Weak Law of Large Numbers and Central Limit Theorem in Lorentz-Bochner Spaces

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Abstract

In this paper, for a Banach space B, we estimate the Lorentz-Bochner norm of a $(L_{p,q}, B)$-valued random variable in terms of its expected value in B and use that estimation to obtain the Weak Law of Large Numbers and the Central Limit Theorem for (L_{pq}, B)-valued random variables.

Keywords: Lorentz-Bochner spaces, Weak Law of Large Numbers, Central Limit Theorem.

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1 Introduction

The Lorentz spaces $L_{p,q}(\Omega, \mathbb{R}, \mu)$ were introduced by G.G. Lorentz in [14, 15] as generalizations of Lebesgue spaces $L_p(\Omega, \mathbb{R}, \mu)$ where $(\Omega, \mathcal{A}, \mu)$ is a measure space on $\mathbb{R}$. The Lorentz-Bochner spaces $L_{p,q}(\Omega, B, \mu)$ differ from original Lorentz spaces in that functions are B-valued where B is a Banach space, and are strongly measurable (or Bochner-measurable) in the sense that each function is a.e the limit of a sequence of countably-valued functions. It is well-established that the Lorentz-Bochner spaces are Banach spaces for $1 < p < \infty,\ 1 \leq q < \infty$. The problem of establishing convergence theorems such as the Weak Law of Large Numbers (WLLN) and the Central Limit Theorem (CLT) in Banach spaces is not a new. In particular, Hoffmann-Jorgensen and Pisier in [10], Araujo and Giné in [1], Jain in [11] have given conditions to obtain the WLLN and the CLT in general Banach spaces. The CLT in $L_p, 0 < p < 1$ was established by Giné in [6] and in $L_p,\ p \geq 2$ by Pisier and Zinn in [17]. Moreover, Theorem 2.9 in [7] establishes a one-to-one and onto correspondence between the WLLN and the CLT in 2-convex Banach lattices. In this paper we establish the WLLN and use the framework of Theorem 2.9 in [7] to obtain the CLT in Lorentz-Bochner spaces. The remaining of the paper is organized as follows: in section 2, we give the necessary preliminary definitions for establishing our results, in section 2, we give the essential results for $L_{p,q}(\Omega, B, \mu)$-valued random variables and establish the WLLN and the CLT.

2 Preliminaries

Throughout this paper, $(B, \|\cdot\|_B)$ will be considered as a Banach space, Ω as the interval $[0,1]$, and $(\Omega, \mathcal{A}, \mathbf{P})$ and $(B, \mathcal{B}, \mathbf{P})$ as probability spaces on Ω and B respectively.

Definition 2.1. For a measurable function $f : \Omega \to B$, we define the *distribution function* of f as

$$D_B(f,y) := \mathbf{P}(\{\omega \in \Omega : \|f(\omega)\|_B > y\}).$$

We define *decreasing rearrangement* of f as the function f^* on Ω by

$$f^*(t) = \inf\{y > 0 : D_B(f,y) \le t\} \ ,$$

For $t > 0$, let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \quad f^{**}(0) = f^*(0).$$

We also define on Ω the function $\|f\|$ by $\|f\|(\omega) = \|f(\omega)\|,\ \omega \in \Omega$.

Definition 2.2. Given a strongly measurable function f, define

$$|||f|||_{p,q} = \begin{cases} \left(\frac{q}{p}\int_\Omega \left((t^{\frac{1}{p}}||f||^{**}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 1 < p < \infty, \quad 1 \le q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}}||f||^{**}(t) & \text{if } 1 < p \le \infty, \quad q = \infty. \end{cases}$$

The set of all functions f with $\|f\|_{p,q} < \infty$ is called *the Lorentz-Bochner Space* with indices p and q and denoted by $L_{p,q}(\Omega, B, \mathbf{P})$. We know that endowed with this norm, the Lorentz-Bochner spaces are Banach spaces. Recall that for $1 < p, q < \infty$, we have $L^*_{p,q}(\Omega, B, \mathbf{P}) = L_{p',q'}(\Omega, B^*, \mathbf{P})$ and $L^*_{p,1}(\Omega, B, \mathbf{P}) = L_{p,\infty}(\Omega, B^*, \mathbf{P})$ where p' and q' are the Hölder conjugates of p and q respectively and B^* is the dual space of B. In the sequel, we will refer to $L_{p,q}(\Omega, B, \mathbf{P})$ as $L_{p,q}$ for simplicity.

