Quantifying Risks of Extremes

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Abstract
We study ruin probability, a commonly used performance measure in actuarial science, as a risk assessment tool for extreme events. Classical ruin problem usually assumes independent, identically distributed steps and a linear drift. However a more realistic setting in the context of climate extremes requires temporal dependence in a given variable. Moreover, empirical evidence showing increased likelihood of extremes necessitates modeling these variables using heavy tailed probability laws. The case of heavy tailed, dependent steps is also interesting from a purely mathematical point of view as it raises the possibility of relating the dependence structure of heavy tailed stochastic processes to the asymptotic behavior of the ruin probability. This becomes particularly intriguing when the variables are highly volatile so that it is not possible to use traditional covariance based measures to quantify the dependence structure of the process. We propose a probability model, which uses dependent variables following infinite-variance stable distributions, and allows for nonlinear drifts. We then analyze the effects of range of dependence on the ruin probability in this setting.

Key Words: Stable, risk, extreme, ruin

1. Introduction

The study of random walks with negative drifts is a popular subject of the the theory of stochastic processes. The tail behavior of the supremum of such a random walk is of particular interest in fields such as actuarial science, economics and time series analysis, queuing theory, biology, and earth sciences, including environmental science, geophysics and climate modelling. For instance, in actuarial science, the probability of the supremum exceeding a certain threshold is known as the *ruin probability*, where the negative drift random walk represents the excess of the total claim amount in an insurance portfolio over the loaded total premium. (See Embrechts et al. (1997), Chapter 1.) There is a vast literature on ruin probabilities and a list of classical references can be found in Asmussen (2000). This probability is also important in the context of solutions to stochastic recurrence equations, including ARCH and GARCH processes. (See Embrechts et al. (1997), Chapter 8.) In queuing theory, this quantity is known as the *overflow probability* and is directly related to the stationary solution of Lindley equation. (See Baccelli and Brémaud (2003).)

Classical models in the literature on random walks with negative drifts usually deal with independent, identically distributed (iid) steps with light tails. In most applications however, the assumption of independent steps is clearly unrealistic. Additionally, the case of dependent steps is of theoretical interest as it introduces the possibility of gaining more insight into the probabilistic structure of the processes underlying the step sizes through the study of such probabilities. This aspect becomes particularly intriguing when the steps are heavy tailed with infinite second moments, in which case many of the classical measures of strength and range of dependence that mainly rely on covariance like functions become meaningless or ambiguous at best. The presence of heavy tails in everyday life has been established through empirical evidence. In the light of recent environmental, geophysical, and financial extremes that affected masses, the concept of heavy tailed phenomena has been attracting even more attention from academia and public alike.

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The asymptotic behavior of the tail probability of the supremum of a random walk with negative drift and iid heavy tailed steps was given in Embrechts and Veraverbeke (1982) in the greatest generality of the family of subexponential step sizes. Later Asmussen et al. (1999) showed that the results in the iid case still hold under a fairly general dependence structure of the heavy tailed steps, yet the results are not valid for all dependent claim sizes, (see, for instance Mikosch and Samorodnitsky (2000b)).

In this paper we study the tail behavior of the supremum of a negative drift random walk with dependent (sum-)stable increments. Stable probability laws are arguably the most important class of heavy tailed laws due to well known generalized central limit results for random sequences with infinite variance. Here we will assume that the steps of the random walk constitute a stationary, ergodic, symmetric $\alpha-$stable (SαS) process with $\alpha \in (1, 2)$. For this particular range of the tail index the step size process has a finite first moment, however the the second moment does not exist. This allows us to focus on the underlying dependence structure when metrics based on covariance like functions do not work. Furthermore, the results for stable steps represent, to some extend, provide a benchmark for what can be expected for more general processes such as stationary infinitely divisible processes.

In the past decade there has been some effort to study the asymptotic behavior of the supremum with stationary, ergodic SαS steps. Mikosch and Samorodnitsky (2000a) established certain results on the order of magnitude of the tail probability of the supremum for negative drift random walks driven by different classes of stationary, ergodic SαS increments. Later Alparslan and Samorodnitsky (2007a,b) extended some of these results in discrete time and established the order of magnitude for continuous time approximations for high frequency increments. All of these papers however, assume a constant drift rate yielding a linear cumulative drift process.

Braverman (2004) established an asymptotic equivalence between a class of functionals acting on the sample paths of continuous time SαS processes and a deterministic process based on the integral representation of the underlying stable process. One special case of these functionals indeed gives the tail probability of the supremum of a continuous time negative drift SαS random walk, where the drift rate is not necessarily constant. However, this study does not provide any explicit results regarding the order of magnitude. Later, Alparslan (2009) established the order of magnitude for certain classes of continuous time SαS processes and accumulated drift processes following power rules.

