Lower Prediction and Tolerance Bounds in Accelerated Life Testing for the Rayleigh Distribution

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Abstract: The problem of obtaining lower prediction and tolerance bounds for a future observation from a Rayleigh population at field use (design) level of stress, using Type II censored accelerated life test data from higher than design stress levels is considered. Maximum Likelihood Predictive Density method to derive a predictive density for a future observation as described by Jayawardhana and Samaranayake (2003) is used for this study. The mean life of the Rayleigh distribution is assumed to have an inverse power relationship with the level of stress. The use of lower percentile points of the predictive density as a lower prediction and tolerance bounds is investigated using Monte Carlo simulation. The results show that reasonable prediction and tolerance bounds can be provided using the predictive density.

Key words: Prediction Intervals, Tolerance Intervals, Rayleigh Distribution, Maximum Likelihood Predictive Density

1. Introduction

Most of the modern products are designed with high quality to last a long time under normal working conditions. These products are subjected to stresses such as humidity, temperature, voltage, pressure, and use rate. Testing under normal working conditions will be time consuming and not very useful in a continuously improving production process. Accelerated life tests (ALT) are designed to collect data in a timely manner under high levels of stress. Information from tests at high levels of stress is extrapolated through a physically reasonable statistical model, to obtain estimates of life at lower, normal levels of stress (Escobar and Meeker 1995). These tests are used to characterize durability properties or the life distribution of materials or sample components (Meeker and Escobar 1998). In production processes, estimating a lower quantile is of interest for reasons such as warranty assurance.

Most parametric ALT models have the following two components: 1) A parametric distribution for the life of a population of units at a particular level(s) of an experimental variable or variables; and 2) A relationship between one (or more) of the distribution parameters and the acceleration or other variables (Meeker and Escobar 1998). We assume a Rayleigh life distribution and the mean life of the Rayleigh distribution to be inversely related to the levels of stress. We also assume that the two factors have no statistical interaction. Escobar and Meeker (2006) describe that the inverse power relationship is generally considered as an empirical model because it has no formal basis from knowledge of the physics/chemistry of the modeled failure modes. They further state that the inverse power relationship is commonly used because the engineers have found that it often provides a useful description of certain kinds of accelerated test data, for example with factors such as pressure, voltage, cycling rate, electric current, and humidity. The Raleigh distribution is widely used in communications engineering (Akhter and Hirai, 2009).

For predicting a lower bound for a single future observation from a Weibull distribution, Jayawardhana and Samaranayake (2003) used the method of Maximum Likelihood Predictive Density (MLPD) with one stress factor and two levels of acceleration under Type II censoring. In 2004, using the same method, Jayawardhana and Samaranayake explored two stress factor ALT experiments using the Exponential life distribution. In the current paper, the results are further extended to two stress factor ALT experiments with two levels of higher than the design-factor levels of stress in each factor using Rayleigh distribution. According to Escobar and Meeker (1995), there is both practical and physical motivation for ALT models without interactions and one should choose factor definitions to minimize statistical interactions among the factors.

2. Maximum Likelihood Predictive Density

Suppose $\tilde{X} = \left(X_1, X_2, ..., X_n\right)'$ is a set of observations from a distribution $f(x;\theta)$ and $\tilde{Y} = \left(Y_1, Y_2, ..., Y_m\right)'$ is a set of future observations, independent of $\tilde{X}$, from the same distribution. Let Z be some statistic based on $\tilde{Y}$. Our objective is to find an estimate of the density of Z based on the observed values $\tilde{X}$.

Lejeune and Faulkenberry (1982) proposed the Maximum Likelihood Predictive Density function $\hat{f}(z\,|\,\tilde{x}) = k(\tilde{x})\sup_{\theta\in\Theta} f(\tilde{x};\theta)g(z;\theta)$ as an estimated density function of Z, where $f(\tilde{x};\theta)$ is the joint probability density of X's, $g(z;\theta)$ is the probability density function of the statistic Z, Θ is the parameter space of unknown parameter θ, and $k\left(\tilde{x}\right)$ is a normalizing constant. Parameter θ is then replaced in function $\hat{f}$ by its Maximum Likelihood Estimate (MLE), $\hat{\theta}$. The estimator $\hat{\theta}$ is a function of both $\tilde{X}$ and Z. Resulting predictive density is the MLPD of Z.

We propose our method in the following section but make a deviate from the true MLPD method. Our MLE, $\hat{\theta}$ has two other unknown parameters. At the beginning we proceed as we know the two parameters. If these parameters are estimated using both $\tilde{X}$ and Z, the resulting predictive distribution does not have a recognizable from. Instead, we use only $\tilde{X}$ to estimate the two extra unknown parameters. This makes our method different from the method proposed by Lejeune and Faulkenberry (1982).

