Using Statistical Moments to Improve the Control of Chaotic Oscillators

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Abstract
A general design strategy is presented using a statistical error-based controller structure to improve the synchronization of chaotic oscillators. Such oscillators arise in both biological and aerodynamic settings where drive tracking is important. Here fuzzy interval-based representations of nonlinear terms associated with the response oscillator are used to provide initial controller estimates that are then updated using a proposed error-based controller design generated from statistical moments of the uncoupled response dynamics. This scheme enhances convergence and reduces the relative error between the drive and response oscillator. Such an approach is particularly applicable to the problem of stabilizing unmanned aerial vehicles (UAVs) in group flight formations where the goal is to maintain a given positional flight configuration until override commands are provided to individual units.

Key Words: chaotic, oscillators, fuzzy interval, controller, synchronization

1. Overview

A fuzzy interval formalism is used to provide necessary and sufficient constraints for stabilizing an error-based controller structure. This current approach leads to a direct estimation scheme for the controller parameters, avoids direct linearization of the drive and response chaotic equations and provides insight about

- A screening algorithm or heuristic that selects the dominant extended system (DES) from a candidate population arising from the response model, (For a small candidate pool this statistical data pre-processing step is not important because all cases can be quickly tested.)
- The applicability of this approach to a broad class of Sprott systems, and
- The controller design strategy that may be applicable to functional equations and hybrid systems.

This research will demonstrate that an arbitrary nonlinear chaotic system can be conveniently decomposed into a linear interval polynomial with uncertain parameters and that statistics of the uncoupled response model can be used to synchronize these systems. Furthermore, this approach provides heuristic justification for why general synchronization systems are not reflective and suggests that a combinatoric approach that avoids direct linearization of nonlinear terms in the ordinary differential equations (odes) is advantageous. For traditional control systems, this method leads to a linear optimization problem for controller parameters. The present method eliminates this requirement.

Prior works by Morgan and Morgan (2011, 2012) on chaotic controls dealt primarily with structured uncertainties that arose in "complete" synchronized systems through coefficients of the equations or fluctuations introduced via controller gains. These statistical uncertainties ultimately manifested themselves as additions to either the diagonal (equation coefficients) or off-diagonal terms (introduced via controller structure) of the overall linear Jacobian control matrix that governs local stabilization.

The impact of stochastic noise introduced via these terms was of primary concern. This was especially important given that the overriding objective was to devise a strategy for the design of reliable control systems for chaotic oscillators (a network) that are robust to such factors. Also, providing a mathematical framework for understanding the impact of these quantities on the performance of large integrated networks was likewise a major goal. The initial phases of the studies were limited to analyzing complete (dual) chaotic system behavior with respect to three control issues:

- controller numbers and control variable selection,
- error propagation patterns arising via either initial condition disturbances or via uncertainties in model parameters, and the
- effectiveness of unit redundancy as a tool for enhancing overall system robustness and reliability.

Generally dynamical states are said to be synchronized if the distance metric between them converges to zero over time. Such systems are classified on the basis of their topological similarities as summarized by four broad types: general synchronization, phase synchronization, lag synchronization, and complete synchronization. General synchronization involves coupling different systems and results in a static functional relationship between the two systems. Phase synchronization, on the other hand, involves locking the phases of two systems, but not their magnitudes. In contrast, lag synchronization links the magnitude but produces a time delay between signals such that one follows or anticipates the other. Complete synchronization connects mathematically equivalent systems that are offset by initial conditions. Regardless of their type, synchronized systems exhibit asymptotical stability where small induced perturbations in these systems damp out rapidly and stabilization is quickly restored.

Pecora and Carroll (1991) were the first to show that a proper decomposition of a system into a drive and response subsystem can lead to successful synchronization and they devised a general framework for stable drive selection. Itoh and Murakami (1994) applied this technique to couple discrete-time dynamic systems where parameter estimation is required and found that synchronization was not successful if the number of subsystems were large. Both of these prior approaches required an explicit or estimated knowledge of the system’s state variables. However, in practical situations, these parameters may change over time and, in such cases, an adaptive control strategy is particularly attractive. Zeng and Singh (1997) developed such a controller using the Lyapunov stability theory to select appropriate state control variables. Only a single state variable was used to accomplish the control that employed an equation that did not contain any singularities.