Our primary objective in this paper is to establish an asymptotic equivalence between the tail probability of the supremum of a negative drift random walk with stationary, ergodic SαS steps and a functional based on the integral representation of the step size process in the fashion of Mikosch and Samorodnitsky (2000a). However, here we will remove the constant drift rate assumption and will allow a fairly general class of cumulative drift processes. A detailed description of the model and the main results will be given in Section 2. In Section 3 we will apply the main result to a class random walks with negative power drifts and stationary SαS steps generated by dissipative flows to explicitly compute the order of magnitude for the tail probability of the supremum. Section 4 will show how a different order of magnitude emerges when one uses SαS steps generated by conservative flows instead.

2. The Model and the Asymptotics of the Tail Probability

Let $\mathbf{X} = \{X_n,\ n \in \mathbb{Z}_{++}\}$, be a measurable (implicitly assumed hereon), ergodic SαS process with $\alpha \in (1,2)$. Since $\mathbf{X}$ is SαS, one can utilize the representation given by

$$X_n = \int_E f_n(x) M(dx), \quad n \in \mathbb{Z}_{++}, \tag{2.1}$$

where M is a SαS random measure on a measurable space $(E, \mathcal{E})$ with a $\sigma-$finite control measure m on $\mathcal{E}$, (i.e. M is an independently scattered random measure on $\mathcal{E}$ such that

$$E\left[\exp\{i\theta M(A)\}\right] = \exp\{-|\theta|^\alpha m(A)\}, \quad \theta \in \mathbb{R} \tag{2.2}$$

for every $A \in \mathcal{E}$ with $m(A)$ finite), and $\{f_n\}_{n\in\mathbb{Z}_{++}} \subset L^\alpha(E, \mathcal{E}, m)$. (See Section 3.3 of Samorodnitsky and Taqqu (1994) for details.)

Further, if $\mathbf{X}$ is stationary then one can choose f_n to be in a more descriptive form given by

$$f_n(x) = a_n(x) \left[\frac{dm \circ \phi_n}{dm}(x)\right]^\alpha f \circ \phi_n(x), \quad x \in E,\ n \in \mathbb{Z}_{++}, \tag{2.3}$$

where $\{\phi_n\}_{n\in\mathbb{Z}_{++}}$ is a non-singular flow, (i.e. a family of measurable maps from E onto E such that $\phi_n = \phi_{n-1} \circ \phi$ for all $n \in \mathbb{Z}_{++}$, ϕ_0 is the identity function on E, and ϕ is a non-singular map on E), $\{a_n\}_{n\in\mathbb{Z}_{++}}$ is a cocycle for this flow (i.e. $a_0(\cdot) = 1$, and $a_{n+1}(x) = a_n(x) a_1(\phi_n(x))$ for $m-$a.a. $x \in E$) taking values in $\{-1, 1\}$, and $f \in L^\alpha(E, \mathcal{E}, m)$. (See Rosiński (1995).)

This representation is noteworthy as it makes it possible to relate the probabilistic structure of a stationary SαS process to the properties of a deterministic flow and a single kernel. (See Alparslan (2009); Alparslan and Samorodnitsky (2007a,b); Mikosch and Samorodnitsky (2000a); Samorodnitsky (2004) for examples in which this idea is utilized.) For instance, Hopf decomposition of the flow $\{\phi_n\}_{n\in\mathbb{Z}_{++}}$ (see, for instance Krengel (1985)), immediately implies that a stationary SαS process, $\mathbf{X}$, can be written (in distribution) as a sum of two independent stationary SαS processes

$$\mathbf{X} = \mathbf{X}^{(\mathbf{d})} + \mathbf{X}^{(\mathbf{c})}, \tag{2.4}$$

where $\mathbf{X}^{(\mathbf{d})}$ is given by representations (2.1) and (2.3) with a dissipative flow, and $\mathbf{X}^{(\mathbf{c})}$ is given by representations (2.1) and (2.3) with a conservative flow. We will focus on step size processes of the form $\mathbf{X}^{(\mathbf{d})}$ in Section 3 and on step size processes of the form $\mathbf{X}^{(\mathbf{c})}$ in Section 4.

