), \text{for all value of x}$$
$$H_1 : F(x) \neq F^*(x), \text{for at least one value of x} \quad ,$$

where $F(x)$ is the true but unknown distribution function of the sample and $F^*(x)$ is the theoretical distribution function, GPD in *(2.2).*

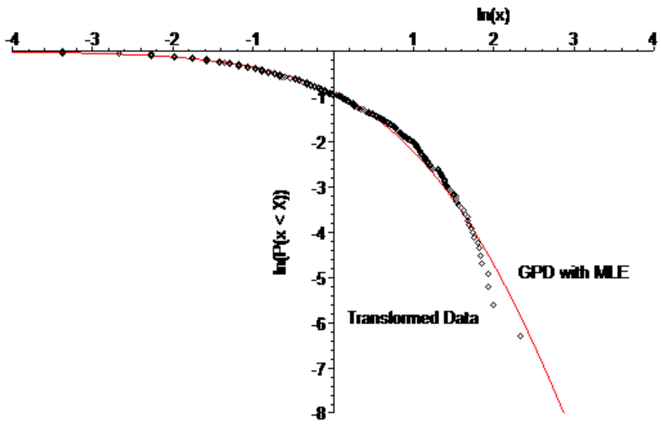

Figure 6. Log-log plot of Alberta Precipitation example, data are in dots, the estimated GPD is in red line.

Table 2. The p-value of all the three tests using MLE for Alberta Rainfall Example

$K-S$		$A-D$		$C-v-M$	
Test Statistics	p-value	Test Statistics	p-value	Test Statistics	p-value
0.04289	0.2507	1.1732	0.2780	0.1594	0.3623

Table 2 shows that, by $K-S$ test, our transformed data fit GPD with probability 25.1%, while by $A-D$ and $C-v-M$, the data fit GPD with probability 27.8% and 36.2% respectively.

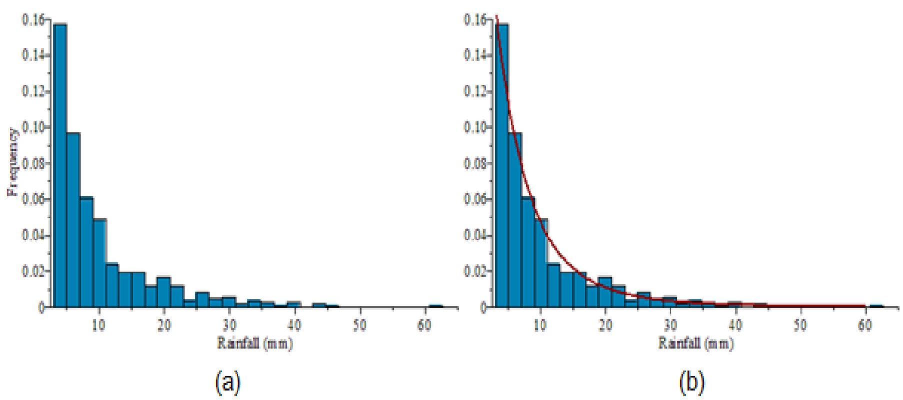

Figure 7. (a) Histogram of Rainfall in Alberta from January 1999 to July 2013. Frequency V.S Rainfall (mm); (b) Histogram and GPD model.

Figure 7(a) shows the frequency of precipitation. Most precipitations are between 0 and 20 millimeter, while there are still some days that have heavy precipitation. The data below 3 millimeter are removed and a new histogram is created. Figure 7(b) indicates that a GPD seems to model the Alberta Rainfall data.

