Processing Blurred Images With Random Data

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Abstract
The objective of image deblurring is to reduce the noise generated when the lens is out of focus, incoming light is bent, or object moves while shutter is open. In this work, we present an abstract analysis of Euler-Lagrange equations associated with the total variation model based on Tikhonov regularization with random input data to reconstruct the original image. The optimizer produces a nonlinear system of elliptic type equations. To this end, we introduce a stochastic smoothing operator and develop a stochastic version of the Euler-Lagrange equations defined on suitable finite dimensional deterministic and probability spaces. We incorporate spectral expansion techniques such as the KL expansion to eliminate the dependency on the random effect.

Key Words: Image deblurring, image reconstruction; stochastic smoothing operator; stochastic blurring operator; KL expansion; total variation.

1. Introduction

Recording an image that is sharp and clear is sometimes challenging, and it seems that perturbations are inevitable. Brain CAT scans, for example, may contain blurry regions, and ultrasound images may have unclear object. This may be a result of various reasons. In many cases the the lens is out of focus lens, the incoming light is bent, or the object moves while shutter is open. The objective of image processing is to reduce the noise generated in the image and to produce a sharper image with a better representation and understanding of the scene.

A lot of times the noise is caused by hardware problems such as malfunctioning pixels in camera sensors, faulty memory locations in hardware or transmission in a noisy channel, in which the noise is called impulse noise [10]. Adaptive and multistate median filtering were among the remedial tools used to treat impulse noise [18]. Noise filtering is an important aspect in image deblurring, for example [16] implemented matrix decomposition and spectral analysis techniques to deblur images.

Alvarez et. al. [3], proposed a nonlinear diffusion model with Gaussian smoothing kernel to detect edges, which consists of a diffusion component acting on the exact image to smooth it out on both sides of an edge; the algorithm is a selective smoothing of the image with enhanced edges.

The theory we develop in this paper extends the models suggested in [1, 12, 13, 22, 28, 33, 34], taking into account the stochastic behavior of the smoothing operator operating on the exact image. It is reasonable to believe that the blurriness is dictated by a random stochastic process, which produces noise. The random effect generates blurry images that may be treated as random data. This paper presents an abstract analysis of the total variation model for blurred images with stochastic components embedded in the smoothing operator operating on the true image.

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In section 4, we discuss the stochastic behavior of the linear blurring operator operating on the exact image producing a blurry random image. In this regards, we extend the spaces and subspaces of the deterministic total variation problem to incorporate the probability space associated with the random effect of the smoothing operator. We define suitable measurable Banach and Hilbert spaces for the stochastic differential equation problem.

In section 5, we utilized spectral expansion methods such as the KL-expansion to eliminate the dependency of the true and blurred image on the random effect. This is done by projecting the probability space onto the space of polynomials. We obtain a semi-discretized version of the stochastic total variation problem with respect to the probability subspace.

Finally, in section 6, we utilize Lagrange polynomials to transform the semi-discretized system to a fully-discretized Euler-Lagrange equations, which can be solved using numerical analysis techniques such as the cell-centered finite difference, finite element method, or finite volume scheme.

2. Literature Review

Rudin et. al. [28] presented a model that optimizes the total variation of an image subject to constraints pertaining to the variability of the noise imposed by Lagrange multipliers, the showed the the solution converges to a steady state, which is the deblurred image.

Acar and Vogel initially suggested a bounded variation method for the ill-posed operator equation $Au = z$ [1]. They showed that under mild conditions the total variation

$$T(u) = ||Au - z||^2 + \alpha J(u)$$

has a unique and stable minimizer with respect to the functional $J(u)$, the blurred data z, the operator A, and the parameter α. The work in [1] addresses the existence, uniqueness, convergence and stability of the non-linear integro-differential system of equations, thus the deterministic minimizer converges and is stable.

