Performance of Processes With Multiple Variables

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Abstract

Most processes in manufacturing and service industries involve more than one quality characteristic. If the characteristic pertains to a finished product, the requirements for it are influenced by the external customer. On the contrary, for sequential processes, the customer is internal and could be the unit that deals with the subsequent operation. Target values and specifications may be specified for the desired quality characteristics. When the average value of the characteristic deviates from the target value, a loss may be incurred by the organization. Similarly, if the variability of the characteristic exceeds certain bounds, it is undesirable as it may lead to production of nonconforming units. The paper considers a loss function that incorporates the bias of the characteristic relative to the target value, the variability of the characteristic, the covariance between characteristics since the characteristics may not necessarily be independent of each other, and the proportion of nonconforming product. Further, all processes face resource and physical constraints. Physical constraints may involve the permissible ranges of the quality characteristics. A mathematical model is formulated as a constrained optimization problem and the optimal settings of the decision variables, which are the means and standard deviations of the respective quality characteristics, are determined.

Key Words: Target value, specifications, loss function, constrained optimization

1. Introduction

Manufacturing and service processes, in reality, have to deal with several quality characteristics when either satisfying an end customer or an intermediate customer. Quality characteristics can be product or process related. Often, certain process-related quality characteristics have to be monitored in order to accomplish a desired level of a product-related quality characteristic.

Satisfaction of the end user of a product may involve meeting requirements or specifications set by the customer. For example, the desired or target tensile strength of a beam may be 2000 kg/cm^2, and the specifications could be 2000 $\pm$ 100 kg/cm^2. Additionally, there could be a target and specifications on the beam width as 0.1 $\pm$ 0.05 m. Along the same lines, there might be specifications on the beam thickness and the beam length.

In order to satisfy product quality characteristics as required by the customer, the manufacturer may have to keep certain process variables under controlled conditions. Equivalently stated, target and specification limits for process-related variables could be necessary. These specifications could be determined by process engineers. Hence, a manufacturer normally may have to satisfy an external customer as well as some internal customers. The internal customer may be different departments or units or divisions within the organization that makes the product. A supplier of a component or a sub-assembly may have certain specifications on the component imposed by the manufacturer. Organizations, therefore, face problems that involve multiple variables and those that necessarily may not be independent of each other. Given the nature of the manufacturing processes and the dependence of outcomes at a certain stage to those at prior stages, it is quite possible for the attained values of a quality characteristic to influence the achieved value of another quality characteristic at a subsequent stage. So, an appropriate loss function must include the degree of co-variance between quality characteristics being investigated.

2. Development of the Loss Function

Suppose that the quality characteristics of interest are represented by $X_1, X_2, \ldots, X_p$. An assumption of multivariate normality of $X_1, X_2, \ldots, X_p$ is made using the following notation for the mean vector $\underline{\mu}$ and the variance-covariance matrix $\underline{\Sigma}$:

$$X_1, X_2, \ldots X_p \sim \text{MVN}(\underline{\mu}, \underline{\Sigma}) \tag{1}$$

where

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix} \quad \text{and} \quad \underline{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \cdots \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 \cdots \sigma_{2p} \\ \vdots & \\ \sigma_{p1} & \sigma_{p2} \cdots \sigma_p^2 \end{pmatrix}$$

Note that the variance-covariance matrix $\underline{\Sigma}$ is symmetric, i.e., $\sigma_{ij} = \sigma_{ji},\ i \neq j$. Let the target values of the p quality characteristics be represented by $\underline{t} = (t_1, t_2, \ldots, t_p)$ and the corresponding specification limits be given by (LSL_i, USL_i,) $i = 1,2, \ldots, p$.

Prior research has investigated multivariate loss functions formulated in a variety of ways. Derringer and Suich (1980) used the concept of a desirability function, first introduced by Harrington (1965). For two-sided specifications, the formulated desirability function for the ith characteristic is given by

$$d_i = \begin{cases} \left[\dfrac{X_i - LSL_i}{USL_i - LSL_i}\right]^{\phi_i}, & LSL_i \le X_i \le t_i \\ \left[\dfrac{X_i - LSL_i}{USL_i - LSL_i}\right]^{\psi_i}, & t_i \le X_i \le USL_i \\ 0, & X_i < LSL_i \ or \ X_i > USL_i \end{cases} \tag{2}$$

