SPC Charts for Detecting Shifts in Variance with Autocorrelated Data

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Abstract

Processes that arise naturally, e.g., from manufacturing or the environment, often exhibit complicated autocorrelation structures. When monitoring such a process for changes in variance, accounting for that autocorrelation structure is critical. While charts for monitoring the variance of processes of independent observations and some specific autocorrelated processes have been proposed in the past, the chart presented in this article can handle any general stationary process. The performance of the proposed chart was examined through simulations for AR(1) processes and demonstrated with an example.

Key Words: Autocorrelated processes, average run length, exponentially weighted mean square, nonstationary processes, statistical process control, variance shift

1. Introduction

Statistical process control charts have been widely used to monitor changes in a process mean and variance. Traditionally, control charts are based on a fundamental assumption that process data are statistically independent. Process data, however, are often not statistically independent; this is especially the case in continuous industries such as the chemical industry. Various control charts have been proposed to monitor process data from an autocorrelated process. Zhang (1998) and (2000) proposed to monitor the process mean with a variant of the EWMA chart, called the EWMAST chart, which can be used for any general stationary process. A related chart was proposed by Jiang et al. (2000).

To monitor process variability, MacGregor and Harris (1993) proposed the exponentially weighted mean square (EWMS) chart. Other charts related to the EWMS chart have also been proposed to monitor the process variance, e.g., Castagliola (2005).The EWMS and exponentially weighted moving variance (EWMV) charts were discussed by Eyvazian et al. (2008) and Huwang et al. (2009), which focus on the case that the process data are independent. Although, MacGregor and Harris (1993) discussed the applications of the EWMS chart to autocorrelated observations, but their discussion was limited to AR(1) plus white noise processes. In this article, we extend the EWMS chart to general stationary processes.

2. The EWMS Chart for a stationary process

Assume that $\{X_n, n=1,2...\}$ is a stationary process with mean μ and variance σ_X^2. Namely, $\mathrm{E}[X_n]=\mu$, $\mathrm{Var}[X_n]=\sigma_X^2$ for $n=1,2,...$, and the autocovariance at lag m is given by

$$R_m = \mathrm{Cov}[X_n, X_{n+m}], \tag{1}$$

where R_m only depends on the lag m. MacGregor and Harris (1993) defined the exponential weighted moving mean square by

$$S_n^2 = (1-r)S_{n-1}^2 + r(X_n - \mu)^2 \qquad n=1,2,..., \tag{2}$$

where r $(0<r\le 1)$ is the weighting parameter. We take $S_0^2=\sigma_X^2$, the process variance. From (2), it is clear that S_n^2 is an estimator of the process mean square error at time n.

$$\begin{aligned} S_n^2 &= (1-r)S_{n-1}^2 + r(X_n-\mu)^2 \\ &= \sum_{j=1}^{n} r(1-r)^{n-j}(X_j-\mu)^2 + (1-r)^n S_0^2 \\ &= T_n^2 + (1-r)^n\sigma_X^2, \end{aligned} \tag{3}$$

where $T_n^2 = \sum_{j=1}^{n} r(1-r)^{n-j}(X_j-\mu)^2$. MacGregor and Harris (1993) proposed to use S_n^2 to monitor the process mean square error (MSE) and called the corresponding chart the EWMS chart. Obviously, when the process mean is a constant, S_n^2 is an estimator of the process variance and the EWMS chart can be used to monitor the process variance. MacGregor and Harris (1993) initially proposed the EWMS chart for independent sequences. Applying the EWMS chart to autocorrelated observations was also discussed in that article. Specifically, an AR(1) process with additional white noise was discussed. The process is described by

$$X_n = \eta_n + e_n, \tag{4}$$

where $\{e_n\}$ is a white noise process and η_n is an AR(1) process with zero mean given by

$$\eta_n = \phi\eta_{n-1} + \alpha_n, \tag{5}$$

where α_n is a white noise process that is independent of e_n and $|\phi|<1$, which guarantees the stationarity of $\{\eta_n\}$ and thus $\{X_n\}$. However, we need to emphasize that although the AR(1) plus white noise processes is popular and reasonably realistic, it is a special case of general stationary processes.