Vogel and Oman [34], later introduced a total variation model based on Tikhonov regularization [30] with additive noise for ill-posed inverse problems to reconstruct noisy, blurred images, also in [13]. Chang et. al. [12] used Gaussian additive white noise.

$$z = Ku + \epsilon, \tag{1}$$

where z is the blurred data, K is the smoothing operator also known as the linear blurring operator, u is the true image to be recovered, and ϵ is the additive Gaussian white noise. Other resources in the literature propose image deblurring models and algorithms with multiplicative noise [26].

For the additive noise model in equation 1, let x be a point in the domain D of x, the smoothing operator K operating on the true image u is defined by

$$(Ku)(x) = \int_D k(x, y) \cdot u(y)dy, \quad x \in D,$$

where $D \subset \mathbb{R}^2$ is a bounded open domain.

The total variation with Tikhonov regularization (objective functional) is given by

$$T(u) = \frac{1}{2}\left\|Ku - z\right\|^2 + \alpha \int_D \sqrt{\left|\nabla u\right|^2 + \beta^2}dx, \quad \alpha, \beta > 0, \quad x \in D,$$

or equivalently,

$$\min_u \int_D \sqrt{\left|\nabla u\right|^2 + \beta^2} \text{ subject to } \left\|Ku - z\right\|^2 = \sigma^2,$$

where $\left\|\cdot\right\|$ denotes the norm in $L^2(D)$ and, for a function $u \in L^2(D)\left(\subset L^1(D)\right)$.

Because the Euclidean norm is not differentiable at zero and to avoid issues with differentiability, Acar and Vogel [1] considered a modified version of the functional to be minimized to derive the Euler-Lagrange nonlinear integro-differential equations of elliptic type. They also established the well-posedness of this optimization problem as well as existence, uniqueness and stability with respect to the perturbations in α and β.

Therefore, the Euler-Lagrange equations associated with the functional are

$$\begin{array}{lll} g(u) \overset{def.}{=} K^*(Ku - z) + \alpha L(u)u = 0, & & x \in D \\ \frac{\partial u}{\partial n} = 0, & & x \in \partial D. \end{array} \tag{2}$$

where, K^* is the operator adjoint and

$$L(u)w = -\nabla \cdot \left(\frac{1}{\sqrt{\left|\nabla u\right|^2 + \beta^2}}\nabla w\right).$$

Finally, the system associated with equation 2 is then

$$\left[K^*K + \alpha L(u)\right]u = b = f(u),$$

which may be discretized and solved using the cell-centered finite difference (CCFD) scheme. There are several techniques to solve the discretized system, these include polynomial preconditioner, product preconditioner [34], cosine preconditioner [11] and primal-dual [4, 15].

3. Working Spaces and Assumptions

For a function $u \in L^1(D)$, denote by ∇u the distributional gradient of u and set

$$\int_D \left|\nabla u\right| = \sup\left\{\int_D u \operatorname{div}\varphi \, dx : \varphi \in C_0^1(D)^2, \left|\varphi\right|_\infty \leq 1\right\}.$$

The space of functions of bounded variation, $BV(D)$ is defined as

$$BV(D) = \left\{u \in L^1(D) : \int_D \left|\nabla u\right| < \infty\right\}.$$

It is a Banach space under the norm

$$\left|u\right|_{BV(D)} = \left|u\right|_{L^1(D)} + \int_D \left|\nabla u\right|.$$

It is customary (see [5]) to work with the weak* topology on $BV(D)$ defined by

$$u_n \overset{w^*}{\rightarrow} u$$

iff $u_n \overset{L^1(D)}{\rightarrow} u$ and $\int_D \varphi \nabla u_n \rightarrow \int_D \varphi \nabla u_n$ for all $\varphi \in C_0(D)^2$.