5.2. Quantile regression method

Figure 8(a) shows there is a quadratic relationship between the maximum temperature and rainfall, so we use polynomial model

$$\widehat{Q}_y(\tau|x) = \widehat{\beta}_0 + \widehat{\beta}_1 x + \widehat{\beta}_2 x^2,$$

where y is rainfall and x is the maximum temperature.Then we have two regressor, one is x and the other is x^2, and we use the weigh in *(3.2)*, which is $||x_i|| / \sum_i ||x_i||$. Figure 8(a) gives the scatter plot, and the fitted μ_{LS}, Q_R and Q_W curves at two high 0.95th and 0.99th quantiles. We note that the Q_R and Q_W curves appears a decent fit of data at $\tau = 0.95$ and 0.99. And Q_W have a heavier rainfall than Q_R in general when quantile is high, these also show in Table 3.

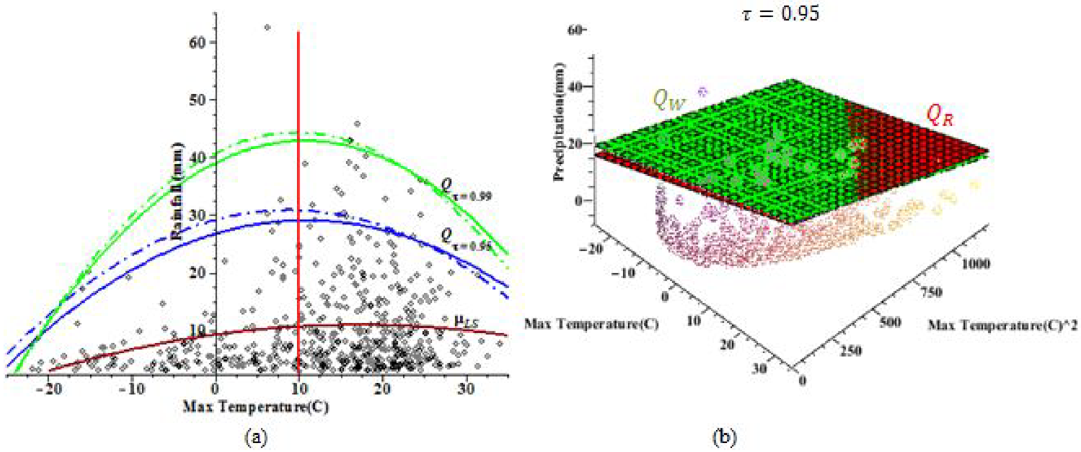

Figure 8. (a) Quantile Regression of Alberta Rainfall data at $\tau = 0.95$ in blue and $\tau = 0.99$ in green using Q_R−solid and Q_W−dash; μ_{LS}−red solid; (b) 3D view of Q_R−red; Q_W−green at $\tau = 0.95$.

Table 3 Alberta Rainfall (mm) at high quantile using Q_R and Q_W

	$\tau = 0.95$		$\tau = 0.99$	
Temperature (^oC)	Q_R	Q_W	Q_R	Q_W
−20	10.22	12.57	11.32	11.52
−15	15.82	18.36	20.75	21.60
−10	20.44	23.05	28.50	29.82
−5	24.06	26.63	34.58	36.20
0	26.70	29.11	38.99	40.73
5	28.35	30.50	41.73	43.41
10	29.01	30.77	42.80	44.24
15	28.68	29.95	42.20	43.23
20	27.36	28.02	39.93	40.36
25	25.05	24.99	35.99	35.65
30	21.76	20.86	30.38	29.08

