Using Novel Distance Metrics to Evaluate Statistical Robustness of Chaotic Control Systems

Morris Morgan[1] and Carolyn Morgan[1]
[1]Hampton University, Queen and Tyler Streets, Hampton, VA 23668

Abstract
Evaluating the robustness of integrated control systems with clusters of unmanned aerial vehicles (UAVs) flying in stable formation is of primary concern in the aerospace industry. This paper describes the use of robust statistical control metrics to assess the level of synchronization existing between chaotic systems or networks comprised of local chaotic subsystems. Distance measures based on chordal and spherical displacements were used to determine the closeness of companion dynamical systems. The success of such metrics for predicting linear control stability is well documented. This paper addresses the extension and statistical limitations of these metrics to highly nonlinear systems where robust statistical control is sensitive to both parameter and disturbance input noises. In this work these systems are modeled as chaotic units where disturbances can arise in a myriad of ways and robustness must be properly characterized. The proposed statistical metrics will aid this study and provide insight about dynamical interactions among UAVs.

Key Words: statistical robustness, control, statistical synchronization, fuzzy systems

1. Introduction

In a prior paper on chaotic control, Morgan and Morgan (2011) dealt primarily with structured uncertainties that arose in "complete" synchronized systems through coefficients of the equations or fluctuations introduced via the controller gains. These statistical uncertainties ultimately manifested themselves as additions to either the diagonal (equation coefficients) or off-diagonal terms (introduced via controller structure) of the overall linearized system control matrix that governs global stabilization. Specifically, the impact of stochastic noise introduced via these terms was of primary concern where the overriding objective was to devise a strategy for the design of reliable control systems for chaotic oscillators (a network) that was robust to such factors. Also, understanding the impact of these quantities on the performance of large integrated networks was likewise a critical concern. The initial phase of that study was limited to analyzing complete (dual) chaotic system behavior with respect to three generic control issues:

- Identifying controller numbers and selecting control variables,
- Determining the error propagation patterns arising via either initial condition disturbances or via uncertainties in model parameters, and

- Assessing the effectiveness of unit redundancy as a tool for enhancing overall system robustness and reliability.

Synchronization bundling was established as a way to enhance overall system reliability in the presence of additive coefficient noise. That approach involved connecting multiple independent and identical receivers (drives) to a single transmitter (response). Thus, an "or gate" connection/parallel processing configuration could be structured in sufficient numbers to guarantee successful signal transmission at a given synchronization probability level (SPL). A quantitative procedure was devised to determine the number of bundling units for a given controller combination given that the SPL requirement varied with both controller number and gain. The present study, however, focuses on the specification of the minimum controller gain level needed to stably synchronize a connected system of dissimilar (general) chaotic oscillators that have fundamental different dynamical behavior. Such problems are intimately linked to the controllability of clusters of unmanned aerial vehicles (UAVs) and are of vital interest to the aerospace industry.

1.1 Basic Methodology

Our current method is based on the premise that it is possible to construct an interval approximation model (crisp fuzzy model) of the system of ordinary differential equations that describes an individual chaotic (response) oscillator. Such systems where there are parameter uncertainties in the coefficients are referred to as extended or fuzzy parameter uncertain systems. The initial step in this process involved constructing interval approximation for each dependent variable in the chaotic model using descriptive statistics shown in Table 1 of the unsynchronized response system. The chaotic oscillators analyzed in Table 1 are all described in detail in Sprott (2003). The volume contraction (C) and (E) expansion values are measured relative to the Halvorsen volume since it serves as the Lead or driver for synchronization. Next, armed with these interval approximations the local stability of each term in the model is assessed and used to determine the global stability requirement for a given drive- response combination. Under this paradigm the stability of the drive system is not necessary for establishing overall system synchronization. An obvious extension of the current effort is to focus on improving such interval estimates by using nearest-interval approximations that have generated much excitement in the fuzzy set community. Specifically, there are plans to use fuzzy number α - cuts to more precisely narrow the interval estimates for stabilization. This approach will necessitate developing a membership function for each dependent term in our response model. Furthermore, it will be assumed that these membership functions mirror the respective dependent variable frequency distributions. Once equipped with this membership function, it will be used to obtain a nearest interval approximation that is best for estimating distances between fuzzy numbers. This mathematical operation has the net effect of converting a fuzzy system with uncertainties into one with interval certainties that is more tractable. The success of the present exploratory inquiry does however justify the merits of the current approach.