The stochastic space $L^\infty_P(\Omega; L^2(D))$ is defined by

$$L^\infty_P(\Omega; L^2(D)) := \left\{ v : \Omega \rightarrow L^2(D) : \ v \text{ is measurable and } P\text{–}\operatorname*{ess\,sup}_{\omega\in\Omega} |v(\cdot,\omega)|_{L^2(D)} < +\infty \right\}.$$

It is also a Banach space under the norm

$$|v|_{L^\infty_P(\Omega; L^2(D))} = P\text{–}\operatorname*{ess\,sup}_{\omega\in\Omega} |v(\omega,\cdot)|_{L^2(D)}.$$

We consider a random smoothing integral operator $K : L^2(D) \rightarrow L^\infty_P(\Omega; BV(D))$ defined by

$$Ku(x,\omega) = \int_D k(x,y,\omega)u(y)dy, \quad (x,\omega) \in D \times \Omega,$$

where the kernel $k : D \times D \times \Omega \rightarrow \mathbb{R}$ is assumed to have enough properties so that the operator K is continuous and does not annihilate constants; i.e. $K(1) \neq 0$. This assumption will be sufficient to guarantee coercivity of a certain energy functional later on.

Our aim is to find a u that minimizes the energy

$$E(u) = \frac{1}{2}\int_D |z - Ku|^2 \, dx + \lambda \int_D \phi(|\nabla u|)\, dx, \tag{3}$$

where $z \in L^\infty_P(\Omega; BV(D))$ is a given blurred and noised image, $\lambda > 0$ is a parameter and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is assumed to satisfy the following hypotheses:

1. ϕ is strictly convex
2. $\phi(0) = 0$, $\lim_{s\rightarrow\infty}\phi(s) = \infty$
3. There exist two constants $c > 0, b > 0$ such that

$$cs - b \leq \phi(s) \leq cs + b \; \forall s \in \mathbb{R}^+.$$

According to (3) and the assumptions on ϕ, it is natural to seek a solution u in V, where

$$V = \left\{ u \in L^2(D) : \ \nabla u \in L^1(D) \right\}.$$

However, this space is not reflexive and we cannot assert compactness of minimizing sequences that are bounded in V. Instead, we observe that sequences that are bounded in V are also bounded in $BV(D)$ and therefore, they are compact for the weak* topology on $BV(D)$ defined above. The following theorem [5] states the relaxed energy to be computed.

Theorem 1. The relaxed functional of (3) for the weak* topology on $BV(D)$ is defined by

$$\begin{aligned} \overline{E}(u) &= \frac{1}{2}\int_D |z - Ku|^2 \, dx + \lambda \int_D \phi(|\nabla u|)\, dx \\ &\quad + \lambda c \int_{S_u} \left(u^+ - u^-\right) d\mathcal{H}^1 + \lambda c \int_{D - S_u} |C_u|, \end{aligned}$$

4. The Stochastic Behavior of $u(x)$

4.1 L^p-Spaces and Measure Theory

Let $X : \Omega \to \mathbb{R}^n$ be a random variable and $1 \leq p < \infty$. Define the L^p-norm of X, $\|X\|_p$ as follows

$$\|X\|_p = \|X\|_{L^p(\mathbb{P})} = \left[\int_\Omega |X(\omega)|^p \cdot dP(\omega)\right]^{\frac{1}{p}}.$$

The L^p-space associated with the L^p-norm is defined by

$$L^p(\mathbb{P}) = L^p(\Omega) = \{X : \Omega \to \mathbb{R}^n; \|X\|_p < \infty\}.$$

L^p-space is a Banach space, i.e. complete normed linear space. If $p = 2$ the space $L^2(\mathbb{P}) = L^2_{\mathbb{P}}$ is a Hilbert space, i.e. complete inner product space.

4.2 Measure Spaces of The Smoothing Operator (Ku) With Random Effect

Consider the relation $z = Ku$ defined on a domain $D \times D \subset \mathbb{R}^2$, with the random smoothing operator $K(x, \omega) : D \times D \times \Omega \to \mathbb{R}$, where $x = (x_1, x_2) \in D \times D,\ \omega \in \Omega$. In this sense, the data $z = z(x, \omega)$ and true image $u = u(x, \omega)$ are random. $u(x, \omega)$ is thought of as stochastic variables even though u is deterministic.

Let the triplet $(\Omega, \mathcal{F}, P)$ be a complete probability space, consisting of the sample space Ω, $\sigma-$algebra of events $\mathcal{F}$, and probability measure $P : \mathcal{F} \to [0, 1]$.