5.3. Compare two estimation methods

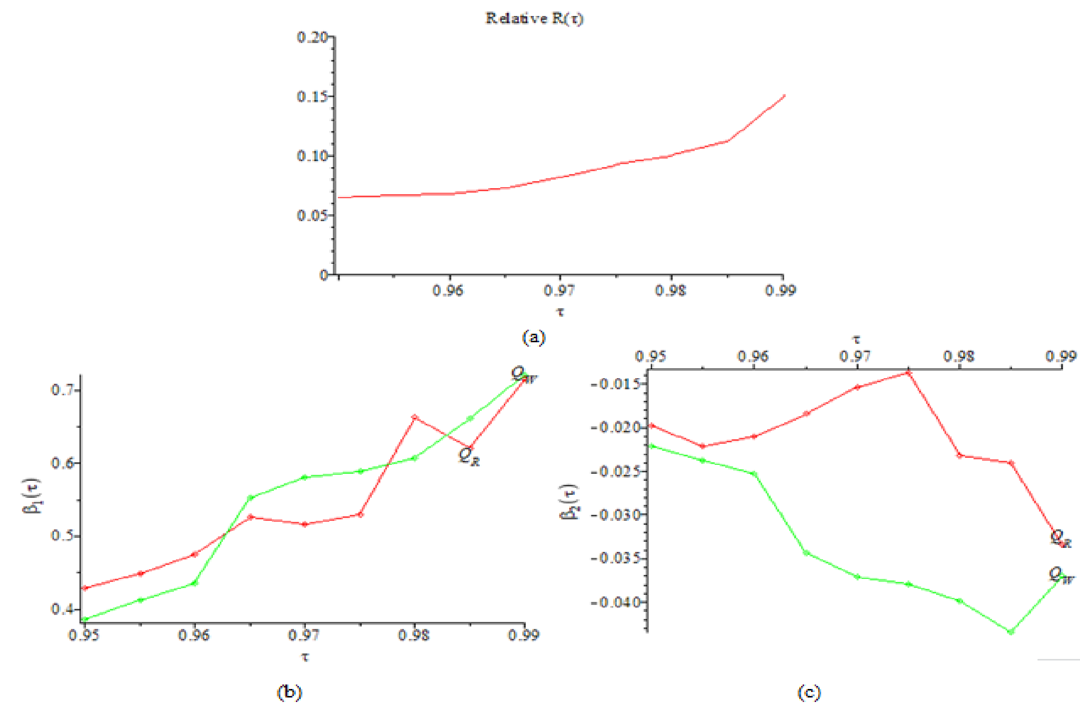

Figure 9. (a) Relative $R(\tau)$; (b) Comparison of $\beta_1(\tau)$; (c) Comparison of $\beta_2(\tau)$; Q_R−red solid and Q_W−green solid.

Figure 9(a) shows $R(\tau) > 0$, which means $V_{weighted}(\tau) < V_{regular}(\tau)$ when $\tau > 0.95$, thus the Q_W curve fits data better than the Q_R curve does when $\tau \geq 0.95$. Figures 9(b) and 9(c) show that the values of $\widehat{\beta}_1(\tau)$ and $\widehat{\beta}_1(\tau)$ which are consistent with Figure 8 and Table 3 which shows Q_W have a heavier rainfall than Q_R in general when quantile is high.

From the estimation of quantile regression, we concludes that when the temperature is moderate, it's more likely to have extreme rainfall. When the weather is too hot or too cold, the rainfall seems to be normal. Thus the government need to pay more attention to the rain forecast when the weather is moderate and maybe use the proposed weighted quantile regression to forecast for extreme heavy rainfall. Moreover, Quantile regression of precipitation, in general, can be useful to predict the rainfall and as a part of decision support systems, especially in the fields of regional planning and environment management.

6. Conclusions

Overall we conclude that:

(1) Least square estimator is for estimating conditional mean, by using L_2−loss function. But in many situation, we need to estimate high conditional quantile for extreme events. The traditional conditional mean regression models are unable to deal with heavy-tailed distributions,

since the measurement of central location can be significantly affected by outliers. Quantile regression proposed an efficient way to estimate high quantiles with a L_1- loss function, and overcomes the limitation of conditional mean models, especially the application in the extreme events. Quantile regression is a new powerful method with a lot of potential.

(2) In this paper, the proposed weighted quantile Regression Q_W with proposed weight scheme perform better goodness-of fit data than the regular Quantile regression Q_R in the computer simulations and the rainfall example. The results show that the Q_W behaves better than the Q_R.

(3) Search for optimal weight with better efficiency of the Q_W relative to the Q_R is a difficult task. Further study on optimal weight is suggested.

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