Definition 2.3. For $1 < p < \infty$ and for $1 \le q < \infty$, we define for a strongly measurable function f two equivalent norms to quantities $|||f|||_{p,q}$ as

$$\|f\|^*_{p,q} = \left(\frac{q}{p}\int_\Omega \left[t^{\frac{1}{p}}(f^*(t))\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{2.1}$$

and

$$\|f\|_{p,q} = \left(p\int_\Omega \left[tD_B(f,t)^{\frac{1}{p}}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}. \tag{2.2}$$

Proposition 2.4 (Proposition 1.4.9 in [8])**.** *For $1 < p < \infty$ and for $1 \le q < \infty$ we have*

$$\|f\|^*_{p,q} \le |||f|||_{p,q} \le \frac{p}{p-1}\|f\|_{p,q}$$

and

$$\|f\|_{p,q} = \|f\|^*_{p,q}.$$

In the sequel, we will use the norm defined in (2.2) as it is more appropriate for our objective.

Definition 2.5. Let X be a B-valued random variable defined on $(\Omega, \mathcal{A}, \mathbf{P})$. For $p \geq 1$, we define

$$E(\|X\|_B^p) := p \int_\Omega t^{p-1} D_B(X, t) dt.$$

Remark 2.6. This definition can be viewed as an extension of the definition of the expected value of a real-valued random variable to normed spaces, see [2, 7]. Indeed, if X is real-valued, then we know classically that for $p \geq 1$,

$$E(|X|^p) = p \int_\Omega t^{p-1} \mathbf{P}(\omega : |X(\omega)| > t) dt.$$

Definition 2.7. Let $X_1, X_2, \cdots, X_n$ be B-valued random variables defined on $(\Omega, \mathcal{A}, \mathbf{P})$. We define the disjoint sum $\sum_{i=1}^{n} \oplus X_i$ of $X_1, X_2, \cdots, X_i$ as a function on Ω such that

$$D_B\left(\sum_{i=1}^{n} \oplus X_i, t\right) = \sum_{i=1}^{n} D_B(X_i, t).$$

Remark 2.8. This definition of the disjoint sum of random variables has been used by authors such as Johnson, Maurey, Schetchtman and Tzafiri in [12], Carothers and Dilworth in [3, 4], Hitzchenko and Montgomery-Smith in [9], mostly in the contest of sums of independent random random variables.

The following definition is a natural extension to Banach spaces of definition 2.1 in [16].

Definition 2.9. A sequence of B-valued random variables $\{X_n\}_{n\in\mathbb{Z}_+}$ is said to be *stochastically bounded* in B if

$$\lim_{C\to\infty} \sup_{n\in\mathbb{Z}_+} \mathbf{P}(\omega : \|X_n(\omega)\|_B > C) = 0.$$

We say that a sequence of $L_{p,q}(\Omega, B, \mathbf{P})$-valued random variables $\{X_n\}_{n\in\mathbb{Z}_+}$ defined on $(\Omega, \mathcal{A}, \mathbf{P})$ is *strongly p-mean stochastically bounded* in B if

$$\lim_{n\to\infty} \sup_{C>0} \mathbf{P}(\omega : E(\|X_n(\omega)\|_B^p) > C) = 0.$$

Note that, in terms of distribution function, this amounts to

$$\lim_{n\to\infty} \sup_{C>0} D_{\mathbb{R}}[E(\|X_n\|_B^p), C] = 0.$$

We say that that Y is a copy of X if X and Y have the same distribution.

Definition 2.10. We say that a random variable is $(L_{p,q}, B)$-valued if

1. $X : (\Omega, \mathcal{A}) \to (L_{p,q}, \mathcal{L}_{p,q})$ is measurable, where $\mathcal{L}_{p,q}$ is the Borel σ-algebra of $L_{p,q}$.

2. For all $\omega \in \Omega,\ X(\omega) := X_\omega : (\Omega, \mathcal{A}) \to (B, \mathcal{B})$ is measurable.

The next definition is borrowed from [7].

Definition 2.11. Let X be an mean zero $(L_{p,q}, B)$ random variable and $\{X_i\}$ be a sequence of independent copies of X.

1. We say that $X \in$ CLT or that the sequence $\{X_i\}$ satisfies the Central Limit Theorem if there is a Gaussian p.m. γ on B such that

$$\mathcal{L}\left(\sum_{i=1}^{n} X_i/n^{\frac{1}{2}}\right) \to_w \gamma,$$

where $\to_w$ denote the weak convergence and $\mathcal{L}$ the law of $\sum_{i=1}^{n} X_i/n^{\frac{1}{2}}$.

2. We say that $X \in$ WLLN in $L_{p,q}$ or that the sequence $\{X_i\}$ satisfies the Weak Law of Large numbers in $L_{p,q}$ if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left(\omega \in \Omega : \left\|\frac{1}{n}\sum_{i=1}^{n} X_i(\omega)\right\|_{p,q} > \epsilon\right) = 0.$$

3 Results

Our first result, is lemma that will pave the way for most of our results.