Let $B(D \times D)$ denote a Banach space of functions $v : D \times D \to \mathbb{R}$. We define the stochastic Banach space by

$L^q_P(\Omega; B(D \times D)) :=$

$$\left\{v : \Omega \to B(D \times D) |\ v \text{ is measurable and } \int_\Omega \|v(\omega, \cdot)\|^q_{B(D\times D)} dP(\omega) < +\infty\right\}$$

for $1 \leq q < \infty$. Also,

$L^\infty_P(\Omega; B(D \times D)) :=$

$$\left\{v : \Omega \to B(D \times D) |\ v \text{ is measurable and } P - \operatorname*{ess\,sup}_{\omega\in\Omega} \|v(\omega, \cdot)\|^2_{B(D\times D)} < +\infty\right\},$$

where $\operatorname{ess\,sup}$ is the essential supremum that is almost everywhere (a.e.) except on a set of measure 0. In particular, we are interested in the stochastic Banach valued functions with finite second moment, namely, $L^2_P(\Omega; B(D \times D))$; i.e. finite mean and variance. In this regard, $(Ku)(x, \omega)$ is assumed to be square integrable with respect to P.

$$\mathcal{V} = \{v : v(\cdot, \omega) \in L^2(D \times D) \quad a.e. \quad \omega \in \Omega\},$$

with $v(\cdot, \cdot)$ being a measurable function.

Define the subspace $V \subset \mathcal{V}$ as follows

$$V = L^2_p(\Omega; L^2(D \times D)) = \left\{v \in \mathcal{V} : E\left(\|v\|^2_{L^2(D\times D)}\right) < \infty\right\} \subset \mathcal{V}.$$

Then,

$$\|v\|_V = E\left(\|v\|^2\right)^{\frac{1}{2}}.$$

We assume the true solution $u(\cdot,\omega) \in B(D \times D)$ a.s. $\forall\omega \in \Omega$. Moreover, we claim that the stochastic solution u is unique and bounded in $L^2_P(\Omega; B(D \times D))$.

For a given ω, the smoothing operator is a random variable, namely,

$$K(x,\cdot) \in L^2_p(\Omega) \quad \forall\, x \in D \times D,$$

that has finite mean and covariance. Hence,

$$E\left[K(x,\cdot)\right] = \int_\Omega K(x,\omega)\, dP(\omega) < \infty \quad \in L^2(D \times D).$$

and

$$\text{Cov}[K(x,\cdot), K(y,\cdot)] < \infty \quad \in L^2(D \times D).$$

In addition, for a given $x \in D \times D$, the smoothing operator represents a path or realization, namely,

$$K(\cdot,\omega) \in L^2_p(D \times D) \quad \forall\, \omega \in \Omega.$$

Ultimately,

$$\left[Ku\right](x,\omega) = \int_{D\times D} K(x,y,\omega)\cdot u(y)\, dy, \quad \in L^2(\Omega \times D \times D).$$

$$E[Ku(x)] = \int_\Omega [Ku](x)\, dP(\omega) = \int_\Omega \int_{D\times D} k(x,y,\omega)\cdot u(y)\, dy\, dP(\omega),$$

where $k(x,y,\omega) = k(\omega)\cdot(x-y)$.

Next, we investigate the total variation minimizer with stochastic terms and derive the Euler-Lagrange integro system of equations associated with the stochastic total variation.

4.3 Total Variation With Random Effect

Let

$$T(u(x,\omega)) = \frac{1}{2}\left\|(Ku)(x,\omega) - z(x,\omega)\right\|^2,$$

be the total variation associated with $z(x,\omega) = (Ku)(x,\omega)$.

