Adaptive Estimators of Process Capability Indices Using Preliminary Test

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Abstract

A process capability index is a statistical measure of a process to produce output within pre specified limits known as specification limits. Process capability indices measure natural variations in processes relative to specification limits determined *a priori*. They allow detection of shifts in the processes. Common process capability indices are C_p and C_{pk} . If LSL and USL are the lower specification limit and upper specification limits of a process, the midpoint of the specification interval is given by m=(USL+LSL)/2. It is well known that when the process mean equals m, C_{pk} reduces to C_p. However the process mean is usually unknown. This makes it difficult to determine whether to use C_p or C_{pk} without any guideline in practice. In this paper, we introduce new estimators of process capability based on a preliminary test of statistical hypothesis. The estimators are known as the preliminary test estimator (PTE) and the weighting function estimator (WFE). Based on mathematical derivations and Monte Carlo experiments, we demonstrate that the PTE and WFE outperform the C_p and C_{pk} indices under the mean square error criterion.

Key Words: process capability; preliminary test estimator; weighting function estimator; mean square error.

1. Introduction

Process capability indices are commonly used in industry to measure natural variations in processes relative to predetermined specification limits. The two basic process capability indices are C_p and C_{pk} . Let LSL and USL be the lower specification limit and upper specification limits of a process respectively, then

$$C_p = \frac{USL-LSL}{6\sigma} = \frac{d}{3\sigma} \tag{1}$$

and
$$C_{pk} = \frac{\min(USL-\mu, \mu-LSL)}{3\sigma} = \frac{d-|\mu-m|}{3\sigma} \tag{2}$$

where μ and σ are the mean and standard deviation of the process, d=(USL−LSL)/2, and m=$\frac{1}{2}(LSL+USL)$. Note that C_p does not take into account the process mean μ. The index C_{pk} that is a function of μ and σ is an extension of C_p.

In practice, the mean and standard deviation are usually unknown; hence they must be estimated by a sample. Let $\bar{X}$ and S be the sample mean and sample standard deviation from a random sample of size *n*. The usual estimators of C_p and C_{pk} are obtained by replacing the unknown parameters by their respective sample estimates. The usual estimators are

$$\hat{C}_p = \frac{USL-LSL}{6S} = \frac{d}{3S} \tag{3}$$

and $$\hat{C}_{pk} = \frac{\min(USL-\bar{X}, \bar{X}-LSL)}{3S} = \frac{d-|\bar{X}-m|}{3S} \tag{4}$$

It is easily seen from Equation (2) that when μ = *m*, the index C_{pk} reduces to C_p . Properties of $\hat{C}_p$ and $\hat{C}_{pk}$ are studied and given in the literature, see e.g. Kotz and Johnson (1993). Since μ is unknown, we do not know whether μ = *m*. However, we can use the sample to test the null hypothesis H_o: μ = *m* against the alternative hypothesis H_1: μ ≠ *m* . We assume that the process has a normal distribution. So the test is a t test. The test statistic is

$$t = \frac{\bar{X}-m}{S/\sqrt{n}} \tag{5}$$

The null hypothesis H_o is rejected if |t| > $t_{\alpha/2}$, where $t_{\alpha/2}$ is the upper 100α/2 percentage point of the t distribution with n – 1 degrees of freedom. The result of the t test can lead us to which estimator to use. The t test becomes a preliminary test. And the estimator depending on the outcome of the preliminary test is known as the preliminary test estimator (PTE). Preliminary test estimator was first studied by Bancroft(1944). Since then there are many papers published in this area, see for example the bibliography by Bancroft and Han(1977), Han, Rao and Ravichandran(1988). We now define the process capability preliminary test estimator as

$$\hat{C}_{pte} = \begin{cases} \hat{C}_p & if\ |t| \le t_{\alpha/2} \\ \hat{C}_{pk} & if\ |t| > t_{\alpha/2} \end{cases} \tag{6}$$

It is seen that when H_o is rejected, the test indicates that μ ≠ *m*, so we use $\hat{C}_{pk}$ as the estimator of the process capability. On the other hand, when H_o is not rejected, we use $\hat{C}_p$ as the estimator.

Another estimator based on the t statistic is the weighting function estimator (WFE), it is defined as

$$\hat{C}_{wfe} = w\hat{C}_p + (1-w)\hat{C}_{pk} \tag{7}$$

where *w* = 1/(1 + t^2), which is the weight. When t^2 is small, the weight *w* is near 1 and $\hat{C}_{wfe}$ is close to $\hat{C}_p$. When t^2 is large, the weight *w* is near 0 and $\hat{C}_{wfe}$ is close to $\hat{C}_{pk}$. We will study these adaptive estimators in this paper. Section 2 will derive the expected value and variance of $\hat{C}_{pte}$. A Monte Carlo study is used to compare the preliminary test estimator and the weighting function estimator with the estimator $\hat{C}_{pk}$.