3. The Proposed Method

Suppose the mean life of a product is dependent on two stress factors, factor 1 and factor 2. Each factor has three levels: design, low, and high, denoted by D, L, and H, respectively. Let the observation X_{ijk}, denotes the product life of k^{th} test item subject to stress level i of factor 1 and stress level j of factor 2. For example, X_{LH4} denotes the 4th test item subject to low level of stress of factor 1 and high level of stress of factor 2. Correspondingly, θ_{ij} denote the mean lifetime of product under stress level i of factor 1 and the stress level j of factor 2. In addition, let V_{1i} denote the stress value at level i of factor 1, and V_{2i} denote stress value at level i of factor 2.

We make the following model assumptions:

1) Product life has a Rayleigh distribution with p.d.f. $f(x) = \frac{x}{\sigma^2} e^{-\left(x^2/2\sigma^2\right)}$, $x > 0$, $\sigma > 0$.
2) Mean lifetime $\theta_{ij} = \sigma_{ij}\sqrt{\pi/2}$ is related to the stress levels by $\ln\theta_{ij} = \ln\eta_0 - \eta_1 \ln V_{1i} - \eta_2 \ln V_{2j}$, where $\eta_0 > 0$, $\eta_1 > 0$, $\eta_2 > 0$, and $i, j \in \{D, L, H\}$.
3) The product lifetimes are independent of each other.
4) Without loss of generality, we assume $V_{1D} = V_{2D} = 1$. Note that under such assumptions $\theta_{DD} = \eta_0$.
5) We assume that $V_{1D} < V_{1L} < V_{1H}$ and $V_{2D} < V_{2L} < V_{2H}$, and hence $\theta_{DD} > \theta_{LH} > \theta_{HH}$ and $\theta_{DD} > \theta_{HL} > \theta_{HH}$.

Suppose that a total of n_{LH} items are subject to test at low level of factor 1 and high level of factor 2. Let $x_{LH1}, x_{LH2}, \ldots, x_{LHr_{LH}}$ be the first r_{LH} ordered failure times among all n_{LH} test items. Similarly, assume n_{HL} items are subjected to test at high level of factor 1 and low level of factor 2 and n_{HH} items are subjected to test at high levels of both factors. And let $x_{HL1}, x_{HL2}, \ldots, x_{HLr_{HL}}$ be the first r_{HL} ordered failure times among all n_{HL} test items, let $x_{HH1}, x_{HH2}, \ldots, x_{HHr_{HH}}$ be the first r_{HH} ordered failure times among all n_{HH} test items. Note that we do not include the case of low stress levels on both factors.

Let Z be a single future observation obtained at designed levels of both factors. The probability density function of Z is: $f(z) = \frac{z}{\sigma_{DD}^2} e^{-\left(z^2 / 2\sigma_{DD}^2\right)}$, $z > 0$. The relationship between σ_{ij} and θ_{ij} is: $\theta_{ij} = \sqrt{\pi / 2}\sigma_{ij}$.

Let $\tilde{x} = \left(x_{LH1}, x_{LH2}, \ldots, x_{LHr_{LH}}, x_{HL1}, x_{HL2}, \ldots, x_{HLr_{HL}}, x_{HH1}, x_{HH2}, \ldots, x_{HHr_{HH}}\right)'$.

Then the joint likelihood function of σ_{DD}, σ_{LH}, σ_{HL}, and σ_{HH} given $\tilde{x}$ and z can be written as:

$$L(\sigma_{DD}, \sigma_{LH}, \sigma_{HL}, \sigma_{HH} \mid \tilde{x}, z)$$

$$= \frac{(n_{LH})!}{(n_{LH} - r_{LH})!} \frac{\prod_{i=1}^{r_{LH}} x_{LH_i}}{\sigma_{LH}^{2r_{LH}}} \exp\left[-\frac{(n_{LH} - r_{LH}) x_{LH_{r_{LH}}}^2 + \sum_{i=1}^{r_{LH}} x_{LH_i}^2}{2\sigma_{LH}^2} \right]$$

$$\times \frac{(n_{HL})!}{(n_{HL} - r_{HL})!} \frac{\prod_{i=1}^{r_{HL}} x_{HL_i}}{\sigma_{HL}^{2r_{HL}}} \exp\left[-\frac{(n_{HL} - r_{HL}) x_{HL_{r_{HL}}}^2 + \sum_{i=1}^{r_{HL}} x_{HL_i}^2}{2\sigma_{HL}^2} \right]$$

$$\times \frac{(n_{HH})!}{(n_{HH} - r_{HH})!} \frac{\prod_{i=1}^{r_{HH}} x_{HH_i}}{\sigma_{HH}^{2r_{HH}}} \exp\left[-\frac{(n_{HH} - r_{HH}) x_{HH_{r_{HH}}}^2 + \sum_{i=1}^{r_{HH}} x_{HH_i}^2}{2\sigma_{HH}^2} \right]$$

$$\times \frac{z}{\sigma_{DD}^2} \exp\left(-\frac{z^2}{2\sigma_{DD}^2} \right)$$

For the Rayleigh distribution, the mean lifetime is given by $\theta_{ij} = \sqrt{\frac{\pi}{2}}\sigma_{ij} = \frac{\eta_0}{V_{1i}^{\eta_1} V_{2j}^{\eta_2}}$. By making a substitution $\eta_0^* = \frac{\eta_0}{\sqrt{\pi/2}}$, we have $\sigma_{ij} = \frac{\eta_0^*}{V_{1i}^{\eta_1} V_{2j}^{\eta_2}}$. Without loss of generality, we let $\eta_0^* = 1$ in simulations which makes $\sigma_{DD} = 1$.