Define a random walk $\mathbf{S} = \{S_n,\ n \in \mathbb{Z}_+\}$ whose steps are governed by $\mathbf{X}$ as

$$S_n = \sum_{k=1}^n X_k,\ n \in \mathbb{Z}_+, \tag{2.5}$$

and let

$$h_n(x) = \sum_{k=1}^n f_k(x),\ n \in \mathbb{Z}_+, \tag{2.6}$$

(with the usual understanding that $\sum_{k=1}^0 \cdot = 0$.) Then it follows from (2.1) and Theorem 11.4.1 of Samorodnitsky and Taqqu (1994) that

$$S_n = \int_E h_n(x) M(dx),\ n \in \mathbb{Z}_+. \tag{2.7}$$

Finally let $\mathbf{D} = \{D_n,\ n \in \mathbb{Z}_+\}$ be a deterministic drift process taking positive values. Then one can express the probability of ultimate exceedance of a positive threshold u by the random walk $\mathbf{S}$ with negative drift $\mathbf{D}$, as a function of u, as

$$\psi(u) = P\left[\sup_{n\in\mathbb{Z}_+} (S_n - D_n) > u\right], \quad u > 0. \tag{2.8}$$

The main result of this paper, which we present below, establishes the asymptotic relation between $\psi(u)$ and the functional

$$\psi_0(u) = \frac{C_\alpha}{2}\int_E \left\{\sup_{n\in\mathbb{Z}_+} \frac{[h_n(x)]_+^\alpha}{(u+D_n)^\alpha} + \sup_{n\in\mathbb{Z}_+} \frac{[-h_n(x)]_+^\alpha}{(u+D_n)^\alpha}\right\} m(dx), \quad u > 0, \tag{2.9}$$

where h_n is given by (2.6) and

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x\, dx\right)^{-1}, \tag{2.10}$$

where $\alpha \in (1,2)$ is the tail index of the step sizes.

THEOREM 2.1. *Let $\mathbf{S} = \{S_n,\ n \in \mathbb{Z}_+\}$, given by (2.5), be a random walk whose steps, given by (2.1), form a sequence of ergodic SαS random variables with $\alpha \in (1,2)$. Also let $\mathbf{D} = \{D_n,\ n \in \mathbb{Z}_+\}$ be a deterministic drift process such that $D_n \geq d\,n^p$, $n \in \mathbb{Z}_+$ for some $d > 0$ and $p \geq 1$.*

(a) *For $\psi(u)$ and $\psi_0(u)$ given by (2.8) and (2.9), respectively,*

$$\liminf_{u\to\infty} \frac{\psi(u)}{\psi_0(u)} \geq 1.$$

(b) *If in addition, $\mathbf{X}$ is stationary and $\|h_n(\cdot)\|_{L^\alpha(E,\mathcal{E},m)} = O(n^\gamma)$ as $n \to \infty$ for some $\gamma \in (0,1)$ then*

$$\psi(u) \sim \psi_0(u), \quad \textit{as} \quad u \to \infty.$$

Proof of Theorem 2.1(a)*.* For the proof of the first part of the theorem we will use the series representation of $\mathbf{X}$. Let m_0 be a probability measure on $\mathcal{E}$ that is equivalent to the control measure m in (2.1) and define $g := dm_0/dm$. Then it follows from Section 3.10 of Samorodnitsky and Taqqu (1994) that $\mathbf{X}$ can also be represented (in distribution) by

$$X_n = C_\alpha^{1/\alpha} \sum_{i=1}^\infty \sigma_i \Gamma_i^{-1/\alpha} g^{-1/\alpha}(Y_i) f_n(Y_i), \quad n \in \mathbb{Z}_{++}, \tag{2.11}$$

where $\{\sigma_i\}_{i\in\mathbb{Z}_{++}}$ is an iid sequence of Rademacher random variables, (i.e. discreet uniform random variables on $\{-1,1\}$), $\{\Gamma_i\}_{i\in\mathbb{Z}_{++}}$ are the arrival times of a unit rate Poisson process on the positive real line, (which are clearly dependent), and $\{Y_i\}_{i\in\mathbb{Z}_{++}}$ is an independent sequence of $E-$valued random variables with common probability law m_0. Furthermore, these three sequences are mutually independent. For $x \in E$ define

$$H_n(x) = C_\alpha^{1/\alpha} g^{-1/\alpha}(x) \sum_{k=1}^n f_k(x),\ n \in \mathbb{Z}_+. \tag{2.12}$$

Then one can rewrite

$$\psi(u) = P\left[\sup_{n\in\mathbb{Z}_+} \left(\sum_{i=1}^\infty \sigma_i \Gamma_i^{-1/\alpha} H_n(Y_i) - D_n\right) > u\right], \quad u > 0. \tag{2.13}$$

and

$$\psi_0(u) = \sum_{i=1}^{\infty} P\left[\sup_{n\in\mathbb{Z}_+}\left(\sigma_i\Gamma_i^{-1/\alpha}H_n(Y_i) - D_n\right) > u\right], \quad u > 0. \tag{2.14}$$

(Conditioning on Γ_is on the right hand side of (2.14) and summing over i gives (2.9).)