In Equation (2), ϕ_i and ψ_i are positive constants selected by the decision maker that indicates the degree of importance of the quality characteristic being close to the target value. Large values of ϕ_i and ψ_i imply a desirability curve with a sharp peak at the target value and a rapid drop as the characteristic moves away from the target. Subsequently, Derringer (1994) formulated an aggregate function that weighted the individual desirability functions based on the degree of importance given to the particular quality characteristic. It is the geometric mean of the individual weighted desirability functions and is given by

$$D = [d_1^{w_1} d_2^{w_2} \ldots d_p^{w_p}]^{1/\Sigma w_i} \qquad (3)$$

For the special case when $w_i = 1$ for all i, Equation (3) yields the geometric mean of the desirability functions.

Some of the other approaches incorporate the concept of the Taguchi loss function (1986) that states that an economic loss is experienced whenever there is any deviation of the quality characteristic from the target value. Taguchi proposed a quadratic loss function given by

$$L(X) = k(X - t)^2 \qquad (4)$$

for a nominal-is-best type characteristic where k is a loss coefficient chosen to best represent the economic loss in the region of interest. While selection of k is not easy, one approach is to find the loss, i.e., cost of rework or scrap associated with a unit of the product, when its quality characteristic value is just outside the acceptable limit, i.e., USL or LSL. The value of k is then calculated using Equation (4).

Artiles-Leon (1996-97) assumed that the relationship between the quality characteristic (Y) and the design variables (X) can be determined. Typically, this is done through regression modeling. They assumed that the target value is at the center of the specifications. Assuming a symmetric loss function around the target value, the constant k in Equation (4) is defined as

$$k = \left(\frac{2}{USL - LSL}\right)^2 \qquad (5)$$

A "standardized" loss function is now given by

$$L(Y) = 4\left[\frac{Y(X) - t}{USL - LSL}\right]^2 \qquad (6)$$

This standardized loss function is dimensionless. Further, if the quality characteristics are uncorrelated, an aggregate standardized loss function may be obtained by summarizing the individual loss functions.

The formulation by Ames et al. (1997) used a general form of the Taguchi loss function and used constraints to capture the specification limits for each characteristic. Their loss function is given by

$$L(Y) = \sum_{i=1}^{p} k_i \{Y_i(X) - t_i\}^2 \tag{7}$$

Kapur and Cho (1996) used a loss function that captures possible correlations between different quality characteristics (Y_i, i = 1,2, … p). The proposed loss function is given by

$$L(Y) = \sum_{i=1}^{p} \sum_{j=1}^{i} k_{ij}(Y_i - t_i)(Y_j - t_j) \tag{8}$$

The loss function formulation by Pignatiello (1993) is similar to that of Kapur and Cho. It allows for the variability of the individual characteristics around the corresponding target values as well as the possible correlations between the quality characteristics. The proposed loss function is given by

$$L(Y) = \sum_{i=1}^{p} k_i [Y_i(X) - t_i]^2 + \sum_{i=1}^{p} \sum_{j=i+1}^{p} k_{ij}[Y_i(X) - t_i][Y_j(X) - t_j] \tag{9}$$

Suhr and Batson (2001) considered a loss function of the quadratic type. They used a quadratic programming approach, with constraints, to find optimal solutions of deterministic decision variables. Ma and Zhao (2004) improved on the loss function proposed by Artiles-Leon and developed loss functions for smaller-is-better and larger-is-better type quality characteristics. They also accounted for the correlation structure among the quality characteristics.

We consider a loss function that captures four components that lead to a loss. These are the following: deviation of the quality characteristic mean from the target value; variability of the characteristic; covariances between pairs of quality characteristics; and the proportion of nonconforming product. Details of these components of the loss function are now developed. Let the half-width of the specifications, considered to be two-sided for each quality characteristic, be given by

$$A_i = (USL_i - LSL_i)/2 , \; i = 1, 2, \ldots , p \tag{10}$$

Let the unit cost to shift the mean of characteristic i be denoted by C_{1i}, $i = 1, 2, \ldots, p$ and the unit cost associated with the variance of characteristic i be given by C_{2i}, $i = 1, 2, \ldots, p$. Further, let the unit cost associated with the covariances between characteristic i and characteristic j be C_{3ij}, $i = 1, 2, \ldots, (p-1)$; $j = i+1, i+2, \ldots, p$. Finally, let the unit cost of a nonconforming product be given by C_4.