Using the inverse power law relationship, the joint likelihood function can be re-written as

$$L(\eta_0^*, \eta_1, \eta_2 \mid \tilde{x}, z) = \frac{n_{LH}!}{(n_{LH} - r_{LH})!} \frac{n_{HL}!}{(n_{HL} - r_{HL})!} \frac{n_{HH}!}{(n_{HH} - r_{HH})!}$$

$$\times \left(\prod_{i=1}^{r_{LH}} x_{LH_i} \right) \left(\prod_{i=1}^{r_{HL}} x_{HL_i} \right) \left(\prod_{i=1}^{r_{HH}} x_{HH_i} \right)$$
$$\times \left[\frac{V_{1L}^{2\eta_1 r_{LH}} V_{2H}^{2\eta_2 r_{LH}} V_{1H}^{2\eta_1 r_{HL}} V_{2L}^{2\eta_2 r_{HL}} V_{1H}^{2\eta_1 r_{HH}} V_{2H}^{2\eta_2 r_{HH}} z}{\left(\eta_0^*\right)^{2(N+1)}} \right]$$

$$\times \exp\left[-\frac{V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH} + z^2}{2\left(\eta_0^*\right)^2} \right] \quad (1)$$

where $N = r_{LH} + r_{HL} + r_{HH}$, $A_{LH} = (n_{LH} - r_{LH}) x_{LH_{r_{LH}}}^2 + \sum_{i=1}^{r_{LH}} x_{LH_i}^2$,

$A_{HL} = (n_{HL} - r_{HL}) x_{HL_{r_{HL}}}^2 + \sum_{i=1}^{r_{HL}} x_{HL_i}^2$ and $A_{HH} = (n_{HH} - r_{HH}) x_{HH_{r_{HH}}}^2 + \sum_{i=1}^{r_{HH}} x_{HH_i}^2$.

Taking natural log of $L(\eta_0^*, \eta_1, \eta_2 \mid \tilde{x}, z)$, we get

$$\ln L\left(\eta_0^*, \eta_1, \eta_2 \mid \tilde{x}, z\right) = C + 2\eta_1 \left(r_{LH} \ln V_{1L} + r_{HL} \ln V_{1H} + r_{HH} \ln V_{1H} \right)$$
$$+ 2\eta_2 \left(r_{LH} \ln V_{2H} + r_{HL} \ln V_{2L} + r_{HH} \ln V_{2H} \right) - 2\left(N+1\right) \ln \eta_0^*$$
$$- \frac{V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH} + z^2}{2\left(\eta_0^*\right)^2},$$

where C is independent of all the parameters. Taking the partial derivative with respect to η_0^* and solving for $\frac{\partial}{\partial \eta_0^*}[\ln L(\eta_0^*, \eta_1, \eta_2 \mid \tilde{x}, z)] = 0$, we obtain the MLE of η_0^*,

$$\hat{\eta}_0^* = \sqrt{\frac{V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH} + z^2}{2(N+1)}}.$$

Substituting the MLE of η_0^* back in the joint likelihood function in equation (1), we get the predictive density function of z, given by

$$\hat{f}(z \mid \tilde{x}, \eta_1, \eta_2) = \frac{k(\tilde{x}) z}{\left(V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH} + z^2\right)^{N+1}},$$

where $k(\tilde{x})$ is a proportionality constant.

By letting $\int_0^\infty \hat{f}(z \mid \tilde{x}, \eta_0, \eta_1, \eta_2) dz = 1$, we can easily show that the proportionality constant $k(\tilde{x}) = 2N\left(V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH}\right)^N$.

Assuming η_1 and η_2 are known, we have

$$\hat{f}(z \mid \tilde{x}, \eta_1, \eta_2) = \frac{2N\left(V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH}\right)^N z}{\left(V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH} + z^2\right)^{N+1}},$$

$$z > 0. \qquad (2)$$

Consider the transformation $u = z^2$, then $du = 2z dz$, $|J| = \left|\frac{dz}{du}\right| = \frac{1}{2z}$ and

$$g(u) = \frac{N\left(V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH}\right)^N}{\left(V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH} + u\right)^{N+1}}, \; u > 0.$$

Which is in standard notation a Pareto distribution with $\kappa = N$ and
$\alpha = V_{1L}^{2\eta_1} V_{2H}^{2\eta_2} A_{LH} + V_{1H}^{2\eta_1} V_{2L}^{2\eta_2} A_{HL} + V_{1H}^{2\eta_1} V_{2H}^{2\eta_2} A_{HH}$.