Lemma 3.1. *Given two positive real numbers a, b and a B-valued random variable X defined on $(\Omega, \mathcal{A}, \mathbf{P})$, we have, for $t \in \Omega$,*

1. *If $a, b \geq 1$, then $D_B(X, t^{\frac{1}{a}})^b \leq D_B(X, t)^b \leq D_B(X, t^a)$.*

2. *If $a \geq 1$ and $b \leq 1$, then $D_B(X, t^{\frac{1}{a}}) \leq D_B(X, t) \leq D_B(X, t)^b$.*

Theorem 3.2. *Let $1 \leq p, q < \infty$ be two reals and let $X \in L_{p,q}$ be a B-valued random variable. Then there is exist an absolute constants $C = C(p, q)$ such that*

1. *If $p \leq q$, then*

$$\|X\|_{p,q} \leq C E^{\frac{1}{q}}(\|X\|_B^p).$$

2. *If $q \leq p$, then*

$$\|X\|_{p,q} \geq C E^{\frac{1}{q}}(\|X\|_B^p).$$

Remark 3.3. Note that if $B = \mathbb{R}$, then

1. for $p = q \geq 1$, we have the classical result

$$\|X\|_{pp} = \|X\|_p = E^{\frac{1}{p}}(|X|^p).$$

2. We also have that

   (a) For $p \leq q$, $\|X\|_{p,q} \leq C\|X\|_p^{\frac{p}{q}}$.

   (b) For for $q \leq p$, $\|X\|_{p,q} \geq C\|X\|_p^{\frac{p}{q}}$.

From this remark, it follows the following result, a different version of Corollary 5.6 in [4].

Corollary 3.4. *Let $1 \leq p, q < \infty$ be two reals.*

1. *If $1 \leq p \leq q$, then $L_{p,q}(\Omega, \mathbb{R}, \mathbf{P})$ contains a complete metric subspace of $L_q(\Omega, \mathbb{R}, \mathbf{P})$.*
2. *If $1 \leq q \leq p$, then $L_{p,q}(\Omega, \mathbb{R}, \mathbf{P})$ is contained in a complete metric subspace of $L_p(\Omega, \mathbb{R}, \mathbf{P})$.*

Corollary 3.5. *Let $1 < p \leq q < \infty$ be two reals and let $X_i \in L_{p,q}(\Omega, B, \mathbf{P})$, $i = 1, \cdots, n$ be B-valued random variables. Then there is C=C(p,q) such that*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \oplus X_i \right\|_{p,q} \leq C \left[\sum_{i=1}^{n} E\left(\left\| \frac{X_i}{n} \right\|_B^p \right) \right]^{\frac{1}{q}}.$$

3.1 Weak Law of large Numbers

Theorem 3.6. *Suppose $2 \leq p \leq q < \infty$, and let and let X be a mean zero $(L_{p,q}, B)$-valued random variable and let $\{X_i, i \geq 1\}$ be independent copies of X. If for all $\omega \in \Omega$, $X_n(\omega)/n^{1-\frac{1}{p}}$ is strongly p-mean stochastically bounded in B, then*

$$X \in WLLN.$$

3.2 Central Limit Theorem

Theorem 3.7. *Suppose $4 < p < q < \infty$, and let X be a mean zero $(L_{p,q}, B)$-valued random variable and let $\{X_i, i \geq 1\}$ be independent copies of X. Then if for all $\omega \in \Omega$, $X_i^2(\omega)/n^{1-\frac{1}{p}}, i \leq n$ is strongly p-mean stochastically bounded in B, then*

$$X \in CLT.$$

Remark 3.8. Note that Theorem 3.6 is true when the indices $2 \leq p \leq q$ where as Theorem 3.7 is true for indices $2 < p < q$. This is due to the result on types and cotypes of Lorentz spaces in Creekmore [5].

3.3 Concluding remarks

We have established that for X a $(L_{p,q}, B), 2 \leq p < q$ random variable, $X \in WLLN$ if $X_i/n^{1-\frac{1}{p}}, i \leq n$ are p-mean stochastically bounded. This condition appears to be a generalization of the condition (iii) of Theorem 4.3 in [7] (or Theorem 5.1 in [17]). This condition yields the CLT, and thus by Theorem 2.9 in [7] implies the WLLN. Indeed, for $p > 2$, and $X_i/n^{1-\frac{1}{p}}$ is p-stochastically bounded implies (iii) in [7]:

$$\lim_{n\to\infty} n\mathbf{P}(\omega : \|X(\omega)\|_B > n^{\frac{1}{2}}) = 0.$$

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