2. Expected Value and Variance of the Preliminary Test Estimator

The expected value and variance of $\hat{C}_{pte}$ are obtained by using the distribution of $\bar{X}$ and S^2. We assume that the process follows a normal distribution. Then $\bar{X}$ ~ N(μ, σ^2) and (n–1)S^2/σ^2 ~ χ^2(n –1); $\bar{X}$ and S^2 are independently distributed. The expected value of $\hat{C}_{pte}$ is

$$E(\hat{C}_{pte}) = E(\hat{C}_p \,|\, |t| \le t_{\alpha/2})P(|t| \le t_{\alpha/2}) + E(\hat{C}_{pk} \,|\, |t| > t_{\alpha/2})P(|t| > t_{\alpha/2}) \quad (8)$$

Now $\hat{C}_{pk}$ can be written as

$$\hat{C}_{pk} = \hat{C}_p - \frac{|t|}{3\sqrt{n}}$$

So we have

$$E(\hat{C}_{pte}) = E(\hat{C}_p) - \frac{1}{3\sqrt{n}} E(|t| \,|\, |t| > t_{\alpha/2})P(|t| > t_{\alpha/2})$$

Let $Z = \sqrt{n}(X-\mu)/\sigma$ and V = (n–1)S^2/σ^2 ; then Z~ N(0, 1), V~ χ^2(n –1), Z and V are independently distributed. The t statistic can be written as

$$t = \frac{Z-\delta}{\sqrt{V/(n-1)}}$$

where δ = $\sqrt{n}$(m – μ)/σ. Using the joint probability density function of Z and V and after some tedious algebra, we obtain

$$E(\hat{C}_{pte}) = \frac{\sqrt{n-1}\,\Gamma((n-1)/2)}{\Gamma((n-2)/2}C_p - \frac{1}{3\sqrt{n}}(G_{1+}G_2) \quad (9)$$

where $$G_1 = \frac{\sqrt{\frac{n-1}{2}}\,\Gamma((n-2)/2)}{\Gamma((n-1)/2)}[\varphi(\delta) - \delta(1-\Phi(\delta))] \quad (10)$$

$$-\frac{\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n-1}{2}\right)}\sum_{i=1}^{\frac{n}{2}-1}\frac{\sqrt{n-1}}{2^{i-1}}\left(\frac{n-1}{t^2_{\frac{\alpha}{2}}}\right)^{i-1}e^{-(n-1)\delta^2/(2(t^2_{\frac{\alpha}{2}}+n-1))}$$

$$\times\sum_{j=0}^{2i-1}\binom{2i-1}{j}(-\delta)^{2i-1-j}\sum_{k=0}^{j}\binom{j}{k}\sigma_A^{k+1}\mu_A^{j-k}B_{k1}$$

and φ(·) and Φ(·) are the probability density and cumulative distribution function of the standard normal distribution respectively,

$$B_{k1} = 1 - \Phi(a)\ ,\ if\ k = 0, \tag{11}$$

$$=\varphi(a)[a^{k-1} + (k-1)a^{k-3} + (k-1)(k-3)a^{k-5} + \cdots + (k-1)(k-3)\ldots$$

$$\times 4 \times 2],\quad if\ \ k = 1, 3, 5, \ldots..(odd)$$

$$=\varphi(a)[a^{k-1} + (k-1)a^{k-3} + (k-1)(k-3)a^{k-5} + \cdots + (k-1)(k-3)\ldots$$

$$\times 3 \times a] + (k - 1(k-3) \times \ldots \times 3 \times 1 \times [1 - \Phi(a)],$$

$$if\ k = 2, 4, 6, \ldots.(even)$$

and $a = \frac{\delta - \mu_A}{\sigma_A},\ \mu_A = \frac{(n-1)\delta}{t^2_{\frac{\alpha}{2}} + n - 1},\ \ \sigma_A = \frac{t^2_{\frac{\alpha}{2}}}{t^2_{\frac{\alpha}{2}} + n - 1}$ (12)

$$G_2 = \frac{\sqrt{\frac{n-1}{2}}\Gamma((n-2)/2)}{\Gamma((n-1)/2)}[\varphi(\delta) + \delta\Phi(\delta)] \tag{13}$$