At this stage, we need to estimate parameters η_1 and η_2 but we use only the past data to obtain the maximum likelihood estimates of η_1 and η_2. Let the maximum Likelihood estimates of the parameters η_1 and η_2 be $\tilde{\eta}_1$ and $\tilde{\eta}_2$ respectively. The likelihood function of η_0^*, η_1 and η_2 given $\tilde{x}_{LH} = \left(x_{LH_1}, x_{LH_2}, \ldots, x_{LH_{r_{LH}}}\right)'$ is

$$L(\eta_0^*,\eta_1,\eta_2 \mid \tilde{x}_{LH}) = \frac{n_{LH}!}{(n_{LH}-r_{LH})!}\left[\prod_{i=1}^{r_{LH}} x_{LH_i}\right]\left[\frac{V_{1L}^{2\eta_1 r_{LH}} V_{2H}^{2\eta_2 r_{LH}}}{\left(\eta_0^*\right)^{2r_{LH}}}\right]\exp\left[-\frac{V_{1L}^{2\eta_1}V_{2H}^{2\eta_2}A_{LH}}{2\left(\eta_0^*\right)^2}\right]$$

.

Taking natural log of the likelihood function above, we get

$$\ln L(\eta_0^*,\eta_1,\eta_2 \mid \tilde{x}_{LH}) = C + 2\eta_1 r_{LH}\ln V_{1L} + 2\eta_2 r_{LH}\ln V_{2H} - 2r_{LH}\ln\eta_0^* - \frac{V_{1L}^{2\eta_1}V_{2H}^{2\eta_2}A_{LH}}{2\left(\eta_0^*\right)^2}$$

,

where C is independent of η_0^*, η_1 and η_2. Similar expressions can be derived using the data $\tilde{x}_{HL} = \left(x_{HL1}, x_{HL2}, \ldots, x_{HL_{r_{HL}}}\right)'$ and $\tilde{x}_{HH} = \left(x_{HH1}, x_{HH2}, \ldots, x_{HH_{r_{HH}}}\right)'$.

Taking Partial derivative with respect to η_0^*, η_1, or η_2 and solving for zero simultaneously , we obtain the three equations

$$2r_{LH}\left(\tilde{\eta}_0^*\right)^2 = V_{1L}^{2\tilde{\eta}_1}V_{2H}^{2\tilde{\eta}_2}A_{LH},$$

(3)

$$2r_{HL}\left(\tilde{\eta}_0^*\right)^2 = V_{1H}^{2\tilde{\eta}_1}V_{2L}^{2\tilde{\eta}_2}A_{HL},$$

(4)

and $$2r_{HH}\left(\tilde{\eta}_0^*\right)^2 = V_{1H}^{2\tilde{\eta}_1}V_{2H}^{2\tilde{\eta}_2}A_{HH}.$$

(5)

Dividing equation (3) by equation (5) we get $\frac{r_{LH}}{r_{HH}} = \left(\frac{V_{1L}}{V_{1H}}\right)^{2\tilde{\eta}_1}\frac{A_{LH}}{A_{HH}}$. Taking natural log on both sides and solving for $\tilde{\eta}_1$, we get

$$\tilde{\eta}_1 = \frac{\ln\left(r_{LH}A_{HH}\right) - \ln\left(r_{HH}A_{LH}\right)}{2\left(\ln V_{1L} - \ln V_{1H}\right)}.$$

(6)

Similarly, dividing equation (4) by equation (5) and solving for $\tilde{\eta}_2$, yields

$$\tilde{\eta}_2 = \frac{\ln\left(r_{HL}A_{HH}\right) - \ln\left(r_{HH}A_{HL}\right)}{2\left(\ln V_{2L} - \ln V_{2H}\right)}.$$

(7)

Then the predictive density of U given in equation (2) can be modified to

$$\tilde{g}(u) = \frac{N\left(V_{1L}^{2\tilde{\eta}_1}V_{2H}^{2\tilde{\eta}_2}A_{LH} + V_{1H}^{2\tilde{\eta}_1}V_{2L}^{2\tilde{\eta}_2}A_{HL} + V_{1H}^{2\tilde{\eta}_1}V_{2H}^{2\tilde{\eta}_2}A_{HH}\right)^N}{\left(V_{1L}^{2\tilde{\eta}_1}V_{2H}^{2\tilde{\eta}_2}A_{LH} + V_{1H}^{2\tilde{\eta}_1}V_{2L}^{2\tilde{\eta}_2}A_{HL} + V_{1H}^{2\tilde{\eta}_1}V_{2H}^{2\tilde{\eta}_2}A_{HH} + u\right)^{N+1}},\ u>0$$

An approximate 100 p^{th} percentile point for $u = z^2$ is given by

$$\hat{u}_p = \left(V_{1L}^{2\tilde{\eta}_1} V_{2H}^{2\tilde{\eta}_2} A_{LH} + V_{1H}^{2\tilde{\eta}_1} V_{2L}^{2\tilde{\eta}_2} A_{HL} + V_{1H}^{2\tilde{\eta}_1} V_{2H}^{2\tilde{\eta}_2} A_{HH}\right)\left[(1-p)^{-\frac{1}{N}} - 1\right].$$