Jiang and Tang (2002) on the other hand developed a novel global chaotic synchronization criterion employing a Lyapunov matrix inequality for choosing the suitable coupling parameters. Sun and Zhang (2004) also developed some simple but generic criteria for the global synchronization of two coupled general time-varying chaotic systems with a unidirectional linear error feedback coupling, along with a simple configuration for the corresponding implementation. Phase synchronization with unidirectional coupling was studied in Guan, Lai, and Lai (2006), where three different chaotic oscillators were considered. It was shown that phase synchronization could be achieved even when there was a large difference between the natural frequencies of the coupled oscillators. The relation between phase and generalized synchronization was analyzed using the conditional Lyapunov exponent. Huang (2006) used a simple adaptive coupling to explore the collective synchronization of weighted networks and showed that under such a control scheme no upper bound existed on the number of network units.

2. Approach and Preliminary Findings

This new method is based on the premise that an interval approximation model (crisp fuzzy system) representation of the system of ordinary differential equations for the (response) oscillator can be used to determine arbitrary 'controller parameters.' The initial step in this process involves constructing an interval approximation for each nonlinear and/or positive linear term in our chaotic model from descriptive statistics of the unsynchronized response system. Thus armed with these interval approximations the (local) stability of each term in the model is assessed and used to determine the global stability requirement for a given drive-response combination. Under this paradigm, the stability of the drive system is not necessary for establishing overall system synchronization only the response is needed.

Specifically, we plan to use fuzzy number α-cuts to more precisely narrow the interval estimates for this current study. This approach will generate a membership function for each nonlinear term in our response model. Here it is assumed that the membership function mirrors the distributional patterns. This mathematical operation has the net effect of converting a fuzzy system with uncertainties into an interval model that is more tractable. The success of the present exploratory inquiry does however justify the merits of this approach.

Figure 1 shows the original uncoupled phase portraits of several dissimilar chaotic oscillators prior to any synchronization. The drive for this system is the Halvorsen's chaotic oscillator that is linked dynamically to four distinct response oscillators (Lorenz, Rossler, Moore-Spiegel and Chen) with different controller stability requirements. Figure 2 highlights the desired state of a uniform synchronization under a minimum controller gain level that results in a high level of fidelity between drive and response systems.

Figure 1: Uncoupled Connectivity for Dissimilar State Models

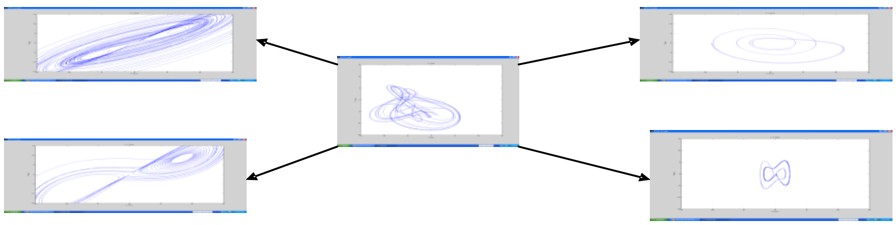
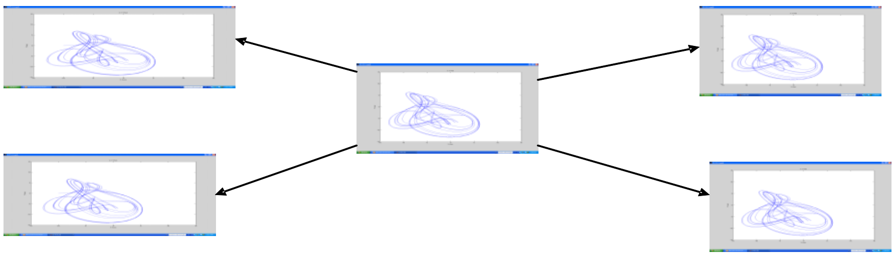