$$- \frac{\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n-1}{2}\right)} \sum_{i=1}^{\frac{n}{2}-1} \frac{\sqrt{n-1}}{2^{i-1}\Gamma(i)} \left(\frac{n-1}{t^2_{\frac{\alpha}{2}}}\right)^{i-1} e^{-(n-1)\delta^2/(2(t^2_{\frac{\alpha}{2}}+n-1))}$$

$$\times \sum_{j=0}^{2i-1} \binom{2i-1}{j} (-1)^j \delta^{2i-1-j} \sum_{k=0}^{j} \binom{j}{k} \sigma_A^{k+1} \mu_A^{j-k} B_{k2}$$

where

$$B_{k2} = \Phi(a)\ ,\quad if\ k = 0, \tag{14}$$

$$=-\varphi(a)[a^{k-1} + (k-1)a^{k-3} + (k-1)(k-3)a^{k-5} + \cdots + (k-1)(k-3)$$

$$\ldots \times 4 \times 2] \qquad if\ \ k = 1, 3, 5, \ldots..(odd)$$

$$=-\varphi(a)[a^{k-1} + (k-1)a^{k-3} + (k-1)(k-3)a^{k-5} + \cdots + (k-1)(k-3)$$

$$\ldots \times 3 \times a] \times (k - 1(k-3) \times \ldots \times 3 \times 1 \times \Phi(a) \quad if\ k = 2, 4, 6, \ldots.(even)$$

In a similar procedure we can derive $E(\hat{C}^2_{pte})$ that is

$$E\left(\hat{C}^2_{pte}\right) = \frac{n-1}{n-3} C_p^2 - \frac{2(n-1)}{3\sqrt{n}} \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} C_p \{2\varphi(\delta) - \delta\big(1 - \Phi(\delta)\big) + \delta\Phi(\delta) \tag{15}$$

$$- \sum_{i=1}^{(n-3)/2} \frac{1}{2^{i-1}\Gamma(i)} \left(\frac{n-1}{t^2_{\frac{\alpha}{2}}}\right)^{i-1} e^{-(n-1)\delta^2/(2(t^2_{\frac{\alpha}{2}}+n-1))} [\sum_{j=0}^{2i-1} \binom{2i-1}{j} (-\delta)^{2i-1-j} \sum_{k=0}^{j} \binom{j}{k} \sigma_A^{k+1} \mu_A^{j-k} B_{k1}$$

$$+\sum_{j=0}^{2i-1}\binom{2i-1}{j}(-1)^j\delta^{2i-1-j}\sum_{k=0}^{j}\binom{j}{k}\sigma_A^{k+1}\mu_A^{j-k}B_{k2}]\}$$

$$+\frac{(n-1)}{9n}\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}[1+\delta^2-\sum_{i=1}^{\frac{n-3}{2}}\frac{n-1}{2^{i-1}\Gamma(i)}(\frac{n-1}{t_{\frac{\alpha}{2}}^2})^{i-1}e^{-(n-1)\delta^2/(2(t_{\frac{\alpha}{2}}^2+n-1))}$$
$$\times\sum_{j=0}^{2i}\binom{2i}{j}(-\delta)^{2i-j}\sum_{k=0}^{j}\binom{j}{k}\sigma_A^{k+1}\mu_A^{j-k}\mu_k']$$

where μ_k' is the k^{th} moment of N(0, 1), i.e.

μ_k' = 1 if k=0

= 0 if k is odd

= (k-1)(k-3)….5$\times$ 3 $\times$ 1 if k is even

Using equations (9) and (15) we obtain the bias and mean square error of the preliminary test estimator.

Bias($\hat{C}_{pte}$)= $E(\hat{C}_{pte})-C_{pk}$

MSE($\hat{C}_{pte}$)= $E\left(\hat{C}_{pte}^2\right)-\ (E\left(\hat{C}_{pte}\right))^2$ + (Bias($\hat{C}_{pte}$))^2

3. Comparisons of the Estimators

We now compare the estimators in terms of bias and mean square error (MSE). Since the exact formulas for the bias and MSE of the weighting function estimator are difficult to obtain, we use a Monte Carlo study to obtain them. In the Monte Carlo study we let USL and LSL be 1 and –1 respectively without loss of generality. Also we let σ = 0.33 and μ = -0.9 (0.1) 0.9. The relative efficiency of the preliminary test estimator to $\hat{C}_{pk}$ is defined as

REPTE=$\frac{1/MSE(\hat{C}_{pte})}{1/MSE(\hat{C}_{pk})}$

Similarly the relative efficiency of the weighting function estimator to $\hat{C}_{pk}$ is defined as