Define $F(\xi) = T(u + \xi v)$ and take $\frac{\partial T}{\partial \xi} = 0$ when $\xi = 0$.

$$F(\xi) = T(u+\xi v) = \frac{1}{2}\|K(u+\xi v) - z\|^2 = \frac{1}{2}\|Ku + \xi Kv - z\|^2$$

Thus,

$$F(\xi) = \frac{1}{2}\left\langle Ku, Ku\right\rangle + \xi\left\langle Ku, Kv\right\rangle + \frac{1}{2}\xi^2\left\langle Kv, Kv\right\rangle - \left\langle Ku, z\right\rangle - \xi\left\langle Kv, z\right\rangle$$

Therefore,

$$\frac{\partial T}{\partial \xi} = \left\langle Ku, Kv\right\rangle + \xi\left\langle Kv, Kv\right\rangle - \left\langle Kv, z\right\rangle$$

$$= \langle Ku, Kv\rangle + \xi \langle Kv, Kv\rangle - \langle Kv, z\rangle$$

Finally,

$$\left.\frac{\partial T}{\partial \xi}\right|_{\xi=0} = \langle Ku, Kv\rangle - \langle Kv, z\rangle = 0$$

But,

$$\langle Ku, Kv\rangle - \langle Kv, z\rangle = \langle K^*Ku, v\rangle - \langle K^*z, v\rangle$$
$$= \langle K^*Ku - K^*z, v\rangle = \langle K^*(Ku - z), v\rangle$$
$$\Rightarrow \langle K^*(Ku - z), v\rangle = 0$$

Therefore, the Euler-Lagrange equations with stochastic terms produces the following system of equations

$$\begin{array}{ll} K^*((Ku)(x,\omega) - z(x,\omega)) = 0, & x \in D \times D,\ \omega \in \Omega \\ \frac{\partial u}{\partial n} = 0, & x \in \partial(D \times D). \end{array} \tag{4}$$

Note that the system does not contain the differential component that is present in the additive models in [1, 12, 13, 22, 28, 33, 34]. In the following section, we discretize the system in equation 4 with respect to the $D \times D$ space and Ω space.

5. Approximation of $(Ku)\,(x, \omega)$

In order to solve $z = Ku$ numerically, we transform the random operator $(Ku)(x, \omega)$ to a deterministic form. Such methods include spectral expansion of the random smoothing operator to separate the stochastic dependence on $\omega \in \Omega$.

$$Ku(x, \omega) = z(x, \omega) \tag{5}$$

5.1 Karhunen-Loève Expansion

We now introduce Karhunen-Loève expansion for the stochastic integro equation 1. The random smoothing operator $(Ku)(x, \omega)$ with a continuous covariance function $cov[(Ku)(x, \cdot)]$ can be represented in terms of an infinite sum of random variables of a Karunen-Loève expansion, originally in [20] and later in [6, 7, 8, 24].

Recall,

$$(Ku)(x, \omega) = \int_{D\times D} k(x, y, \omega) \cdot u(y)\, dy,$$

where $k : BV(D \times D) \to L^\infty(D \times D \times \Omega)$, i.e. $k \in L^\infty(D \times D \times \Omega)$.

For a given realization $\omega \in \Omega$, we define the self-adjoint integral operator (transformation) $\mathcal{K} : L^2(D \times D) \to L^2(D \times D)$ by

$$\mathcal{K}[v(\cdot)] = \int_{D\times D} cov[k(x, \cdot)] \cdot v(x)\, dx, \qquad \forall v \in L^2(D \times D),$$

which produces a set of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ associated with orthonormal eigenvectors $\{v_n\}_{n=1}^{\infty}$ according the relation

$$\mathcal{K}[v(\cdot)] \cdot v_n = \lambda_n \cdot v_n.$$

The ordered eigenvalues of the integral operator decay in magnitude, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ [24].

Moreover, we need a set of uncorrelated identically distributed random variables $Y_n \overset{i.i.d.}{\sim} N(0,1)$ defined by

$$Y_n(\omega) = \frac{1}{\sqrt{\lambda_n}} \int_{D\times D} [K(x,\omega) - E[K](x)] \cdot v_n(x) dx, \qquad n = 1, 2, \cdots .$$

This transformation centers the random smoothing operator $K(x,\omega)$ so that $E[Y_i] = 0$ and $E[Y_i \cdot Y_j] = \delta_{ij}, \quad \forall i,j = 1,2,3,\cdots$, where δ_{ij} is the Kronecker's delta.

