Process Control with Quality Gradations and Classification Errors

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Abstract
We consider a process control model in which the quality of the inspected items is classified as being in one of three states rather than simply conforming or nonconforming. Items may be judged to be superior, acceptable, or unacceptable and the percentages associated with these categories will differ based on whether the process is in control or out of control. There is, however, a possibility of misclassifying the quality of an item. Thus, when using any rule for deciding whether or not the process is in control, there is the possibility of being mistaken. Various probabilistic quantities of interest are derived for our rule.

Keywords: Inspection, quality

1. Introduction

Taguchi, Elsayed, and Hsiang(1989) and Taguchi, Chowdhury, and Wu (2004) consider on-line process control by attributes involves inspecting every h^{th} item produced. Initially the process is assumed to have some high fraction of conforming items, say 100 % or close to it. That is, an item conforms to specifications with probability p_1, equal to or very close to 1 when the process is in control. At some random time the process goes out of control and there is a shift to p_2 ($< p_1$) for the fraction conforming, or the probability that the selected item is really conforming. When an inspected item is considered nonconforming, the process is stopped for adjustment.

Variations on this theme have been introduced by a number of authors. Nayebpour and Woodall (1993) consider the random time until the shift from p_1 to p_2 to follow a geometric distribution. That is, the items produced are modeled as independent and identically distributed trials with a constant probability π for each item to be the first item produced after the shift of the fraction conforming. Since only every h^{th} item is inspected, the first item produced after the shift may not be inspected and thus there may be some initial number of items produced before the possibility of the detection of this shift even exists.

Borges, Ho, and Turnes (2001) argue that the inspection process itself may be subject to possible diagnostic errors. That is, in a single classification, a conforming item might be mistakenly classified as nonconforming, let p_{CN} be the probability of this misclassification. In addition, a nonconforming item might mistakenly be judged as conforming, let p_{NC} be the probability of this misclassification. We will also define probability p_{CC} (p_{NN}) of the correct classification that a conforming (nonconforming) item is classified as conforming (nonconforming). This leads to the notion of making repeated classifications of each inspected item before making the final determination as to whether to judge the item as conforming or nonconforming. If the item has been judged in this final determination to be nonconforming, the process is judged out of control and is stopped for adjustment. Otherwise, the process is considered in control and is not stopped for adjustment. Because of the possibility of diagnostics errors in the repeated classifications, it is possible that an item is judged to be nonconforming and thus that the process is judged out of control, when it actually is not. Still it is stopped for adjustment. However, in that case, no cause can be found and the process then is restarted and has not, in some way, been put out of control by the stopping and searching for a cause. On the other hand, it is also possible that the process goes out of control, but is not detected. In that case, it remains out of control until this is detected at a later time, when it will be adjusted and be put back in control.

In Trindade, Ho, and Quinino (2007), the rule for the final determination of whether the inspected item is conforming, and thus whether the process is in control, is based on a pre-specified number of repeated classifications and using majority rule. Quinino, Colin, and Ho (2009) consider a rule in which the item is determined to be conforming and the process to be in control if and only if there are k classifications as conforming before f classifications as nonconforming, where k and f are some pre-specified positive integers. The acronym TCTN is used to describe this rule since the decision is based on the total number of classifications as conforming and nonconforming. Smith and Griffith (2009) studied this rule.

Griffith and Smith (2011) studied an alternative rule in which the final determination that an item is conforming, and thus the process is in control, if and only if k consecutive classifications as conforming occur before a total of f classifications as nonconforming and called this rule by the acronym CCTN. Smith and Griffith (2012) studied the rule in which the final determination that an item is conforming, and thus the process is in control, if and only if k consecutive classifications as conforming occur before a f consecutive classifications as nonconforming and termed this CCCN.