Let $Y_j = X_j - \mu$, $j=1,...,n$. We express the quadratic form T_n^2 in matrix notation by

$$T_n^2 = \mathbf{Y}^{\mathrm{T}}\mathbf{M}\mathbf{Y} = \left(Y_1\, Y_2 \cdots Y_n\right)\begin{pmatrix} r(1-r)^{n-1} & 0 & 0 & \cdots & 0 \\ 0 & r(1-r)^{n-2} & 0 & \cdots & 0 \\ 0 & 0 & r(1-r)^{n-3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & r \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}. \tag{6}$$

We assume that the X_j, $j=1,\dots,n$ are jointly normally distributed with the correlation matrix $\mathbf{P}$. Since $\mathbf{M}$ is non-singular, Theorem 2.1 of Box (1954) gives us,

$$T_n^2 \sim \sigma_X^2 \sum_{j=1}^{n} \xi_j \chi_j^2(1)\,, \tag{7}$$

where ξ_j are the n non-zero eigenvalues of $\mathbf{U} = \mathbf{PM}$. Recall that $\mathbf{P}$ is the correlation matrix of $\{X_j, j=1\dots,n\}$, i.e.,

$$\mathbf{P} = \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{n-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{n-1} & \rho_{n-2} & \rho_{n-3} & \cdots & 1 \end{pmatrix}, \tag{8}$$

where ρ_m is the autocorrelation of $\{X_n\}$ at lag m , i.e., $\rho_m = R_m/R_0$. When $\{X_n\}$ is jointly normally distributed, Box (1954) proposed that T_n^2 is approximately distributed (in the sense of the two moments approximation) as

$$T_n^2 \sim \sigma_X^2 g_n \chi^2(\nu_n) \tag{9}$$

with

$$g_n = \frac{\sum_{j=1}^{n} \xi_j^2}{\sum_{j=1}^{n} \xi_j} \quad \text{and} \quad \nu_n = \frac{\left(\sum_{j=1}^{n} \xi_j\right)^2}{\sum_{j=1}^{n} \xi_j^2}. \tag{10}$$

It can be shown that

$$g_n = \frac{r}{2-r}\,\frac{1-(1-r)^{2n} + 2\sum_{m=1}^{n-1} \rho_m^2 (1-r)^m [1-(1-r)^{2(n-m)}]}{1-(1-r)^n} \tag{11}$$

and

$$\nu_n = \frac{2-r}{r} \frac{[1-(1-r)^n]^2}{1-(1-r)^{2n} + 2\sum_{m=1}^{n-1} \rho_m^2 (1-r)^m [1-(1-r)^{2(n-m)}]}. \tag{12}$$

From (3) and (9), we get the following approximate distributional relationship:

$$\frac{S_n^2}{\sigma_X^2} \sim g_n \chi^2(\nu_n) + (1-r)^n, \tag{13}$$

where g_n and the degrees of freedom ν_n are functions of n, r, and ρ_m, the autocorrelation function of $\{X_n\}$. For a given confidence level α, the upper and lower control limits are given by

$$U_n = \sigma_X^2 \left(g_n \chi^2_{1-\alpha/2}(\nu_n) + (1-r)^n \right) \tag{14}$$

and

$$L_n = \sigma_X^2 \left(g_n \chi^2_{\alpha/2}(\nu_n) + (1-r)^n \right). \tag{15}$$

The central line for the chart of plotted S_n^2 is at σ_X^2. When $\{X_n\}$ is an i.i.d. sequence, $\sum_{j=1}^{n} \xi_j^2 \approx \frac{r}{2-r}$ leading $g_n \approx r/(2-r)$ and $\nu_n \approx (2-r)/r$ as indicated in MacGragor and Harris (1993).