It follows from (2.11) and (2.12) that for $n \in \mathbb{Z}_+$, the random walk $\mathbf{S}$ can be written as

$$S_n = \sigma_1\Gamma_1^{-1/\alpha}H_n(Y_1) + \sum_{i=2}^{\infty}\sigma_i\Gamma_i^{-1/\alpha}H_n(Y_i) =: S_n^{(1)} + S_n^{(2)}. \tag{2.15}$$

Now fix $\theta \in (0,1)$. Set $K^{\theta,u} := \inf\left\{n \in \mathbb{Z}_{++} :\ S_n^{(1)} - (1+\theta)D_n > (1+\theta)u\right\}$ and define

$$\psi^{(1)}(\theta, u) := P\left[\sup_{n\in\mathbb{Z}_{++}}\left(S_n^{(1)} - \theta D_n\right) > \theta u\right] \tag{2.16}$$

Then conditioning on $K^{\theta,u}$, recalling the dependence structure of $\{\Gamma_n\}_{n\in\mathbb{Z}_{++}}$, and using the symmetry properties of $\left\{S_n^{(1)}\right\}_{n\in\mathbb{Z}_{++}}$ and $\left\{S_n^{(2)}\right\}_{n\in\mathbb{Z}_{++}}$ we have

$$\begin{aligned}
\psi(u) &\geq \sum_{k=1}^{\infty} P\left(S_k^{(2)} + \theta D_k > -\theta u \,\middle|\, K^{\theta,u} = k\right) P(K^{\theta,u} = k) \\
&\geq \left[\inf_{n\in\mathbb{Z}_{++}} P\left(S_n^{(2)} + \theta D_n > -\theta u\right)\right]\sum_{k=1}^{\infty} P(K^{\theta,u} = k) \\
&\geq \left[\inf_{n\in\mathbb{Z}_{++}} P\left(S_n^{(2)} + \theta D_n > -\theta u\right)\right]\psi^{(1)}(1+\theta, u) \\
&= \psi^{(1)}(1+\theta,u)\inf_{n\in\mathbb{Z}_{++}}\left[1 - P\left(S_n^{(2)} - \theta D_n > \theta u\right)\right] \\
&= \psi^{(1)}(1+\theta,u)\inf_{n\in\mathbb{Z}_{++}}\left[1 - 2P\left(S_n^{(2)} - \theta D_n > \theta u,\, \sigma_1 H_n(Y_1) \geq 0\right)\right] \\
&= \psi^{(1)}(1+\theta,u)\inf_{n\in\mathbb{Z}_{++}}\left[1 - 2P\left(S_n^{(2)} - \theta D_n > \theta u,\, S_n^{(1)} \geq 0\right)\right] \\
&\geq \psi^{(1)}(1+\theta,u)\inf_{n\in\mathbb{Z}_{++}}\left[1 - 2P\left(S_n - \theta D_n > \theta u\right)\right].
\end{aligned} \tag{2.17}$$

Notice that the ergodicity of $\mathbf{X}$ and the assumptions on $\mathbf{D}$ imply that the stochastic process $\{S_n/(\theta D_n)\}_{n\in\mathbb{Z}_{++}}$ is almost surely bounded and

$$\lim_{n\to\infty}\frac{S_n}{\theta D_n} = 0 \ \text{ a.s.}$$

Thus

$$\limsup_{u\to\infty} P\left[\sup_{n\in\mathbb{Z}_{++}}(S_n - \theta D_n) > \theta u\right] = 0,$$

and consequently, (2.17) gives

$$\liminf_{u\to\infty}\frac{\psi(u)}{\psi^{(1)}(1+\theta,u)} \geq 1. \tag{2.18}$$

Next, define two functions on $E \times \mathbb{R}_+$ by

$$G_+(y,u) := \sup_{n\in\mathbb{Z}_{++}}\left[\frac{H_n(y)}{u + D_n}\right]_+, \quad G_-(y,u) := \sup_{n\in\mathbb{Z}_{++}}\left[-\frac{H_n(y)}{u + D_n}\right]_+.$$