Table 1: Statistical Data Summary of Chaotic Model Variables

	Type of Oscillator (see Sprott, 2003)							
Metric	**Halvorsen**	**Lorenz**	**Rossler**	**Diffuse Lorenz**	**Moore-Spiegel**	**Chen**	**Burke-Shaw**	**Rucklidge**
Volume	3882	52,092	4558	253	3397	65,413	----	2134
x-mean	-3.440	0.296	0.152	0.2632	-0.005	0.1564	----	0.007
y-mean	-3.440	0.304	-0.947	-0.246	-0.005	0.1624	----	-0.433
z-mean	-3.540	24.300	0.925	0.0065	0.052	25.200	----	6.403
x-range	19.560	38.310	20.500	5.740	4.810	50.900	----	24.62
y-range	19.560	52.700	18.630	9.370	12.800	58.300	----	10.64
z-range	19.560	49.260	22.800	9.050	105.600	42.100	----	15.56
x-Fuzzy	-13.3,6.3	-19.,19	-9.1,11.	-3.1,2.7	-2.5,2.3	-25,25.	----	-9.6,15
y-Fuzzy	-13.3,6.3	-26.,26.	-11.,7.8	-4.2,5.2	-6.8,6	-29.,29	----	-5.2,5.5
z-Fuzzy	-13.3,6.3	2.1,47.2	0,22.8	-4.8,4.2	-46,60	6.3,48.	----	0.56,15
Lyapunov Sum	-3.810	-13.667	-5.323	-1.000	-1.000	-12.000	-11.000	-3.000
Contraction (C) - Expansion (E)	Lead	C	C	E	E	C	C	E
Distance Norm	8.983	26.746	7.237	1.891	8.931	-----	-----	-----
Chordal Norm	0	0.080	0.066	0.8473	0.164	-----	-----	-----

Table 1 provides a summary of the specific statistical metrics used to evaluate and assess process dynamics. Estimates of the respective uncoupled oscillator dependent variable means (x, y and z), the corresponding ranges and chordal distance norm were obtained using a sample size of two hundred. This latter distance metric represented the mean spatial displacement between the Halvorsen (driver) and a given responder. These various statistical quantities were also used to construct fuzzy intervals estimates for each of the response system dependent variables and to determine the equivalent ellipsoidal volume occupied by each chaotic oscillator. Both the ellipsoidal volume estimates and the Lyapunov sums (trace) were used to determine whether a given system synchronization process involved an expansion or contraction of size.

2. Discussion

As noted earlier, previous work by Morgan and Morgan (2011) addressed the related problems of controller selection and number. The current effort focuses on individual controller stabilization and the estimation of the minimum gain needed to synchronize all units of a system as shown in Figure 1.