3. Simulation Study

The intention of this simulation study is to explore choices of the tuning parameter *r* that lead to good performance of the chart. We use the average run length (ARL) to measure the performance. MacGregor and Harris (1993) used simulations to compare ARLs for different *r* for independent sequences. Their results showed that *r* = 0.05 is a good choice. In this simulation study, AR(1) processes were considered.

The definition of a stationary AR(1) process is found in (5). That is, $\{X_n, n = 1, 2...\}$ is an AR(1) process with the parameter ϕ and mean μ given by

$$X_n - \mu = \phi(X_{n-1} - \mu) + a_n, \tag{16}$$

where $\{a_n\}$ is a white noise process. Generating realizations from such a process when it is stationary is easily done with existing software, e.g., the function `arima.sim` of the R (R Core Team, 2012) package `stats` (R Core Team, 2012). However, since we must generate processes with variance shifts, i.e., nonstationary AR(1) processes, this section is broken into two parts. The first part describes how realizations of an AR(1) process were generated, and the second presents the results.

3.1 Generating Realizations

Without loss of generality, take $\mu = 0$ and the variance of $\{X_n\}$, σ_X^2, before time n_0, to be one. Denote the variance of the white noise process, $\{a_n\}$ at time *n* as $\sigma_a^2(n)$. Let $k\sigma_X^2 = k$ be the variance of $\{X_n\}$ at time n_0 and after. Namely, the process variance shifts from 1 to k starting at n_0. From (16), we have

$$\mathrm{Var}(X_n) = \phi^2 \mathrm{Var}(X_{n-1}) + \sigma_a^2(n).$$

Thus, when $n < n_0$, $\sigma_a^2(n) = 1 - \phi^2$, when $n = n_0$, $\sigma_a^2(n) = k - \phi^2$ and when $n > n_0$, $\sigma_a^2(n) = k(1-\phi^2)$. Note that it is possible for $k - \phi^2$ to be negative, and when that is the case, the corresponding AR(1) process does not exist.

Given this information, the following algorithm is used to generate an AR(1) process with parameter ϕ and whose variance shifts from 1 to *k* at time n_0:

1. Specify the total length of the process and the length of the warm-up period, $T \geq n_0 + 1$ and $B \geq 2$, respectively
2. $x_{-B+1} = 0$; note that x_n denotes a realization of the random variable X_n
3. While $-B + 2 \leq n < n_0$, $x_n = \phi x_{n-1} + a_n$ where $a_n \sim iid\ \mathrm{N}(0, 1-\phi^2)$
4. When $n = n_0$, if $k - \phi^2 > 0$, then $x_{n_0} = \phi x_{n_0 - 1} + a_{n_0}$, where $a_{n_0} \sim \mathrm{N}(0, k - \phi^2)$
5. When $n_0 < n \leq T$, $x_n = \phi x_{n-1} + a_n$, where $a_n \sim \mathrm{N}(0,\ k(1-\phi^2))$
6. The realization of the process is then $\{x_n, n = 1, \ldots, T\}$.