Therefore, 100 p^{th} percentile point for z can be obtained by

$$\hat{z}_p = \left\{\left(V_{1L}^{2\tilde{\eta}_1} V_{2H}^{2\tilde{\eta}_2} A_{LH} + V_{1H}^{2\tilde{\eta}_1} V_{2L}^{2\tilde{\eta}_2} A_{HL} + V_{1H}^{2\tilde{\eta}_1} V_{2H}^{2\tilde{\eta}_2} A_{HH}\right)\left[(1-p)^{-\frac{1}{N}} - 1\right]\right\}^{\frac{1}{2}}. \quad (8)$$

In previous studies, it has been observed that $\hat{z}_p$ is slightly overestimated and therefore we propose an adjustment to the equation (8) by replacing N by $N+6$.

$$\tilde{z}_p = \left\{\left(V_{1L}^{2\tilde{\eta}_1} V_{2H}^{2\tilde{\eta}_2} A_{LH} + V_{1H}^{2\tilde{\eta}_1} V_{2L}^{2\tilde{\eta}_2} A_{HL} + V_{1H}^{2\tilde{\eta}_1} V_{2H}^{2\tilde{\eta}_2} A_{HH}\right)\left[(1-p)^{-\frac{1}{N+6}} - 1\right]\right\}^{\frac{1}{2}}. \quad (9)$$

4. Monte Carlo Simulation for the Prediction Interval

We limited our simulation study to investigate the coverage probabilities of 99th, 95th, and 90th lower percentile points. Using Monte Carlo simulation, we calculated $E[P(Z > \hat{z}_p \mid \tilde{x})]$, where $\hat{z}_p$ is the $100p^{th}$ percentile point of the predictive density. Without loss of generality we assumed $\eta_0^* = 1$. We simulated data using $\eta_0^* = 1$, $\eta_1 = 1.25$ and $\eta_2 = 1.25$; $\eta_0^* = 1$, $\eta_1 = 1.5$ and $\eta_2 = 1.5$; and $\eta_0^* = 1$, $\eta_1 = 2.0$ and $\eta_2 = 2.0$. Since the different values of η_1 and η_2 produced similar coverages, we only report the results for $\eta_0^* = 1$, $\eta_1 = 1.5$ and $\eta_2 = 1.5$. Combinations of higher and lower levels of acceleration $V_{1L} = V_{2L} = 1.125(0.125)2.000$ and $2V_{1L} \le V_{1H} \le V_{2H}$ $= 2.0(0.5)5.0$ were used in the simulations. For each set of stress levels, the censoring schemes $n_{LH} = n_{HL} = n_{HH} = 10(10)50$ with 50%, 20% and no censoring were used.

For each combination of stress levels V_{1L}, V_{1H}, V_{2L} and V_{2H}, sample sizes n_{LH}, n_{HL}, n_{HH}, and preplanned censoring values r_{LH}, r_{HL}, r_{HH} Rayleigh random numbers were generated. Using equations (6) and (7) $\tilde{\eta}_1$ and $\tilde{\eta}_2$ were calculated. Then for each combination of V_{1L}, V_{1H}, V_{2L}, V_{2H}, n_{LH}, n_{HL}, n_{HH}, r_{LH}, r_{HL}, r_{HH} the percentile estimate $\hat{z}_p$ of the predictive distribution was estimated using the equation (8). Using the theoretical Rayleigh distribution we calculated the probability a Rayleigh random variable is greater than $\hat{z}_p$. This process was repeated 5000 times and the average coverage was calculated to estimate $E\left[P\left(Z > \hat{z}_p \mid \tilde{x}\right)\right]$. Simulation studies reveal that $\hat{z}_p$ is slightly over estimated and the coverages are slightly lower than expected. When the lower level of the stress is as close as possible to the design level of stress and the higher level of stress is as high as possible the results are reasonable. There are practical limitations to have the lower level of stress as close as to the design level of stress because the items may not fail during a reasonable period of time. On the other hand raising the higher level of stress may not physically possible in an experimental situation. This observation can be summarized as the coverage is reasonable when V_{iH} / V_{iL}^2 is large. We used Equation (9) to estimate the percentile $\tilde{z}_p$ and the respective coverages are reported as modified coverages within square brackets in Table 1 in Appendix A.

5. Lower Tolerance Intervals

Lower tolerance intervals can be represented as a lower confidence interval for a percentile point. Let the $(1-p)$-content γ-level lower tolerance bound for the future observation Z is given by $\hat{z}_p$ where $P\left(\int_{\hat{z}_p}^{\infty} f(z)\,dz \geq 1-p\right) \geq \gamma$. One can derive the lower tolerance limit as a lower confidence limit for a percentile point. Using the definition of the $\beta-$ content $\gamma-$confidence tolerance interval