Figure 2: Uniform Connectivity for Dissimilar State Models



In this paper a general error-based controller procedure is devised for an interval (fuzzy) system. Unlike our prior work in Morgan and Morgan (2012) that incorporated a design procedure suggested by Bhiwani and Patre (2011) for a classical proportional-integral-derivative (PID) controller, the current approach eliminates a cumbersome optimization step encountered with the former. Two constraints imposed in the present design are that all error-based moment gains must be positive and all generated errors are bounded by the initial (zero) state difference between the original drive and response systems. Three design cases are possible with this approach: under specified, uniquely specified and over specified. Here only the uniquely specified case, where the number of controller parameters matches the number of state equations, is considered.

The test model for this analysis is the same Halvorsen-Lorenz system examined in Morgan and Morgan (2012) where the synchronization objective was to make the Lorenz system track the Halvorsen within a given level of accuracy. The basic design philosophy is summarized in Table 1 below.

Table 1. Basic Controller Design Philosophy				
Step 1	Step 2	Step 3	Step 4	Step 5
Original Ordinary Differential Equation	Extended System	Dominant Extended System	Interval Polynomial	Controller Parameter Estimation

This approach removes local nonlinearities via construction of fuzzy intervals for each nonlinear term appearing in the response system equation. Thus a system of ordinary differential equations are converted into a system of linear interval equations that can be used to estimate error-based moment controller gains and yields a gain-error characteristic polynomial that addresses local and global system stability (see Table 2).

Table 2. Extended Model of Lorenz System

Original Model	Extended/Fuzzy Model
dx/dt= 10(y-x)	$dx/dt=[10-\sum k_{xi}\, \varepsilon_{xi}^{\;i-1}]y-10x$
dy/dt= -xz +28x-y	$dy/dt=[28-\sum k_{yi}\, \varepsilon_{yi}^{\;i-1}]-z_S]x-y$
dz/dt= xy-2.67z	$dz/dt= [\, y_S-\sum k_{zi}\, \varepsilon_{zi}^{\;i-1}]x-2.67z$
The gain-error characteristic equation for the Lorenz system takes the following form $p(\varepsilon) = (k_1 - C_{max}) + k_2\ \varepsilon + k_3\, \varepsilon^2$	

The C_{max} term appearing in the gain-error characteristic equation of Table 2 is the maximum value observed among the bracketed terms of the extended fuzzy model. Two distinct solutions are possible based upon the sign of the discriminant associated with the gain-error characteristic polynomial. The sign of the discriminant also dictates the type of image produced. A negative value of this quantity generates overlapped images while a positive one produces displaced images. It was also observed that the k_1 gain controlled the error level between the synchronized states (Figure 3) as reported in Morgan and Morgan (2012) and that the regression model (Figure 4) developed in that study was valid for the current investigation. There the relationship between controller gain and the correlation coefficient revealed the presence of two distinct zones (unstable and stable regions) separated by a critical gain value. That single regression model was adequate for describing the general dependency of synchronization fidelity to controller gain for a wide class of chaotic oscillators. The size of the instability region was found to be bounded by the length of the maximum fuzzy interval while the minimum fuzzy interval enclosed the un-entangled (critical) point.

The boxplots in Figure 3 shows the decrease in synchronization error with k_1 gain. The length of the respective outlier regions of the individual boxplots are related to the total time or iteration steps needed to synchronized system behavior.