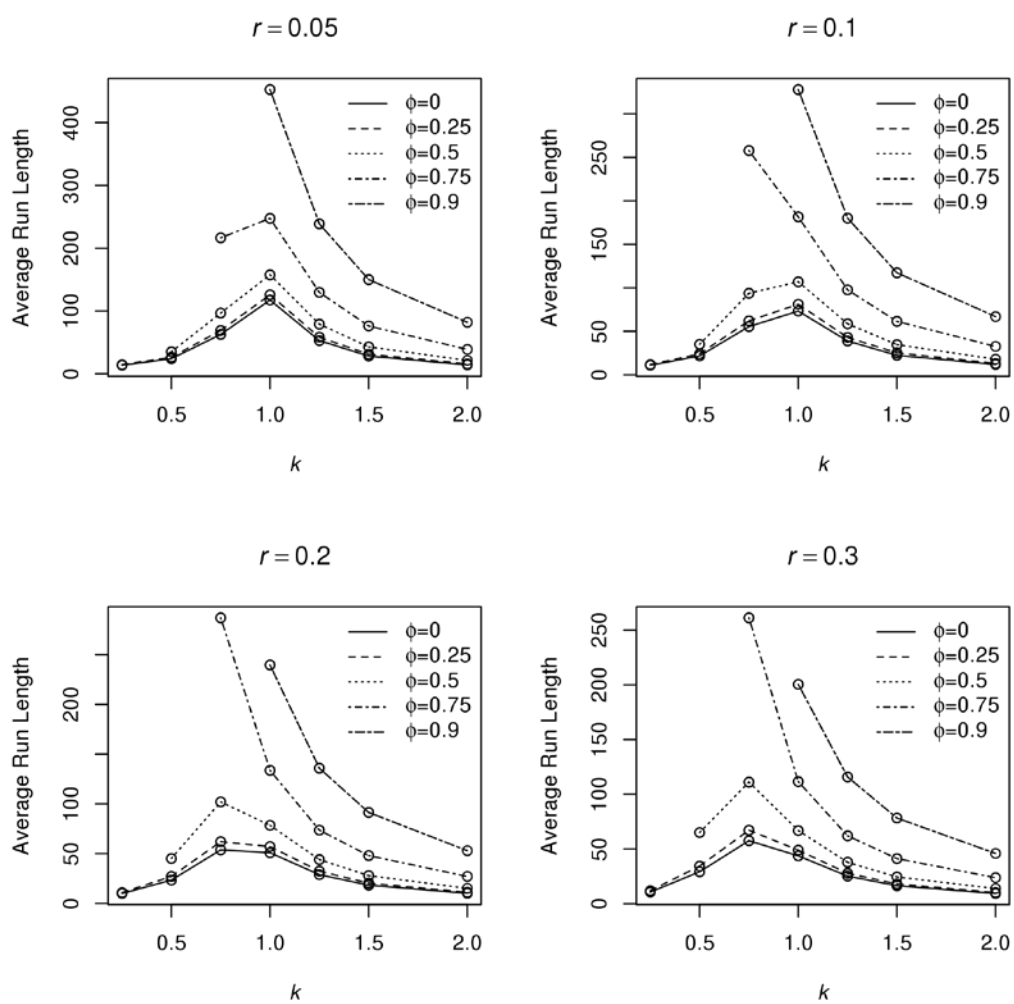
3.2 Results

For each combination of the parameter values $n_0 \in \{3, 6, 11, 21, 51, 501\}$, $\phi \in \{0, 0.25, 0.5, 0.75, 0.9\}$, $k \in \{0.25, 0.5, 0.75, 1, 1.25, 1.5, 2\}$ and $r \in \{0.05, 0.1, 0.2, 0.3\}$, when possible, the RL of 50,000 AR(1) processes was calculated. Of course, for example, when $\phi = 0.9$ and $k = 0.25$ such a process does not exist. Note that n_0 affects the run length because the bounds in Equations (15) and (16) are functions of *n*. From each set of 50,000 RLs, the average was calculated, which is depicted graphically in Figure 1 for $n_0 = 51$. The actual average run length changes for the other values of n_0, but the general pattern is nearly identical.

From Figure 1, we see that r = 0.05 is a good choice for an AR(1) process since it is the only choice presented for which *k* = 1 leads to the highest average run length for all values of ϕ. It is well known that as a general rule for performance of a control chart, larger in-control (i.e., *k* = 1in our case) ARL is desired. This implies that for r = 0.05 the chart is unbiased in the language of Huwang et al. (2009), but that for the other choices of r the chart is biased. Also note that for r = 0.05 as the variance shift gets larger ($k > 1$) or smaller ($k < 1$), the average run length decreases monotonically, an important and desirable property.