$P\left[\int_{L(x_1,x_2,..,x_n)}^{\infty} f(x;\theta)\,dx \geq 1-p\right] \geq \gamma$ a simple derivation will produce the result $P\left[L(x_1,x_2,..,x_n) \leq Z_p\right] \geq \gamma$. If one can find out the $1-\gamma$ th percentile of the distribution of the p th percentile of Z, it will be equal to the required tolerance limit. An estimate of the p th percentile of Z is given by

$$\hat{Z}_p \approx \left\{ \left(V_L^{\hat{\eta}\hat{\beta}} \hat{A}_L + V_H^{\hat{\eta}\hat{\beta}} \hat{A}_H \right) \left[(1-p)^{-1/(r_L + r_H + 4)} - 1 \right] \right\}^{1/\hat{\beta}}.$$

Calculation of the lower confidence limit for this percentile point is difficult due to the complexity of the estimates. In this study we use the content corrected tolerance interval proposed by Fernholz & Gillespie (2001). For parameter values p and γ on (0, 1), a γ-confidence, $(1-p)$-content corrected lower tolerance interval is an interval of the form $[L, \infty]$ if $P\{1 - F(L) \geq 1 - p^*\} \geq \gamma$ holds for some data dependent p^* in which the sample comes from the distribution function F. Our approach will be to find p^* through simulation.

6. Monte Carlo Simulation for the Tolerance Interval

Using the same parameter combinations used in Section 4, we generated $\hat{Z}_p$ for $p = 0.0025(0.0025)0.1$ and calculated $P\left(\int_{\hat{z}_p}^{\infty} f(z)\,dz \geq 0.90 \right)$ and $P\left(\int_{\hat{z}_p}^{\infty} f(z)\,dz \geq 0.95 \right)$.

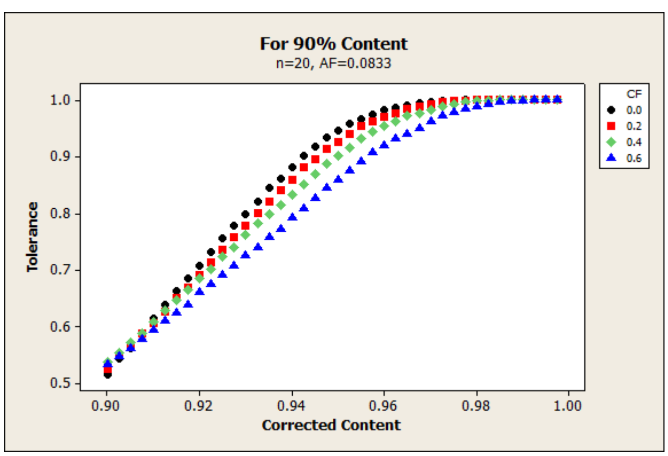

Figure 1

Figure 1, is an example of the coverage level for different values of corrected content and the corresponding level of tolerance for sample size of 20 and acceleration factor of 0.0833. If one wants a γ of 80%, a 94% corrected content value (or a p^* value of 0.06) is required when censoring factor is 0.6. But if the censoring factor is 0, then a 93% corrected content (or a p^* value of 0.07) is required to achieve a γ of 80%. Simulation results are summarized as p^* is the positive solution to the quadratic equation $\gamma = \text{Constant} + a\text{Cen} + b\text{Acc} + cp^* + d\left(p^*\right)^2$, where $Cen = \text{Censoring Factor}$ and Acc = Acceleration Factor. Simulation study was limited to same acceleration in both factors and same censoring in each combination of levels. Tables 2 and 3 in Appendix A provide the coefficients for the quadratic equation.

7. Conclusion

A simple method of calculating a lower prediction interval for a single future observation using an accelerated experiment of two factors is proposed. For larger samples, the modified method gives very reasonable results. Simulation results show that the results are not good for smaller sample sizes especially when Type II censoring is done. Simulation results also demonstrate that the coverage probabilities are liberal when $V_{iH} < 1.5\, V_{DD}$ and $6\left(V_{iL} - V_{iD}\right) < \left(V_{iH} - V_{iL}\right)$. One has to make sure that the inverse power rule works for the range of acceleration of both factors. Proper care should be taken to select the lower and higher levels of stress to have enough failures and not to exceed the physical limits of the units under the stress.

When the Acceleration Factor is greater than 0.17, this method for tolerance interval does not work well. This is mostly due to the estimation errors of the parameters of the predictive distribution. When the Censoring Factor is greater than 50%, this method does not work well either. Tolerance intervals with $0.850 \le \gamma \le 0.975$, the proposed method works well.