$K(x,\omega)$ can be expressed in terms of the KL-expansion to separate its dependence on $x \in D \times D$ and $\omega \in \Omega$

$$K(x,\omega) := E[K](x) + \sum_{n=1}^{\infty} \sqrt{\lambda_k} \cdot v_n(x) \cdot Y_n(\omega). \tag{6}$$

We require the infinite series in 6 be finite and convergent. The convergence is in the L^2 sense because the smoothing operator $K(x,\omega)$ is a second-order random field with finite mean and variance; i.e. $K(x,\cdot) \in L^2_{\mathbb{P}}(\Omega)\ \forall x \in D \times D$.

Next, we work on the discretization of $K(x,\omega)$ to obtain the truncated KL-series $K_N(x,\omega)$ of the stochastic operator $K(x,\omega)$. To serve this purpose, define $K_N(x,\omega)$ as follows

$$K(x,\omega) \approx K_N(x,\omega) = E[K](x) + \sum_{n=1}^{N} \sqrt{\lambda_n} \cdot v_n(x) \cdot Y_n(\omega), \qquad \forall N \in \mathbb{N}.$$

But, Mercer's theorem states that

$$\lim_{N\to\infty} \left\{ \sup_{D\times D} E\left[(K - K_N)^2\right] \right\} = \lim_{N\to\infty} \left\{ \sup_{D} \left[\sum_{n=N+1}^{\infty} \lambda_n \cdot v_n^2 \right] \right\} = 0.$$

Due to the fact that the eigenvalues and eigenvectors decay the coefficients of the truncated KL-expansion have different weights [14]. These coefficients can be described by a finite set N of random variables. Consider the random field $u_N : D \times D \times \Omega \to \mathbb{R}$, $u_N \in L^2_P(\Omega; B(D))$, such that

$$(Ku_N)(x,\omega) = z_N(x,\omega), \qquad \text{a.e. in } D \times D. \tag{7}$$

The stochastic solution $u_N(x,\omega)$ of 7 can be expressed in terms of

$$u_N(x,\omega) = u_N(x, Y_1(\omega), Y_2(\omega), \cdots, Y_N(\omega))$$

using Doob-Dynkin's lemma [9, 25, 27], which is an approximation of the exact solution $u(x,\omega)$.

Assume the random smoothing operator $(Ku)(\vec{x},\omega) : D \times D \times \Omega \to \mathbb{R}$ is a second order stochastic field, i.e. the second moment $E\left[(Ku)^2\right] < \infty$ is finite and $(Ku)(x, \cdot) \in L^2_{\mathbb{P}}(\Omega)\ \forall\ x \in D \times D$. This means that the first moment is also finite. Thus, the mean function is given by

$$E[(Ku)(x)] = \int_\Omega (Ku)(x, \omega) d\mathbb{P}(\omega) < \infty \in L^2(D \times D).$$

And the covariance function is given by

$$Cov[(Ku)(x_1, x_2)] = Cov\left[(Ku)(x_1, \cdot), (Ku)(x_2, \cdot)\right]$$

$$= \int\int \left\{(Ku)(x_1, \omega) - E[(Ku)(x_1)]\right\}\left\{(Ku)(x_2, \omega) - E[(Ku)(x_2)]\right\} d\mathbb{P}(\omega) < \infty,$$

which means that $Cov[(Ku)(x_1, x_2)] \in L^2(D \times D)$.

Suppose the stochastic smoothing operator K is a second-order random field, then KL expansion converges in the L^2 sense. Furthermore, by Mercer's Theorem the convergence will be uniform if the domain $D \times D$ is bounded and the covariance of K is continuous over the domain $D \times D$.

Based on our assumptions, it is naturally to believe that the random data $z_N(x, \omega)$ and the parameters in the random smoothing operator $K_N(x, \omega)$ are independent, i.e. uncorrelated.