Another interesting observation is that when the process is highly autocorrelated, such as when ϕ = 0.9, the performance of the chart degrades with larger out-of-control ARL . This is observed from Figure 1 by comparing the ARLs for $\phi = 0.75$ to $\phi = 0.9$, which are substantially longer. This degradation in performance is expected because when ϕ is near 1, the process is nearly nonstationary.

Figure 1. The ARL plotted against k, the variance jump, for several values of ϕ the AR(1) parameter, and r the weight in the EWMS chart.



4. Constructed Example

To demonstrate the EWMS chart, we use a constructed example. In practice, if the EWMS chart should be used to detect a change in the process variance, a chart to look for changes in the process mean should be applied first because as indicated earlier, the EWMS chart monitors the mean square error of a process. When the process mean is stable, it will demonstrate possible shifts in the process variance. However, when the process mean is unstable, the EWMS chart will show changes in the mean square error of the process even if the process variance does not change. Zhang (1998) proposed to use the EWMA chart for monitoring changes in the process mean when the process is autocorrelated and stationary. The chart is called the EWMAST chart. Therefore, we can apply both the EWMAST and EWMS charts in that order to a stationary process. When an EWMAST chart signals no mean shifts, an EWMS chart can be applied to the data to monitor the process variance.

We generated a realization from an AR(1) process. The process mean is $\mu = 0$, the dependence parameter is $\phi = 0.5$ and the process variance is $\sigma_X^2 = 1$ from $t = 1$ to $t = 150$, $\sigma_X^2 = 0.5$ from $t = 151$ to $t = 300$, $\sigma_X^2 = 2$ from $t = 301$ to $t = 450$, and $\sigma_X^2 = 1.5$ from $t = 451$ to $t = 600$. The observed process is displayed in Figure 2.

An EWMAST chart is applied to the simulated data with the chart parameter $\lambda = 0.2$. The standard deviation of the EWMA statistic in the EWMAST chart based on (10) in Zhang (1998) is 0.51. The chart, with 3-sigma control limits, shown in Figure 3, demonstrates that the process mean is essentially stable.

For the EWMS chart, we choose $r = 0.05$ and $\alpha = 0.05$, which give the asymptotical lower and upper control limits to be 0.52 and 1.64, respectively. Decreases in mean square error were detected at $t = 167$ to 302 and increases at $t = 314$ to 470, $t = 497$ to 530 and other points as shown in Figure 4. Since it is shown in Figure 3 that the process mean is stable, we conclude that the process vairance changed.

Figure 2. The realization of the AR(1) process used to illustrate the EWMS procedure where the process mean is fixed at 0, but the process variance changes 3 times, 1 to 0.5 to 2 to 1.5.

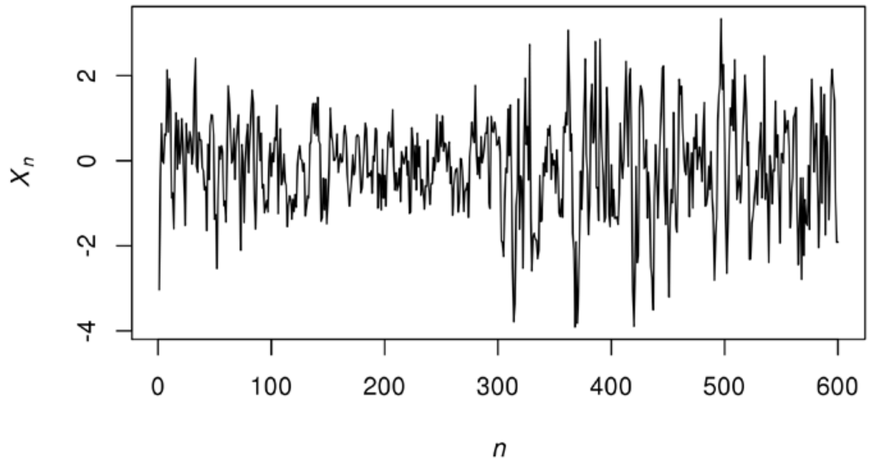


Figure 3. The EWMAST chart, with control limits for the time series displayed in Figure 2.