In the present paper, we shall consider a multistate model in which items are not simply conforming or nonconforming. Rather, items produced are superior, acceptable, or unacceptable. When the process is in control, the percentages are p_1, p_2, and $1 - p_1 - p_2$ respectively and when the process is out of control, the percentages are p_1*, p_2*, and $1 - p_1^* - p_2^*$ ($> 1 - p_1 - p_2$). The process of inspecting an item is imperfect and hence it is subject to possible misclassification. That is, an item which is actually superior might be judged to be only acceptable or even possibly unacceptable. Similarly, misclassifications can occur for an item which is actually acceptable or unacceptable. We shall let p_{ss}, p_{sa}, and p_{su} be the respective probabilities that an item which is superior is judged to be superior, acceptable, and unacceptable respectively. For acceptable items, p_{as}, p_{aa}, and p_{au} will denote these probabilities and for unacceptable items, p_{us}, p_{ua}, and p_{uu} will represent these probabilities.

Every h^{th} item is inspected repeatedly and the process is judged to be in control if either l items are classified as superior or k items are classified as either superior or acceptable prior to f being classified as unacceptable. We call this rule $TC_1TC_{12}TN$ and we will use the notion $TC_1TC_{12}TN(\nu,\rho)$ to represent the probability that there are l classified as superior or k as superior or acceptable prior to f classified as unacceptable where ν is the probability that the inspected item is judged superior and ρ is the probability that the inspected item is judged as acceptable. The method for calculation of this probability for a given ν and ρ will be given in matrix form using a Markov chain approach in section 3.

2. Probabilistic Analysis

1) A) Given that the item being inspected is superior, what is the probability that process is judged to be control?

ANSWER: $\mathrm{TC}_1\mathrm{TC}_{12}\mathrm{TN}(p_{ss}, p_{sa})$

B) Given that the item being inspected is acceptable, what is the probability that process is judged to be control?

ANSWER: $\mathrm{TC}_1\mathrm{TC}_{12}\mathrm{TN}(p_{as}, p_{aa})$

B) Given that the item being inspected is unacceptable, what is the probability that process is judged to be control?

ANSWER: $\mathrm{TC}_1\mathrm{TC}_{12}\mathrm{TN}(p_{us}, p_{ua})$

2) A) Given that the process is in control, what is the probability that the process is judged to be in control?
ANSWER: If it is in control, then the inspected item is superior with probability p_1, acceptable with probability p_2, and unacceptable with probability $1 - p_1 - p_2$. In light of the answer to question 1 and using the law of total probability,

$$\begin{aligned} P_{II} = {} & P(judged\ in\ control|\ in\ control) \\ & = p_1\mathrm{TC}_1\mathrm{TC}_{12}\mathrm{TN}(p_{ss}, p_{sa}) + p_2\mathrm{TC}_1\mathrm{TC}_{12}\mathrm{TN}(p_{as}, p_{aa}) \\ & + (1 - p_1 - p_2)\mathrm{TC}_1\mathrm{TC}_{12}\mathrm{TN}(p_{us}, p_{ua}) \end{aligned}$$

B) Given that the process is out of control, what is the probability that the process is judged to be in control?
ANSWER: If it is out of control, then the inspected item is superior with probability p_1^*, acceptable with probability p_2^*, and unacceptable with probability $1 - p_1^* - p_2^*$. In light of the answer to question 1 and using the law of total probability,

$$\begin{aligned} P_{OI} = {} & P(judged\ in\ control|\ out\ of\ control) \\ & = p_1^*\mathrm{TC}_1\mathrm{TC}_{12}\mathrm{TN}(p_{ss}, p_{sa}) + p_2^*\mathrm{TC}_1\mathrm{TC}_{12}\mathrm{TN}(p_{as}, p_{aa}) \\ & + (1 - p_1^* - p_2^*)\mathrm{TC}_1\mathrm{TC}_{12}\mathrm{TN}(p_{us}, p_{ua}) \end{aligned}$$

3) Once it goes out of control, what is the distribution of the number of inspections needed to determine it is out of control?
ANSWER:
This is geometric distribution with success parameter $1 - P_{OI}$.