We next define $\Gamma_n = Y_n(\Omega)$, and assume $Y_n(\omega)$ is bounded and $\Gamma_n = [-1, 1]$. We are excluding the situation when $Y_n(\omega)$ is unbounded that includes Gaussian and exponential distributions. Moreover, let $\Gamma^N = \prod_{n=1}^N \Gamma_n = \Gamma_1 \times \Gamma_2 \times \cdots \Gamma_{n-1} \times \Gamma_{n+1} \cdots \times \Gamma_N$. Assume the random variables $\{Y_k\}_{k=1}^n$ have joint probability density function $f : \Gamma^N \to \mathbb{R}$, with $f \in L^\infty\left(\Gamma^N\right)$. This means that now we are approximating $u_N(x, y)\ \forall y \in \Gamma^N$ and $x \in \overline{D \times D}$.

$u_N(x, y) = u_N\left(x, Y_1(\omega), Y_2(\omega), \cdots, Y_N(\omega)\right)$ represents a finite set of random variables of the random fields, and the semi-discretized version of the stochastic model is now deterministic in ω and finitely dimensional in ω as well. The stochastic total variation problem is has become an $N-$dimensional deterministic in ω.

The truncated equation in problem 7 can written as

$$\left(Ku_N\right)(x, y) = z_N(x, y), \qquad \text{a.e. in } D \times D. \tag{8}$$

Equation 8 is extended a.e. in Γ^N with respect to the measure $\rho(y)dy$, alternatively, one may use the Lebesgue measure.

6. Discretizing $(Ku_N)(x, y)$ in x

It is time to discretize the semi-discretized $u_N(x, y)$ in x to obtain $u_{N,h}(x, y)$.

6.1 Generating Suitable Finite Dimensional Subspace

In order to proceed with the discretization process, we need to have adequate deterministic and probabilistic subspaces. Let $V_h \subset V = L^2_p(L^2(D \times D))$ be the finite deterministic subspace of dimension N_h. Likewise, let $W_P\left(\Gamma^N\right) \subset W\left(\Gamma^N\right) = L^2_\rho\left(\Gamma^N\right)$ be the finite probabilistic subspace of polynomials with maximum degree of P. The dimension of $W_P\left(\Gamma^N\right)$ is $N_p = \prod_{n=1}^{N}(p_n+1)$, where

$$W_{p_n}(\Gamma_n) = span\left(y_n^k, k=0,\cdots,p_n\right), \quad n=1,\cdots,N.$$

and $W_P\left(\Gamma^N\right) = W_{p_1}(\Gamma_1) \times W_{p_2}(\Gamma_2) \times \cdots \times W_{p_N}(\Gamma_N)$.

For the set of abscissas $y_k \in \Gamma^N$ the semi discretized approximation $(Ku_N)(x,y)$ admits the solution $(Ku_{N,h})(y_k) \in V_h(D \times D)$ in the finite subspace V_h.

We project equation 8 onto the subspace $V_h(D \times D)$ for all $y \in \Gamma^N$ to obtain the semi-discrete approximation $(Ku_{N,h})(y_k)$, where $u_{N,h} : \Gamma^N \to V_h(D \times D)$. Thus,

$$\left(Ku_{N,h}\right)(x,y) = z_{N,h}(x,y), \qquad \forall x \in V_h(D \times D) \text{ a.e. for } y \in \Gamma^N. \tag{9}$$

6.2 Lagrange Polynomials

To obtain the fully discrete version of equation 9, namely, $u_{N,h,k} \in L^2\left(\Gamma^N; V_h(D \times D)\right)$, we implement Lagrange polynomials to interpolate the sample points. The solution is then

$$u_{N,h,k}(\cdot,y) = \sum_i u_{N,h}(\cdot,y_i) \cdot L_i^k(y), \tag{10}$$

where L_i^k is the bases of Lagrange polynomials of degree $= k$.