In all realistic processes, quality characteristics have to be maintained within certain bounds. The bounds are defined as

$$a_i \leq X_i \leq b_i, \; i = 1,2, \ldots , p \tag{11}$$

Utilizing the form of the Taguchi loss function given by Equation (4), the expected loss due to deviation of the process mean, μ_i, from the target value, t_i, is given by

$$E[L(X_i)] = k_i[(\mu_i - t_i)^2 + \sigma_i^2] \quad (12)$$

Standardizing the deviation of the process mean from the target value, the expected loss due to this deviation is expressed as

$$E[L_1(X_i)] = C_{1i} \left(\frac{\mu_i - t_i}{A_i} \right)^2, \quad i = 1, 2, \ldots, p \quad (13)$$

Hence, summing over all characteristics, the expected loss for this component is

$$E[L_1(X)] = \sum_{i=1}^{p} E[L_1(X_i)] \quad (14)$$

To model the loss in customer satisfaction due to variability in the characteristic, we again use a form of standardization relative to the half-width of the specifications as previously defined. Hence, the loss due to process variance is expressed as

$$E[L_2(X)] = \sum_{i=1}^{p} C_{2i} \left(\frac{\sigma_i^2}{A_i^2} \right) \quad (15)$$

The third component of the loss function incorporates a standardized form of the covariance between pairs of quality characteristics and is given by

$$E[L_3(X)] = \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} C_{3ij} \frac{(\mu_i - t_i)(\mu_j - t_j)}{A_i A_j} \quad (16)$$

To derive the proportion nonconforming of the product, we first normalize the multivariate normal density function. The usual multivariate normal density function, with no constraints on the quality characteristics, is given by

$$f(\underline{X}) = \frac{1}{(2\pi)^{1/2}} \frac{1}{|\underline{\Sigma}|^{1/2}} \exp[-(\underline{X} - \underline{\mu})' \underline{\Sigma}^{-1} (\underline{X} - \underline{\mu})] \quad (17)$$

where $\underline{\mu}$ represents the process mean and $\underline{\Sigma}$ is the variance-covariance matrix associated with the characteristics.

Given that each quality characteristic is contained within the bounds given by Equation (11), consider the truncated multivariate normal distribution. We define the following:

$$\text{TR} = \int_{X_1 = a_1}^{b_1} \int_{X_2 = a_2}^{b_2} \cdots \int_{X_p = a_p}^{b_p} f(\underline{X}) dX_1 \, dX_2 \cdots dX_p \quad (18)$$

Now, the truncated multivariate normal density is given by

$$f_{TR}(\underline{X}) = \frac{1}{TR} f(\underline{X}), \tag{19}$$

from which the proportion of nonconforming product is obtained as

$$\Phi_{NC} = 1 - \left[\int_{X_1 = LSL_1}^{USL_1} \int_{X_2 = LSL_2}^{USL_2} \cdots \int_{X_p = LSL_p}^{USL_p} f_{TR}(\underline{X}) dX_1 \, dX_2 \cdots dX_p \right] \tag{20}$$

Hence, the fourth component of the loss function representing the cost of nonconforming product is

$$E[L_4(\underline{X})] = C_4 \, \Phi_{NC} \tag{21}$$

The overall expected cost is obtained as the sum of the four components and expressed as

$$E[L(\underline{X})] = E[L_1(\underline{X})] + E[L_2(\underline{X})] + E[L_3(\underline{X})] + E[L_4(\underline{X})] \tag{22}$$

3. Process Constraints and Model Formulation

Let us discuss typical constraints associated with processes. First, deviations of the process mean from the target value for a given quality characteristic are usually maintained within some desirable bounds. Assuming deviations can take place on either side of the target value, the constraints are expressed as

$$-\alpha_i \leq \frac{\mu_i - t_i}{A_i} \leq \alpha_i, \quad i = 1, 2, ..., p \tag{23}$$

where α_i represents the maximum standardized deviation that the process may drift.

Second, there are feasible bounds on the process variance for each quality characteristic. Process variability is influenced by many factors such as tolerance of the equipment or machinery used in the manufacturing process, operator variability, and variability in incoming raw material or components. Budget limitations create a lower bound on the process variance associated with the quality characteristic. Also, since the characteristic has to meet certain customer tolerances, an upper bound is also mandated. Hence, the relevant constraints on process variance are

$$\sigma_{i\min}^2 \leq \sigma_i^2 \leq \sigma_{imax}^2, \; i = 1, 2, \ldots, p \tag{24}$$

The third set of constraints is on the individual quality characteristics. Based on customer tolerances, the characteristics are constrained between certain lower and upper bounds as given by Equation (11).