Appendix A

Table 1: Average of the Coverage Probabilities $P(Z > \hat{z}_p)$ and $P(Z > \tilde{z}_p)$

n_{LH}	n_{HL}	n_{HH}	r_{LH}	r_{HL}	r_{HH}	$P(Z > \hat{z}_{0.10})$	$P(Z > \hat{z}_{0.05})$ Unmodified [Modified]	$P(Z > \hat{z}_{0.05})$
$V_{1L} = 1.125$ $V_{1H} = 2.00$ $V_{2L} = 1.125$ $V_{2H} = 2.00$								
10	10	10	5	5	5	0.866 [0.901]	0.929 [0.948]	0.985 [0.989]
10	10	10	8	8	8	0.878 [0.899]	0.937 [0.948]	0.987 [0.989]
10	10	10	10	10	10	0.883 [0.900]	0.940 [0.949]	0.988 [0.990]
20	20	20	10	10	10	0.882 [0.900]	0.940 [0.949]	0.988 [0.990]
20	20	20	16	16	16	0.890 [0.902]	0.944 [0.951]	0.989 [0.990]
20	20	20	20	20	20	0.891 [0.901]	0.945 [0.950]	0.989 [0.990]
50	50	50	25	25	25	0.894 [0.901]	0.947 [0.950]	0.989 [0.990]
50	50	50	40	40	40	0.896 [0.900]	0.948 [0.950]	0.990 [0.990]
50	50	50	50	50	50	0.897 [0.899]	0.948 [0.950]	0.990 [0.990]
$V_{1L} = 1.125$ $V_{1H} = 3.00$ $V_{2L} = 1.125$ $V_{2H} = 3.00$								
10	10	10	5	5	5	0.869 [0.905]	0.931 [0.951]	0.986 [0.990]
10	10	10	8	8	8	0.883 [0.905]	0.940 [0.952]	0.988 [0.990]
10	10	10	10	10	10	0.888 [0.905]	0.943 [0.952]	0.988 [0.990]
20	20	20	10	10	10	0.885 [0.904]	0.941 [0.952]	0.988 [0.990]
20	20	20	16	16	16	0.891 [0.902]	0.945 [0.951]	0.989 [0.990]
20	20	20	20	20	20	0.893 [0.903]	0.946 [0.951]	0.989 [0.990]
50	50	50	25	25	25	0.895 [0.903]	0.947 [0.951]	0.989 [0.990]
50	50	50	40	40	40	0.897 [0.901]	0.948 [0.950]	0.990 [0.990]
50	50	50	50	50	50	0.897 [0.901]	0.948 [0.950]	0.990 [0.990]
$V_{1L} = 1.375$ $V_{1H} = 3.00$ $V_{2L} = 1.375$ $V_{2H} = 3.00$								
10	10	10	5	5	5	0.846 [0.878]	0.914 [0.933]	0.980 [0.985]
10	10	10	8	8	8	0.863 [0.887]	0.927 [0.941]	0.985 [0.988]
10	10	10	10	10	10	0.872 [0.892]	0.933 [0.944]	0.986 [0.988]
20	20	20	10	10	10	0.872 [0.891]	0.933 [0.944]	0.986 [0.988]
20	20	20	16	16	16	0.882 [0.895]	0.939 [0.947]	0.988 [0.989]
20	20	20	20	20	20	0.886 [0.896]	0.942 [0.948]	0.988 [0.989]
50	50	50	25	25	25	0.890 [0.897]	0.945 [0.948]	0.989 [0.990]
50	50	50	40	40	40	0.893 [0.898]	0.946 [0.949]	0.989 [0.990]
50	50	50	50	50	50	0.895 [0.899]	0.947 [0.949]	0.989 [0.990]

n_{LH}	n_{HL}	n_{HH}	r_{LH}	r_{HL}	r_{HH}	$P(Z > \hat{z}_{0.10})$	$P(Z > \hat{z}_{0.05})$ Unmodified [Modified]	$P(Z > \hat{z}_{0.05})$
$V_{1L} = 1.375$	$V_{1H} = 4.00$	$V_{2L} = 1.375$	$V_{2H} = 4.00$					
10	10	10	5	5	5	0.857 [0.893]	0.922 [0.943]	0.983 [0.988]
10	10	10	8	8	8	0.872 [0.896]	0.933 [0.946]	0.986 [0.989]
10	10	10	10	10	10	0.879 [0.897]	0.938 [0.947]	0.987 [0.989]
20	20	20	10	10	10	0.875 [0.896]	0.936 [0.947]	0.987 [0.989]
20	20	20	16	16	16	0.887 [0.898]	0.943 [0.949]	0.988 [0.990]
20	20	20	20	20	20	0.890 [0.898]	0.944 [0.949]	0.989 [0.990]
50	50	50	25	25	25	0.891 [0.900]	0.945 [0.950]	0.989 [0.990]
50	50	50	40	40	40	0.895 [0.899]	0.947 [0.949]	0.989 [0.990]
50	50	50	50	50	50	0.896 [0.899]	0.948 [0.949]	0.990 [0.990]
$V_{1L} = 1.50$	$V_{1H} = 3.00$	$V_{2L} = 1.50$	$V_{2H} = 3.00$					
10	10	10	5	5	5	0.829 [0.864]	0.899 [0.923]	0.975 [0.982]
10	10	10	8	8	8	0.854 [0.876]	0.921 [0.933]	0.983 [0.986]
10	10	10	10	10	10	0.859 [0.880]	0.925 [0.937]	0.984 [0.987]
20	20	20	10	10	10	0.863 [0.880]	0.927 [0.937]	0.984 [0.987]
20	20	20	16	16	16	0.878 [0.888]	0.937 [0.942]	0.987 [0.988]
20	20	20	20	20	20	0.879 [0.892]	0.938 [0.945]	0.987 [0.989]
50	50	50	25	25	25	0.884 [0.893]	0.941 [0.946]	0.988 [0.989]
50	50	50	40	40	40	0.891 [0.894]	0.945 [0.947]	0.989 [0.989]
50	50	50	50	50	50	0.893 [0.896]	0.946 [0.948]	0.989 [0.989]
$V_{1L} = 1.50$	$V_{1H} = 4.00$	$V_{2L} = 1.50$	$V_{2H} = 4.00$					
10	10	10	5	5	5	0.844 [0.878]	0.913 [0.934]	0.980 [0.985]
10	10	10	8	8	8	0.869 [0.891]	0.931 [0.943]	0.986 [0.988]
10	10	10	10	10	10	0.873 [0.890]	0.934 [0.944]	0.986 [0.988]
20	20	20	10	10	10	0.873 [0.890]	0.934 [0.943]	0.986 [0.988]
20	20	20	16	16	16	0.884 [0.894]	0.941 [0.946]	0.988 [0.989]
20	20	20	20	20	20	0.886 [0.897]	0.942 [0.948]	0.988 [0.989]
50	50	50	25	25	25	0.888 [0.897]	0.943 [0.948]	0.989 [0.989]
50	50	50	40	40	40	0.893 [0.898]	0.946 [0.949]	0.989 [0.990]
50	50	50	50	50	50	0.894 [0.898]	0.947 [0.949]	0.989 [0.990]