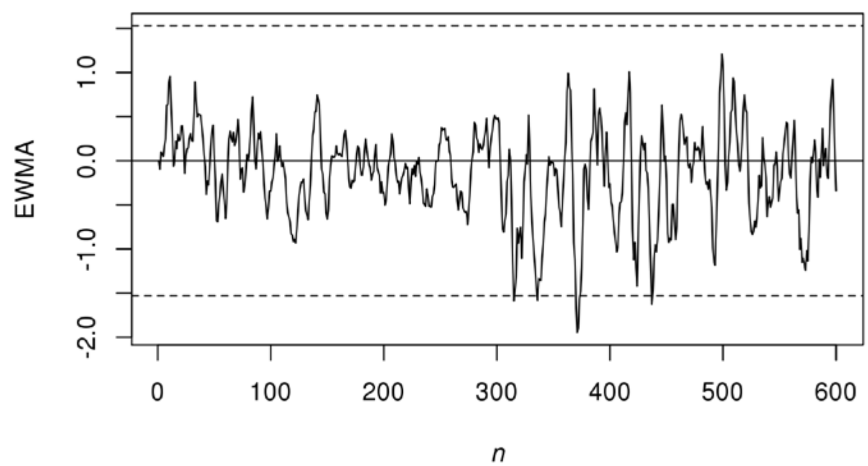
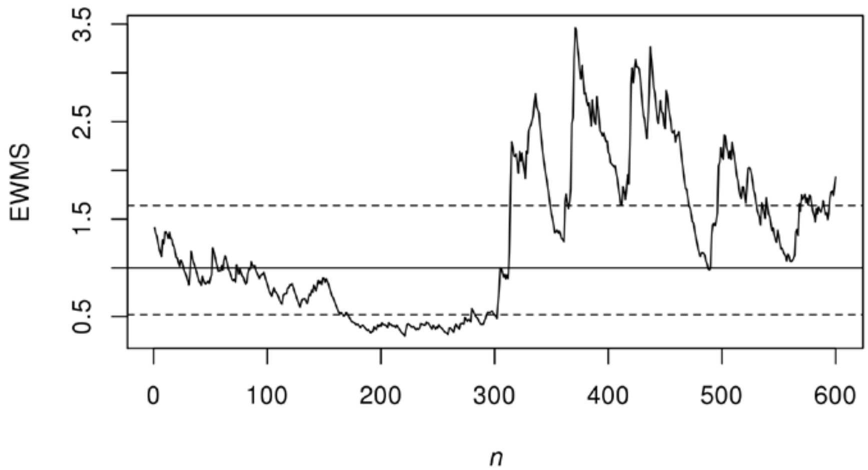


Figure 4. The EWMS chart, with control limits, for the time series displayed in Figure 3.



5. Conclusions

Processes that arise naturally, e.g., from manufacturing or the environment, often exhibit complicated autocorrelation structures. When monitoring such a process for changes in variance, accounting for that autocorrelation structure is critical. While charts for monitoring the variance of processes of independent observations and some specific autocorrelated processes have been proposed in the past, the chart presented in this article can handle any general stationary process, which is the major contribution of the work. The performance of the EWMS chart (in terms of ARL) was examined for AR(1) processes through simulation.

The chart was demonstrated with one example. The example was based on an underlying AR(1) process with the dependence parameter $\phi = 0.5$, process mean $\mu = 0$, and three changes of variance. All three true changes in variance were found.

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