4) Let Y = time measured in decision time until the process is actually out of control and π is the probability of a shift on any item produced then the
$P(Y = y) = [(1-\pi)^h]^{y-1}[1 - (1-\pi)^h] = \theta(1-\theta)^{y-1}$, $y = 1, 2, 3, \ldots$.
So Y has a geometric distribution with parameter $\theta = 1 - (1-\pi)^h$.

5) Let X = time measured in decision time until the process is judged out of control.
$P(X = x) = \sum_{y=1}^{\infty} P(X = x|Y = y)P(Y = y)$ where

$$P(X = x|Y = y) = \begin{cases} [P_{II}]^{x-1}[1 - P_{II}], \ x < y \\ [P_{II}]^{x-1}[1 - P_{OI}], \ x = y \\ [P_{II}]^{y-1}[P_{OI}]^{x-y}[1 - P_{OI}], \ x > y \end{cases}$$

$$P(X = x) = \sum_{y=1}^{x-1}[P_{II}]^{y-1}[P_{OI}]^{x-y}\,[1 - P_{OI}]\,(\theta(1-\theta)^{y-1}) + [P_{II}]^{x-1}[1 - P_{OI}](\theta(1-\theta)^{y-1}) + \sum_{y=x+1}^{\infty}[P_{II}]^{x-1}[1 - P_{II}](\theta(1-\theta)^{y-1})$$

Section 3: Markov Chain Analysis

Consider the Markov Chain $\{X_n\}$ with state space

$$\{(r, s, t):\ 0 \le r \le l, 0 \le s \le \mathrm{k}, \mathrm{r} \le \mathrm{s}, 0 \le \mathrm{t} < \mathrm{f}\} \cup \{(r, s, f):\ 0 \le r < l, 0 \le s < \mathrm{k}, \mathrm{r} \le \mathrm{s},\}$$

where $X_{n=}(r,s,t)$ means that after the n^{th} classification there are a r total superior, s total superior or acceptable, and a total of t unacceptable. The transition probabilities are of the form

$P(X_n = (r + 1,s + 1,t)|\ X_{(n-1)} = (r,s,t)) = \nu$,
$P(X_n = (r,s + 1,t)|\ X_{(n-1)} = (r,s,t)) = \rho$,
$P(X_n = (r,s,t + 1)|\ X_{(n-1)} = (r,s,t)) = 1- (\nu + \rho)$

Using this information and the recognition that the absorbing states are the states such that $r = l$, $s = k$, or $t = f$ we can easily determine the one-step probability matrix **P** which is described as follows.

For each Markov chain there are absorbing (recurrent) states, which correspond to the end of the repetitive classifications for a single item and the consequent decision. Let A denote the set of absorbing states and a denote the number of absorbing states. In fact, the singleton sets consisting of each of these absorbing states are recurrent classes. The remaining states are transient which we will denote by T and likewise the number of transient states by t. Written in canonical form, the one-step transition probability matrix **P** for the Markov chain is $\begin{bmatrix} \mathbf{P_1} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}$, where $\mathbf{P_1}$ is the $a \times a$ identity matrix for the absorbing states, **R** is a $t \times a$ matrix containing the one-step probabilities of the transient states to the recurrent (absorbing) states, **Q** is a $t \times t$ matrix containing the one-step probabilities among the transient states, and **0** is the $a \times t$ zero matrix. The one-step probabilities of **R** and **Q** are determined by the transition probabilities given for each test. The first row of **Q** contains the one step transition probabilities from state (0,0,0).