n_{LH}	n_{HL}	n_{HH}	r_{LH}	r_{HL}	r_{HH}	$P(Z > \hat{z}_{0.10})$	$P(Z > \hat{z}_{0.05})$	$P(Z > \hat{z}_{0.05})$
							Unmodified [Modified]	
$V_{1L} = 2.00$	$V_{1H} = 4.00$	$V_{2L} = 2.00$	$V_{2H} = 4.00$					
10	10	10	5	5	5	0.787 [0.825]	0.863 [0.889]	0.959 [0.966]
10	10	10	8	8	8	0.824 [0.848]	0.897 [0.911]	0.975 [0.978]
10	10	10	10	10	10	0.831 [0.850]	0.904 [0.916]	0.978 [0.981]
20	20	20	10	10	10	0.833 [0.856]	0.906 [0.919]	0.978 [0.981]
20	20	20	16	16	16	0.857 [0.872]	0.923 [0.932]	0.983 [0.986]
20	20	20	20	20	20	0.866 [0.873]	0.929 [0.933]	0.985 [0.986]
50	50	50	25	25	25	0.872 [0.880]	0.934 [0.938]	0.986 [0.987]
50	50	50	40	40	40	0.883 [0.888]	0.941 [0.943]	0.988 [0.988]
50	50	50	50	50	50	0.885 [0.891]	0.942 [0.945]	0.988 [0.989]
$V_{1L} = 2.00$	$V_{1H} = 5.00$	$V_{2L} = 2.00$	$V_{2H} = 5.00$					
10	10	10	5	5	5	0.805 [0.846]	0.879 [0.909]	0.966 [0.976]
10	10	10	8	8	8	0.836 [0.868]	0.908 [0.927]	0.979 [0.984]
10	10	10	10	10	10	0.851 [0.870]	0.919 [0.930]	0.982 [0.985]
20	20	20	10	10	10	0.853 [0.870]	0.919 [0.930]	0.982 [0.985]
20	20	20	16	16	16	0.868 [0.882]	0.931 [0.938]	0.986 [0.987]
20	20	20	20	20	20	0.876 [0.884]	0.936 [0.940]	0.987 [0.988]
50	50	50	25	25	25	0.878 [0.886]	0.937 [0.942]	0.987 [0.988]
50	50	50	40	40	40	0.887 [0.893]	0.943 [0.946]	0.988 [0.989]
50	50	50	50	50	50	0.891 [0.894]	0.945 [0.946]	0.989 [0.989]

Table 2

Coefficients of the Quadratic Equation for 90% Content					
$n_{LH} = n_{HL} = n_{HH}$	$(p^*)^2$	p^*	**Censoring Factor**	**Acceleration Factor**	**Constant**
10	-12.659	-3.947	-0.084	-0.359	1.105
20	-46.541	-0.551	-0.082	-0.337	1.083
30	-65.732	1.568	-0.076	-0.298	1.056
40	-77.253	2.944	-0.072	-0.244	1.032
50	-85.214	3.914	-0.068	-0.266	1.021

Table 3

Coefficients of the Quadratic Equation for 95% Content					
$n_{LH} = n_{HL} = n_{HH}$	$(p^*)^2$	p^*	**Censoring Factor**	**Acceleration Factor**	**Constant**
10	-19.468	-9.385	-0.081	-0.284	1.108
20	-147.848	-2.847	-0.073	-0.289	1.088
30	-225.129	1.412	-0.067	-0.259	1.060
40	-279.080	4.586	-0.071	-0.230	1.038
50	-320.590	6.963	-0.061	-0.237	1.020

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