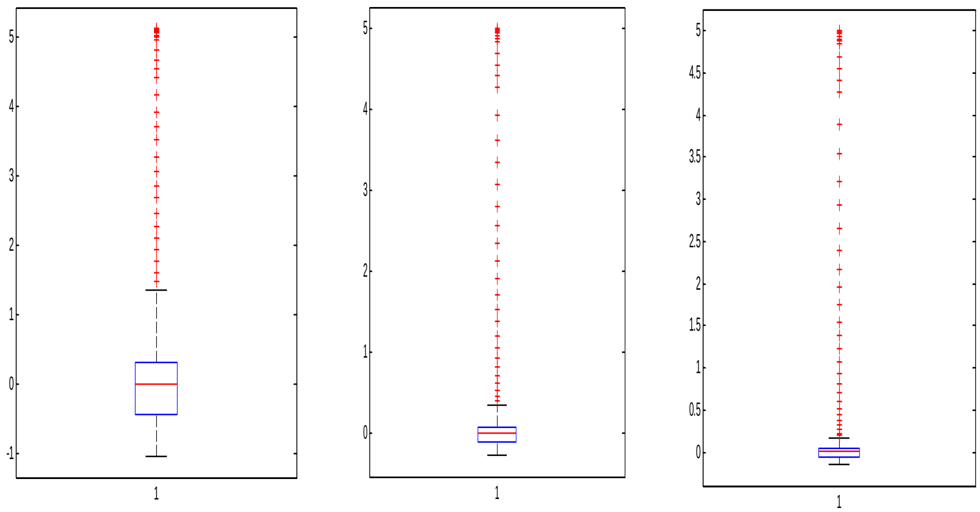


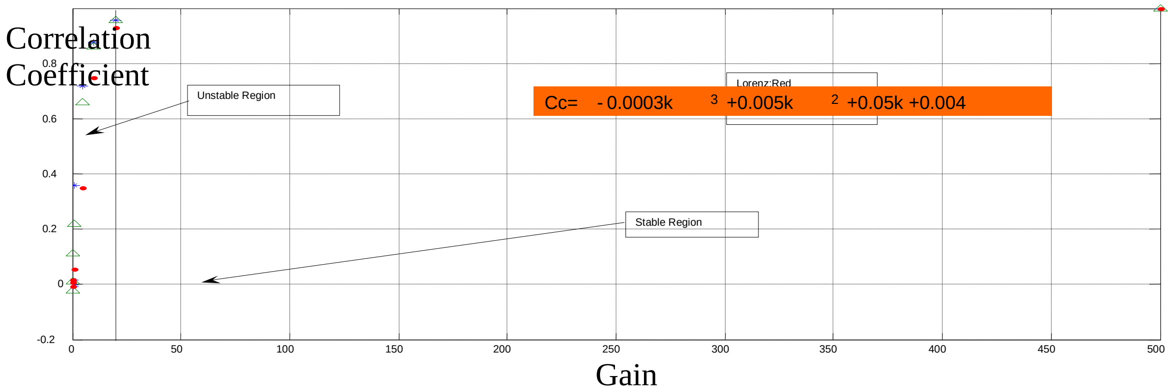
Figure 3. Boxplots of Response System Error with k_1 Gain

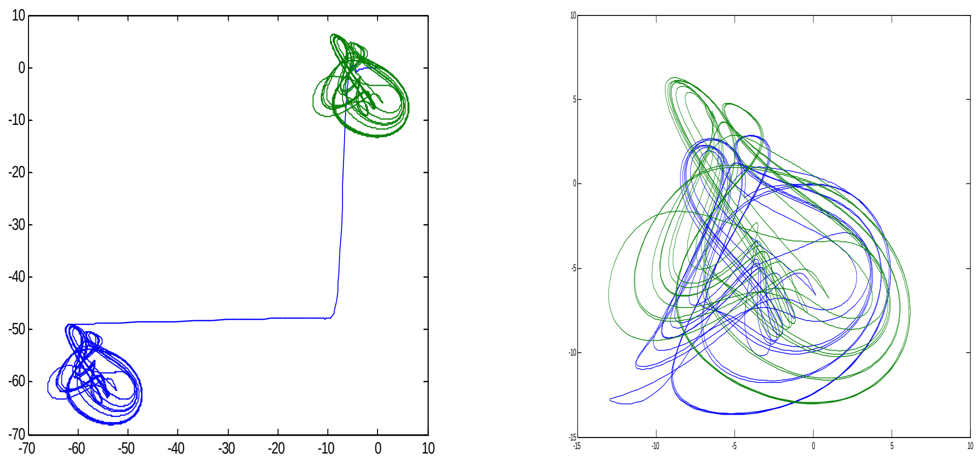
Figures 5 and 6 also capture these aspects of the data. The other two gains (k_2 and k_3) only affected convergence rates and image displacements. Table 3 summarizes the dependency of convergence rates and synchronization errors with k_2 and k_3.