Then conditioning on σ_1 gives

$$\begin{aligned}2\psi^{(1)}(1+\theta,u) &= P\left[\Gamma_1 < (1+\theta)^{-\alpha}G_+^\alpha(Y_1)\right] + P\left[\Gamma_1 < (1+\theta)^{-\alpha}G_-^\alpha(Y_1)\right]\\ &= \int_E \left\{1-\exp\left\{-(1+\theta)^{-\alpha}G_+^\alpha(y)\right\}\right\} m_0(dy)\\ &\quad + \int_E \left\{1-\exp\left\{-(1+\theta)^{-\alpha}G_-^\alpha(y)\right\}\right\} m_0(dy)\\ &\geq \int_E (1+\theta)^{-\alpha}G_+^\alpha(y)\,\exp\left\{-(1+\theta)^{-\alpha}G_+^\alpha(y)\right\} m_0(dy)\\ &\quad + \int_E (1+\theta)^{-\alpha}G_-^\alpha(y)\,\exp\left\{-(1+\theta)^{-\alpha}G_-^\alpha(y)\right\} m_0(dy),\end{aligned} \tag{2.19}$$

where the last inequality follows from the fact that $e^x - 1 \geq x$ for $x \geq 0$. For $M > 0$ define

$$E_M := \left\{ y \in E : \sup_{n\in\mathbb{Z}_{++}} \frac{|H_n(y)|^\alpha}{D_n^\alpha} < M \right\},$$

and note that

$$E_M \uparrow E_\infty := \left\{ y \in E : \sup_{n\in\mathbb{Z}_{++}} \frac{|H_n(y)|^\alpha}{D_n^\alpha} < \infty \right\}, \quad \text{as } M \to \infty.$$

But the ergodicity of $\mathbf{X}$ implies that

$$\int_E \sup_{n\in\mathbb{Z}_{++}} \frac{|H_n(y)|^\alpha}{n^\alpha} m_0(dy) < 0,$$

(see e.g. Section 10.2 in Samorodnitsky and Taqqu (1994)), and hence

$$m_0(E_\infty^c) = 0. \tag{2.20}$$

It is not hard to see that for an arbitrary $\epsilon \in (0,1)$, one can find u_ϵ such that for any $u > u_\epsilon$ and $y \in E_M$

$$\exp\left\{ -\frac{|H_n(y)|^\alpha}{(1+\theta)^\alpha(u+D_n)^\alpha} \right\} > 1-\epsilon.$$

Then for any such u, it follows from (2.19) that

$$2\psi^{(1)}(1+\theta,u) \geq \frac{1-\epsilon}{(1+\theta)^\alpha} \int_{E_M} \left(G_+^\alpha(y) + G_-^\alpha(y)\right) m_0(dy) \tag{2.21}$$

Now letting $u \to \infty$, $\epsilon \to 0$, $M \uparrow \infty$, and using monotone convergence we conclude

$$\liminf_{u\to\infty} \frac{\psi^{(1)}(1+\theta,u)}{\frac{1}{2}\int_{E_\infty}\left(G_+^\alpha(y)+G_-^\alpha(y)\right) m_0(dy)} \geq (1+\theta)^{-\alpha}. \tag{2.22}$$

But then it follows from (2.9), (2.12), and (2.20) that

$$2\psi_0(u) = \int_E \left(G_+^\alpha(y)+G_-^\alpha(y)\right) m_0(dy) = \int_{E_\infty} \left(G_+^\alpha(y)+G_-^\alpha(y)\right) m_0(dy), \tag{2.23}$$

and we have

$$\liminf_{u\to\infty} \frac{\psi^{(1)}(1+\theta,u)}{\psi_0(u)} \geq (1+\theta)^{-\alpha}. \tag{2.24}$$

Combining this with (2.18) and letting $\theta \downarrow 0$ establish the desired result.

□

Proof of Theorem 2.1(b). Clearly, one could try to establish the desired result by proving

$$\limsup_{u\to\infty} \frac{\psi(u)}{\psi_0(u)} \le 1,$$

in the fashion of the proof of Theorem 2.1(a). However, instead of a direct approach, here we prefer a shortcut.

First note that under the given assumptions Proposition 7.4 of Braverman (2004) implies the existence of some $\varepsilon_0 > 0$ such that the process $\left\{n^{\varepsilon_0-1}S_n\right\}_{n\in\mathbb{Z}_{++}}$ is bounded almost surely. Now fix such an ε_0.