To compute the moments of the decision time, we will define the following notation. Since elements of T appear as subscripts, we will use i and j as typical elements of T. However, it should be noted that when we do so, each of i and j refer to an ordered triple such as (r,s,t). Let,

- $\mathbf{I}_{t \times t}$ = identity matrix of dimension $t \times t$
- $\mathbf{M}_{t \times t} = (\mathbf{I}_{t \times t} - \mathbf{Q}_{t \times t})^{-1}$- the fundamental matrix of dimension $t \times t$
- $\mathbf{e}_m$ = column vector of length t where the m^{th} element is one and the remaining elements are zero.
- $\boldsymbol{e_m}'$ is defined to be the transpose of $\mathbf{e}_m$

- $\mathbf{u}_{\{RS\}}$ = column vector where all the elements corresponding to the rejection states are one, and the remainder of the elements are zero.
- $\mathbf{1}_z$ = column vector of ones of length z
- N_{ij} = random variable that represents the number of times the process visits state j before it eventually enters a recurrent state, having initially started from state i ($i,j \in$ T).
- $\mu_{ij} = E(N_{ij})$ for $i,j \in$ T.
- $\mathbf{M}_\rho = \left[\sum_{j\in T} \mu_{ij}\right] = \mathbf{M1}_t$ - column vector such that the m^{th} element is the sum of the m^{th} row of $\mathbf{M}$
- $\mathbf{M}_{\rho^2} = \left[\left(\sum_{j\in T} \mu_{ij}\right)^2\right] = diag(\mathbf{M}_{\boldsymbol{\rho}})\mathbf{M}_{\boldsymbol{\rho}}$ - column vector such that the m^{th} element is the square of the sum of the m^{th} row of **M.** Note: diag ($\mathbf{M}_\rho$) is a diagonal matrix whose entries are the corresponding entries of $\mathbf{M}_{\rho.}$

The results below are given without proof and based on formulas in Bhat (1984).

Given that the item being inspected is superior, what is the probability the process is judged to be in control, the mean, the variance, and probability mass function of the time until a decision is reached? What if the item is acceptable? What if the item is unacceptable?

Consider the decision time for a single item for i.i.d. classifications according as stated in section 1.

1) $TC_1TC_{12}TN(\nu,\rho) = 1 - \mathbf{e}_1' \mathbf{MR}\, \mathbf{u}_{\{RS\}}$
2) Expected decision times
   E(Decision time| superior) = $\mathbf{e}_1'\mathbf{M1}_t$ where $(\nu,\rho) = (p_{ss}, p_{sa})$
   E(Decision time| acceptable) = $\mathbf{e}_1'\mathbf{M1}_t$ where $(\nu,\rho) = (p_{as}, p_{aa})$
   E(Decision time| unacceptable) = $\mathbf{e}_1'\mathbf{M1}_t$ where $(\nu,\rho) = (p_{us}, p_{ua})$

   E(Decision time|in control) = p_1E(Decision time |superior) + p_2E(Decision time |acceptable) + (1-p_1 - p_2) E(Decision time |unacceptable)

   E(Decision time|out of control) = p^*_1E(Decision time |superior) + p^*_2E(Decision time |acceptable) + (1-$p^*_1 - p^*_2$) E(Decision time |unacceptable)

3) The variances of decision time are
   Var(Decision time |superior) = $e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}} - \mathbf{M}_{\rho^2}\right]$ where $(\nu,\rho) = (p_{ss}, p_{sa})$
   Var(Decision time |acceptable) = $e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}} - \mathbf{M}_{\rho^2}\right]$ where $(\nu,\rho) = (p_{as}, p_{aa})$
   Var(Decision time |unacceptable) = $e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}} - \mathbf{M}_{\rho^2}\right]$ where $(\nu,\rho) = (p_{us}, p_{ua})$

4) The probability mass function of the decision time
   P(decision time = m | superior) = $\mathbf{e}_1' \mathbf{Q}^{m-1}\mathbf{R}\, \mathbf{1}_a$ where $(\nu,\rho) = (p_{ss}, p_{sa})$
   P(decision time = m | acceptable) = $\mathbf{e}_1' \mathbf{Q}^{m-1}\mathbf{R}\, \mathbf{1}_a$ where $(\nu,\rho) = (p_{as}, p_{aa})$
   P(decision time = m | unacceptable) = $\mathbf{e}_1' \mathbf{Q}^{m-1}\mathbf{R}\, \mathbf{1}_a$ where $(\nu,\rho) = (p_{us}, p_{ua})$

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