Table 3. Effect of Gains on the Correlation Coefficient and Number of Iterations

Gains	k_1 = 91	k_1 = 154
k_2 = 8	0.9949 (30,317)	0.9982 (43,981)
k_2 = 15	0.9291 (165,353)	0.9982 (44,449)
k_2 = 25	0.9417 (547,145)	0.9031 (501,101)

Figure 4. Correlation Coefficient versus Controller





a. Displaced Images

b. Overlapped Images

Figure 5. Images Produced by Discriminant Effect

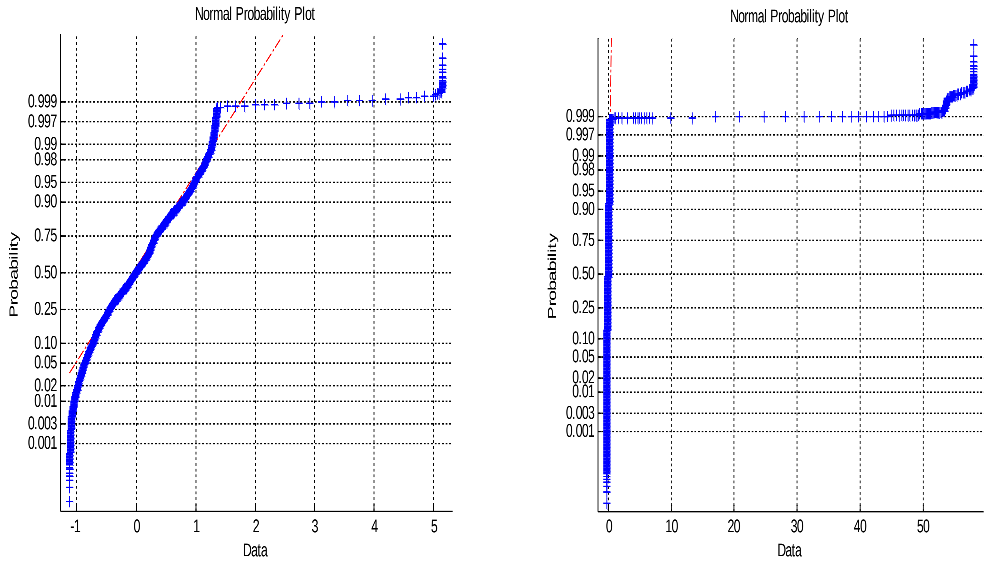


Figure 6. Normal Probability Plot of Discriminant Produced Images

3. Conclusions

An error-based controller design method is shown to be successful for synchronizing an arbitrary chaotic system. This method uses interval estimates obtained from boxplots of the response state variables to formulate an interval representation of the original response system. For the case where the controller parameters and state variables are matched, a system of linear constraint equations are directly solvable for the controller gains. A unique characteristic equation provides stability requirements for the controller gains that can produce two uniquely different solutions that depend upon the sign of the discriminant embedded in that polynomial. A simple regression model devised for predicting the effect of proportional control gain on synchronization fidelity in Morgan and Morgan (2012) also predicts the limiting case behavior for our error-based controller design.

Chaotic systems were studied here because they represent highly complex systems that are inherently very difficult to synchronize. Thus, being able to devise an approach that is robust for such systems insures a high likelihood of success for extension to hybrid designs where mixed inherent time delays are encountered. Such delays are ubiquitous in many physical and biological systems. Surprisingly, most past studies on coupled oscillator systems have avoided such systems. It would be interesting to determine whether the present technique can be successfully extended to time-delay systems.

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