For an arbitrary $\varepsilon > 0$ define a new process by

$$Y_n := \frac{(\log n)^{2+\varepsilon-1/\alpha} S_n}{D_n}, \quad n \in \mathbb{Z}_{++}.$$

Now it is not very hard to verify that for any $\varepsilon > 0$

$$\frac{(\log n)^{2+\varepsilon-1/\alpha}}{D_n} \le \frac{(\log n)^{2+\varepsilon-1/\alpha} S_n}{d\, n^p} = o\left(n^{\varepsilon_0-1}\right), \quad \text{as } n\to\infty,$$

and since ε_0 is chosen such that $\left\{n^{\varepsilon_0-1}S_n\right\}_{n\in\mathbb{Z}_{++}}$ is bounded almost surely, we conclude that $\{Y_n\}_{n\in\mathbb{Z}_{++}}$ is also bounded almost surely. Then it follows from Theorem 10.2.3 of Samorodnitsky and Taqqu (1994) that

$$\sup_{n\in\mathbb{Z}_{++}} \left| \frac{(\log n)^{2+\varepsilon-1/\alpha} h_n(\cdot)}{D_n} \right| \in L^\alpha(E,\mathcal{E},m).$$

Finally, the desired result follows from Theorem 7.2 and Remark 7.3 of Braverman (2004). □

3. Steps Generated by a Dissipative Ergodic Flow

Here we provide an application of the main theorem given in the previous section. We consider step sizes governed by process of the form $\mathbf{X}^{(\mathbf{d})}$ given in (2.4). Rosiński (1995) showed that stationary SαS processes generated by dissipative flows are automatically ergodic and they have a *mixed moving average* representation, i.e. for any such process, $\mathbf{X}^{(\mathbf{d})} = \left\{X_n^{(d)},\ n\in\mathbb{Z}\right\}$, there exists a Borel space W, a σ-finite measure ν on W and a function $f \in L^\alpha(W\times\mathbb{R},\ \mathcal{W}\times\mathcal{B},\ \nu\otimes\lambda)$ such that

$$\left\{X_n^{(d)}\right\}_{n\in\mathbb{Z}} \stackrel{d}{=} \left\{\int_W\int_{\mathbb{R}} f(w,x-n)M(dw,dx)\right\}_{n\in\mathbb{Z}}, \tag{3.1}$$

where M is a SαS random measure on the product space $W\times\mathbb{R}$ with the control measure $\nu\otimes\lambda$, where λ is the Lebesgue measure. Moreover, if the dissipative flow itself is ergodic, (recall that a null-preserving transformation ϕ on $(E,\mathcal{E},m)$ is called ergodic if all ϕ-invariant sets A have the property that $m(A) = 0$ or $m(A^c) = 0$), then W becomes a singleton, and the representation given by a mixed moving average reduces to the more familiar, usual moving average,

$$\left\{X_n^{(d)}\right\}_{n\in\mathbb{Z}} \stackrel{d}{=} \left\{\int_{\mathbb{R}} f(x-n)M(dx)\right\}_{n\in\mathbb{Z}}. \tag{3.2}$$

To illustrate the use of our main results from Section 2, suppose the step sizes, $\mathbf{X} = \{X_n,\ n \in \mathbb{Z}_{++}\}$, form a stationary S$\alpha$S process given by

$$X_n = \int_{\mathbb{R}} f(x-n)\, M(dx),\ n \in \mathbb{Z}_{++}, \tag{3.3}$$

where $f \in L^\alpha(\mathbb{R}, \mathcal{B}, \lambda)$, and M is a SαS random measure on $(\mathbb{R}, \mathcal{B})$ with Lebesgue control measure, λ.

Let the random walk $\mathbf{S}$ be defined as before but now for $n \in \mathbb{Z}_+$, $h_n(\cdot)$ becomes

$$h_n(x) = \sum_{k=1}^{n} f(x-k),\ x \in \mathbb{R}.$$

Further suppose the deterministic drift process $\mathbf{D}$ is given by

$$D_n = d\, n^p,\ n \in \mathbb{Z}_+, \tag{3.4}$$

for some $d > 0$ and $p > 1$. The proof for the main result of this section does not work when $p = 1$. However, the relevant results for the case of $p = 1$ can be found in Alparslan and Samorodnitsky (2007a), which coincide with the findings presented here "in the limit" (as $p \downarrow 1$).

Lastly, for $u > 0$ let $\psi(\cdot)$ be as in (2.8), however note that $\psi_0(u)$ now becomes

$$\psi_0(u) = \frac{C_\alpha}{2} \int_{\mathbb{R}} \left\{ \sup_{n \in \mathbb{Z}_+} \frac{\left[\sum_{k=1}^n f(x-k)\right]_+^\alpha}{(u + d\, n^p)^\alpha} + \sup_{n \in \mathbb{Z}_+} \frac{\left[-\sum_{k=1}^n f(x-k)\right]_+^\alpha}{(u + d\, n^p)^\alpha} \right\} dx. \tag{3.5}$$

THEOREM 3.1. *Let $\mathbf{S} = \{S_n,\ n \in \mathbb{Z}_+\}$ be a random walk whose steps are given by (3.3) with $\alpha \in (1,2)$ and $f : \mathbb{R} \mapsto \mathbb{R}_+$. Also let $\mathbf{D} = \{D_n,\ n \in \mathbb{Z}_+\}$ be a deterministic drift process given by (3.4).*