REWFE=$\frac{1/MSE(\hat{C}_{wfe})}{1/MSE(\hat{C}_{pk})}$

The Monte Carlo results are given in Table 1 and Table 2. From the tables it is seen that both $\hat{C}_{pte}$ and $\hat{C}_{wfe}$ have smaller absolute bias than $\hat{C}_{pk}$. Also the absolute bias of $\hat{C}_{pte}$ is smaller than $\hat{C}_{wfe}$ when μ is equal or close to m; but when μ moves away from m, the absolute bias of $\hat{C}_{wfe}$ is smaller. Now let us compare the relative efficiencies. Both $\hat{C}_{pte}$ and $\hat{C}_{wfe}$ have good relative efficiency when μ is equal or close to m. The relative efficiency of $\hat{C}_{pte}$ is larger than that of $\hat{C}_{wfe}$ when the null hypothesis is true. but when μ moves away from m, the relative efficiency of $\hat{C}_{wfe}$ is higher.

Table 1 Biases and Relative Efficiencies n=7, α =0.05

μ	BiasCpk	BiasPTE	BiasWFE	REPTE	REWFE
-0.9	-0.0115	-0.0115	0.0012	1.00	1.00
-0.8	-0.0230	-0.0230	-0.0086	1.00	1.03
-0.7	-0.0350	-0.0342	-0.0188	0.99	1.04
-0.6	-0.0427	-0.0366	-0.0236	0.97	1.05
-0.5	-0.0638	-0.0476	-0.0415	0.99	1.08
-0.4	-0.0760	-0.0348	-0.0486	1.02	1.09
-0’3	-0.0904	-0.0204	-0.0581	1.11	1.09
-0.2	-0.0935	-0.0041	-0.0572	1.15	1.08
-0.1	-0.1348	-0.0520	-0.0986	1.15	1.09
0.0	-0.2059	-0.1327	-0.1706	1.19	1.11
0.1	-0.1420	-0.0594	-0.1052	1.19	1.11
0.2	-0.0967	-0.0073	-0.0609	1.13	1.08
0.3	-0.0809	-0.0089	-0.0486	1.09	1.08
0.4	-0.0668	-0.0274	-0.0399	1.00	1.07
0.5	-0.0623	-0.0460	-0.0398	0.98	1.07
0.6	-0.0456	-0.0411	-0.0266	0.98	1.05
0.7	-0.0343	-0.0337	-0.0179	0.99	1.04
0.8	-0.0166	-0.0165	-0.0024	1.00	1.02
0.9	-0.0102	-0.0102	0.0026	1.00	0.99

Table 2 Biases and Relative Efficiencies n=7, α =0.10

μ	BiasCpk	BiasPTE	BiasWFE	REPTE	REWFE
-0.9	-0.0115	-0.0115	0.0012	1.00	1.00
-0.8	-0.0230	-0.0230	-0.0086	1.00	1.03
-0.7	-0.0350	-0.0350	-0.0188	1.00	1.04
-0.6	-0.0427	-0.0416	-0.0236	0.99	1.05
-0.5	-0.0638	-0.0597	-0.0415	1.00	1.08
-0.4	-0.0760	-0.0602	-0.0486	1.00	1.09
-0.3	-0.0904	-0.0524	-0.0581	1.05	1.09
-0.2	-0.0935	-0.0337	-0.0572	1.12	1.08
-0.1	-0.1348	-0.0702	-0.0986	1.14	1.09
0.0	-0.2059	-0.1411	-0.1706	1.18	1.11
0.1	-0.1420	-0.0767	-0.1052	1.18	1.11
0.2	-0.0967	-0.0384	-0.0609	1.10	1.08
0.3	-0.0809	-0.0420	-0.0486	1.05	1.08
0.4	-0.0668	-0.0512	-0.0399	1.00	1.07
0.5	-0.0623	-0.0587	-0.0398	0.99	1.07
0.6	-0.0456	-0.0445	-0.0266	0.99	1.05
0.7	-0.0343	-0.0343	-0.0179	1.00	1.04
0.8	-0.0166	-0.0166	-0.0024	1.00	1.02
0.9	-0.0102	-0.0102	0.0026	1.00	0.99

4. Conclusion

Measurement of process capability is an important aspect of manufacturing with significant economic impact. The whole idea of the 6-sigma programs evolved around measuring capability via suitable indexes. Therefore, a reliable estimator of process capability is an essential business imperative. This paper extends the estimation of process capability which is based on a preliminary test of hypothesis leading to adaptive estimators; the preliminary test estimator and the weighting function estimator. We demonstrated that our proposed estimators perform better than the existing estimators when the null hypothesis is true. In the Big Data environment today, where the distributions are affected by man, machine, and materials; estimators that adapt to the underlying data structure are needless to say, essential.

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