The constrained optimization model is given by: Minimize E[L($\underline{X}$)] given by Equation (22), subject to the set of constraints given by Equations (11), (23), and (24).

4. Results

For illustrative purposes, the following parameter values are utilized. The number of quality characteristics (p) is 2 and the target values are $t_1 = 2$, $t_2 = 4$. The specifications on the first quality characteristic are: $USL_1 = 3$, $LSL_1 = 1$, while those for the second quality characteristic are $USL_2 = 6$, $LSL_2 = 2$. These selected values yield half-widths of the specifications as $A_1 = 1$ and $A_2 = 2$. The standardized bounds on the process means from the target values are selected as $\alpha_1 = 0.5$ and $\alpha_2 = 1.0$, respectively. The bounds on the quality characteristics are as follows: $a_1 = 0.5$, $b_1 = 6$, $a_2 = 1.5$, $b_2 = 12$.

The bounds on process variances are chosen as $\sigma^2_{1\min} = 1, 2$; $\sigma^2_{1\max} = 8$; $\sigma^2_{2\min} = 2, 3$; $\sigma^2_{2\max} = 12$. Rather than treat the co-variance between the two variables independently, we select a value of the correlation between the two characteristics since such may be obtained from historical data from the process. Since $\sigma_{12} = \rho_{12}\sqrt{\sigma_1^2}\ \sqrt{\sigma_2^2}$, by selecting values of ρ_{12}, the covariance is expressed as a function of the variances. The chosen values of ρ_{12} are – 0.2 and – 0.8, respectively. Unit costs associated with the mean shift from the target are chosen to be as follows: $C_{11} = 10$, $C_{12} = 16$. For the process variances, the unit costs are $C_{21} = 20, 30$; $C_{22} = 50, 60$. For the covariance between the two characteristics, the unit cost is chosen as $C_{312} = 50$. The unit cost of nonconforming product is selected as $C_4 = 200, 400$.

The decision variables in the constrained optimization problem are the process means, μ_1 and μ_2, and the process variances, σ_1^2 and σ_2^2. For the following unit cost values: $C_{11} = 10$, $C_{12} = 16$, $C_{21} = 20$, $C_{22} = 50$, $C_{312} = 50$, $C_4 = 200$, the lower bounds on the process variances: $\sigma^2_{2\min} = 1$, $\sigma^2_{2\min} = 2$, and the correlation between the two quality characteristics $\rho_{12} = -0.2$, the optimal solution obtained is as follows: $\mu_1 = 2.05$, $\mu_2 = 4.10$, $\sigma_1^2 = 1$, and $\sigma_2^2 = 2$, with an objective function value of 443.42. For all other combinations of the unit cost parameter values, the optimal solution point did not change.

It seems that the binding constraints are those related to the process variances associated with the two quality characteristics. The optimal solution has the process variances at their lower bound. The high unit cost associated with nonconforming product, relative to the other unit costs, could be a reason. Creating a reduction in the process variability is typically resource intensive. Further, detailed study and analysis of the various process steps and those that contribute more to variability are necessary. The necessary actions require much effort relative to changing the process mean associated with the quality characteristics.

5. Conclusions

A process with multiple variables has been considered where the decision variables are the means and variances of the associated quality characteristics. Four components are considered in developing a loss function. These are the deviation of the process mean from a desired target value, the process variability, the covariance between quality characteristics, and the proportion of nonconforming product, respectively. All processes operate in an environment of resource constraints. Hence, constraints on the quality characteristics, permissible deviation from the target value, and variance of the quality characteristic, respectively, are integrated into a constrained optimization model.

The unit costs associated with changing the process mean, process variability, and proportion nonconforming influence the optimal solution. Under the selected values of the unit cost parameters that influence the loss function, the variances of the quality characteristics determine the binding constraints. Variability in the characteristics also influences the proportion of nonconforming product. Hence, the ability to monitor and control variability of the individual characteristics through better equipment, raw material and components, streamlined methods and procedures, and consistent operator performance, specially in processes that are predominantly manual, will lead to minimizing loss.

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