(a) *If $f \notin L^1(\mathbb{R}, \mathcal{B}, \lambda)$ then $u^{\alpha-(1/p)}\psi(u) \to \infty$ as $u \to \infty$, i.e. $\psi(u)$ decays slower than $u^{(1/p)-\alpha}$.*

(b) *Let $K^- \in \mathbb{Z}_-$ and $K^+ \in \mathbb{Z}_+$ be such that the function*

$$f_*(x) := \sup_{t \geq 0} f(x-t)\mathbf{1}_{\{x \leq K^-\}} + f(x)\mathbf{1}_{\{K^- < x < K^+\}} + \sup_{t \geq 0} f(x+t)\mathbf{1}_{\{x \geq K^+\}}$$

belongs to $L^1(\mathbb{R}, \mathcal{B}, \lambda)$. Further assume that there exists a monotone function g, regularly varying at infinity with index $-q$, $q > 1$, which dominates f_* on $[K_+, \infty)$. Then as $u \to \infty$

$$\psi(u) \sim \frac{C_\alpha}{2pd^{1/p}} \mathfrak{B}\left(\frac{1}{p}, \alpha - \frac{1}{p}\right) I(f) u^{(1/p)-\alpha},$$

where $\mathfrak{B}(\cdot,\cdot)$ is the beta function and

$$I(f) := \int_0^1 \left[\sum_{k=-\infty}^{\infty} f(x-k)\right]^\alpha dx < \infty.$$

Proof. See Alparslan (2013) for a detailed proof. ☐

4. Ergodic Steps Generated by a Conservative Flow

In this section we demonstrate another application of the main theorem of Section 2, which also serves to show how drastically different the order of magnitude of $\psi(u)$ can be when one uses stable steps generated by a conservative flow instead of steps generated by a dissipative flow such as in the setting studied in Section 3. To achieve this, it will be sufficient to focus on a particular class of stationary SαS whose control measure is constructed using null recurrent Markov chains. We now proceed with the setup.

Consider an irreducible, null-recurrent Markov chain, $\mathbf{Y} = \{Y_n,\ n \in \mathbb{Z}_{++}\}$ on $\mathbb{Z}$ with law $P_s(\cdot)$ on

$$\mathbb{Z}^{\mathbb{Z}_+} = \{\mathbf{y} = (y_0, y_1, y_2, \dots) :\ y_i \in \mathbb{Z}, i \in \mathbb{Z}_+\}$$

corresponding to the initial state $y_0 = s \in \mathbb{Z}$.

Let $\pi = \{\pi_s,\ s \in \mathbb{Z}\}$ be the $\sigma-$finite invariant measure corresponding to the family $\{P_s,\ s \in \mathbb{Z}\}$ satisfying $\pi_0 = 1$, and define a $\sigma-$finite measure on $\mathcal{Z}^{\mathcal{Z}_+}$, the cylindrical $\sigma-$field of $\mathbb{Z}^{\mathbb{Z}_+}$ by

$$m(\cdot) = \sum_{i=-\infty}^{\infty} \pi_i P_i(\cdot). \tag{4.1}$$

We will model the steps of the random walk, $\mathbf{X} = \{X_n,\ n \in \mathbb{Z}_{++}\}$, with a S$\alpha$S process defined by

$$X_n = \int_{\mathbb{Z}^{\mathbb{Z}_+}} f_n(\mathbf{y}) M(d\mathbf{y}), \quad \mathbf{y} \in \mathbb{Z}^{\mathbb{Z}_+},\ \ n \in \mathbb{Z}_{++} \tag{4.2}$$

where M is a SαS random measure on $\mathbb{Z}^{\mathbb{Z}_+}$ with control measure m given in (4.1) and the kernels f_n are chosen to be of the form

$$f_n(\mathbf{y}) = 1_{[y_n=0]}, \quad n \in \mathbb{Z}_{++},\ \ \mathbf{y} = (y_0, y_1, y_2, \dots) \in \mathbb{Z}^{\mathbb{Z}_+}. \tag{4.3}$$

The results of this section can be extended to a more general family of kernels f_n, as in for instance Resnick et al. (2000) or Alparslan and Samorodnitsky (2007b). However, a kernel as simple as the one given above is enough to accomplish our goal of demonstrating the difference in the order of magnitude of $\psi(u)$ in the conservative case compared to the dissipative case.

It follows from Rosiński and Samorodnitsky (1996) that the process $\mathbf{X}$ given by the stochastic integral representation (4.2) is a stationary mixing process, and in particular is ergodic. Furthermore $\mathbf{X}$ is associated with a conservative flow.