We now discuss the mechanism to choose the interpolation nodes for the Lagrange polynomial. There are several methods in the literature that address this issue. Such methods include Newton-Cotes, Gaussian and Clenshaw-Curtis [2, 17, 19, 23, 24, 29, 31, 32]. These techniques are the building blocks for discretization. Newton-Cotes formula relies on equally spaced knots, while the number of knots increases indefinitely the method becomes unreliable. However, Gauss quadrature optimizes the degree of the polynomial by selectively choosing the interpolation knots and weights. Gauss quadrature converges faster than Newton-Cotes as the number of nodes increases. In this paper, we implement the Gauss quadrature formulas to construct the interpolation using the full tensor product in the space of polynomials.

Consider the Lagrange interpolation on the interval $[-1,1]$ having nodes $\left\{y_1^k, y_2^k, \cdots, y_{n_k}^k\right\}$, where $k = 1,2,3,\cdots$. Let $N=1$ and define the Lagrange interpolation operation $\mathcal{L}^k$ as follows

$$\mathcal{L}^k(Ku)(y) = \sum_{i=1}^{n_k}(Ku)\left(y_i^k\right) \cdot l_i^k(y), \quad \forall u_{1,h,k} \in L^2\left(\Gamma^1; V_h(D \times D)\right), \tag{11}$$

where $l_i^k \in W_{n_k-1}\left(\Gamma^1\right)$ are the Lagrange polynomials with degree $n_k - 1$, $n =$ number of nodes.

$$l_i^k = \prod_{\substack{m=1 \\ m \neq i}}^{n_k} \frac{y - y_m^k}{y_i^k - y_m^k}$$

For $N > 1$, and $(n_{k_1}, n_{k_2}, \cdots, n_{k_N})$ sample points and function values on the grid with a permutation $(k_1, k_2, \cdots, k_N)$, define the $N-$dimensional Lagrange interpolation operator as follows

$$\mathcal{L}_k^N(Ku)(y) = \left(\mathcal{L}^{k_1} \otimes \mathcal{L}^{k_2} \otimes \cdots \otimes \mathcal{L}^{k_N}\right)(Ku)(y)$$

$$= \sum_{i_1=1}^{n_{k_1}} \sum_{i_2=1}^{n_{k_2}} \cdots \sum_{i_N=1}^{n_{k_N}} (Ku)\left(y_{i_1}^{k_1}, y_{i_2}^{k_2}, \cdots, y_{i_N}^{k_N}\right) \cdot \left(l_{i_1}^{k_1} \otimes l_{i_2}^{k_2} \otimes \cdots \otimes l_{i_N}^{k_N}\right).$$

Once we obtain the Lagrange polynomial by interpolating the nodes using the aforementioned method, we may rewrite equation 9 as follows

$$\left(Ku_{N,h,k}\right)(x,y) = z_{N,h,k}(x,y), \qquad \forall x \ \in V_h(D \times D) \text{ a.e. for } y \in \Gamma^N. \tag{12}$$

The system is now fully discretized, which may be solved numerically using proper differential equation solvers.

Conclusion

In many situations, it is difficult not to avoid the noise generated in the image for various reasons. The purpose of image processing is to reduce the perturbations and noise produced when there are instrumental or atmospheric problems and to produce a sharper image that better represents of the actual scene. In this paper, we presented an abstract analysis of Euler-Lagrange equations associated with the total variation model with random input data to restore the exact image. We developed a stochastic smoothing operator operating on the true image defined on suitable finite dimensional deterministic and probabilistic spaces. We implemented the KL expansion to eliminate the dependency on the random effect. It is possible for the noise an impulse noise generated by hardware problems or failures.

In this paper we developed a methodology based on models suggested in [1, 12, 13, 22, 28, 33, 34], that involves a stochastic blurring operator.

We studied the stochastic behavior of the smoothing operator operating on the exact image that results in generating a random blurry image. Moreover, suitable probability and measure spaces and subspaces were introduced to tackle the stochastic total variation problem.

KL-expansion was implemented in the system of stochastic total variation to eliminate the dependency on the random effect, which was performed through projecting the probability space onto the space of polynomials. And a semi-discretized version of the stochastic model was obtained with respect to the probability subspace.

Finally, we applied Lagrange polynomials to transform the semi-discretized system to a fully-discretized Euler-Lagrange equations, which can be solved numerically.

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