Now let $\mathbf{S} = \{S_n,\ n \in \mathbb{Z}_+\}$ be a random walk whose steps are given by (4.2) and define $\psi(u)$ as in (2.8) with $\mathbf{D} = \{D_n,\ n \in \mathbb{Z}_+\}$ given by (3.4). For an integer s and $\mathbf{y} \in \mathbb{Z}^{\mathbb{Z}_+}$, define the number of visits to state s in n steps to be

$$N_n^{(s)} = N_n^{(s)}(\mathbf{y}) := \sum_{j=1}^{n} 1_{[y_j=s]}(\mathbf{y}). \tag{4.4}$$

With this notation (2.9) reduces to

$$\psi_0(u) = \frac{C_\alpha}{2} \int_{\mathbb{Z}^{\mathbb{Z}_+}} \left(\sup_{n \in \mathbb{Z}_+} \frac{N_n^{(0)}}{u + n^p d} \right)^\alpha m(d\mathbf{y}), \quad u > 0. \tag{4.5}$$

For a given $\mathbf{y} \in \mathbb{Z}^{\mathbb{Z}_+}$ and $s \in \mathbb{Z}$, define the number of steps until the chain returns to state s for the first time as

$$\tau^{(s)} = \tau^{(s)}(\mathbf{y}) := \inf\{n \in \mathbb{Z}_{++} :\ y_n = s\}. \tag{4.6}$$

Note that by the null-recurrence of the Markov chain $E_s\tau_s = \infty$, for any $s \in \mathbb{Z}$. We will further assume that there is a constant $\gamma \in (0,1)$ and a slowly varying function L such that for any $n \in \mathbb{Z}_{++}$

$$P_0\left(\tau^{(0)} \geq n\right) = n^{\gamma-1}L(n). \tag{4.7}$$

Further let $\mathbf{Z}_{\mathbf{1}-\gamma} = \{Z_{1-\gamma}(t),\ t \in \mathbb{R}_+\}$ be a $(1-\gamma)$-stable subordinator, i.e. a positive, increasing, strictly stable Lévy motion with characteristic function given by

$$E\left[\exp\{i\theta Z_{1-\gamma}(1)\}\right] = \exp\left\{-\frac{|\theta|^{1-\gamma}}{C_{1-\gamma}}\left[1 - i\tan\frac{\pi(1-\gamma)}{2}\right]\right\}, \quad \theta \in \mathbb{R}. \tag{4.8}$$

Now we are ready to state the main result of this section.

THEOREM 4.1. *If the relation (4.7) holds then*

$$\liminf_{u\to\infty} u^{\frac{\alpha(p-1)+\gamma(\alpha-1)}{p}} L^{\alpha-1}\left(u^{1/p}\right)\psi(u) \geq 2^{-\frac{(p-1)[\alpha-\gamma(\alpha-1)]}{p}} I_{\alpha,p,\gamma} \tag{4.9}$$

and

$$\limsup_{u\to\infty} u^{\frac{\alpha(p-1)+\gamma(\alpha-1)}{p}} L^{\alpha-1}\left(u^{1/p}\right)\psi(u) \leq I_{\alpha,p,\gamma} \tag{4.10}$$

where

$$I_{\alpha,p,\gamma} = \frac{C_\alpha \mathfrak{B}\left(\frac{\gamma}{p}, \frac{\alpha(p-1)+\gamma(\alpha-1)}{p}\right)}{2pd^{\frac{\alpha-\gamma(\alpha-1)}{p}}} E\left[\left(\sup_{t\geq 1}\frac{(t-1)^{1/p}}{Z_{1-\gamma}(t)}\right)^{\alpha(1-\gamma)}\right], \tag{4.11}$$

and $\mathfrak{B}(\cdot,\cdot)$ is the beta function.

In particular, in the setting described above

$$\psi(u) = O\left(u^{-\frac{\alpha(p-1)+\gamma(\alpha-1)}{p}} L^{-(\alpha-1)}\left(u^{1/p}\right)\right), \quad \textit{as } u \to \infty.$$

Proof. See Alparslan (2013) for a detailed proof. □

REMARK 4.2. A closer look at the result above reveals that in the case of stable steps generated by conservative flows $\psi(u)$ could decay very slowly for values of p near 1 and values of γ arbitrarily close to 0 regardless of the value of α. Further observe that this is true under the assumption of a kernel as "nice" as the one given by (4.3). A comparative look at Theorem 3.1(b) shows that this is not the case when one considers $\psi(u)$ for stable steps generated by dissipative flows.

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