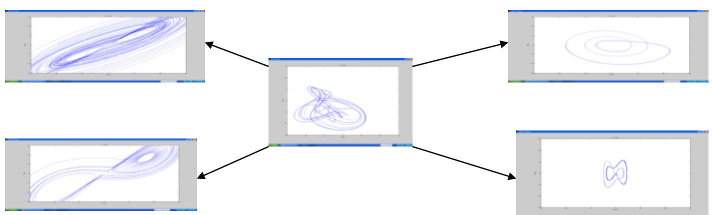


Figure 1: Uncoupled Connectivity in Dissimilar State Models

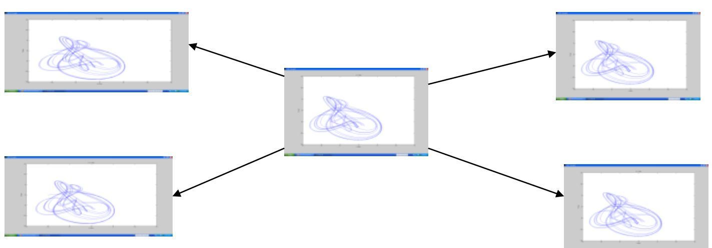
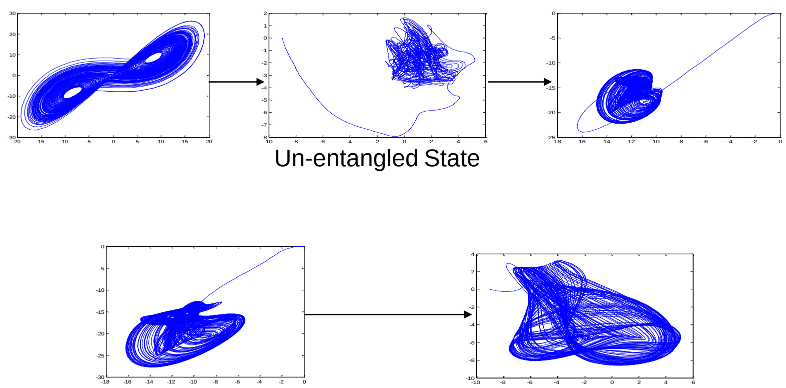


Figure 2: Uniform Connectivity of Dissimilar State Models

Figure 1 shows the original uncoupled phase portraits of several dissimilar chaotic oscillator states prior to any synchronization. The driver for this system was the Halvorsen's chaotic oscillator that was linked dynamically to four distinct response oscillators (Lorenz, Rossler, Moore-Spiegel and Chen) and controller stability ranges inferred from statistical metrics obtained from the uncoupled dynamics of individual response oscillators. Figure 2 highlights the desired goal of a uniform synchronization under a minimum gain requirement that specifies a high level of fidelity based on the correlation coefficient between driver and responder dynamics.

The phase portraits of Figure 3 describe the physical states manifested as a typical oscillator system transforms from an uncoupled to coupled configuration. Interestingly, this type of transition pattern was observed across each of the oscillator systems investigated in this study. Each transition process started with the formation of an un-entangled volume state followed by a morphing into the final desired configuration. It was always noted that the beginning state configuration had to be destroyed, go through a local critical point and reassembled into a final form. The nature of this critical point – whether it was a local minimum or maximum - appears to be related to its classification as a contraction or expansion system which could be predicted *a prior* from either the Lyapunov sum (trace) or the ellipsoidal volumes of the respective chaotic oscillators.

Figure 3: Typical Synchronization Transition Process



Based on the data of Table 1, two methods were developed for predicting the controller stabilization gain needed to achieve general synchronization across these chaotic oscillators. The first one involved using the respective fuzzy intervals of Table 1 to construct a fuzzy model representation of the original chaotic oscillator system. The notion of fuzzy numbers was first introduced by Dubois and Prade (1980) to model and handle imprecise information. The details of this current approach, as outlined in Table 2 using the Lorenz model, begins with the identification of all positive coefficient terms in each ordinary differential equation of the response system and requires adding a controller to each of these respective terms to address or overcome an instability that could arise via that term. Next, the requirement that each of these bracketed terms should be less than or equal to zero for stability provides numerical estimates of the controller stabilization gain for each. Since overall stability is demanded, the most stringent requirement for stability must be exceeded. An upper bound stabilization value estimate for the Lorenz model using this approach was 26 while a lower bound was 10.

Table 2: Fuzzy Model of Original Lorenz System

Original Model	Fuzzy Model
$dx/dt = 10(y-x)$	$dx/dt = [10-k]y-10x$
$dy/dt = -xz + 28x-y$	$dy/dt = [28-k-z_s]x-y$
$dz/dt = xy-2.67z$	$dz/dt = [y_s-k]x-2.67z$

Selection Rule:

Identify positive terms add controller and construct fuzzy interval with z_s =2.1 and y_s =26.

Maximum Interval, [0,26] and Minimum, [0,10]

The second approach simply involves using the largest absolute bounds obtained directly from the fuzzy interval arising from the dependent variables. There is some indication that the lower bound values for controller stabilization are linked with the formation of the un-entangled state identified in Figure 3. In addition it was also observed that increasing the controller gain in this region shifted all initial frequency distributions toward a limiting gamma distribution. Although not investigated as part of this study, it would be interesting to see if there is a unique frequency distribution pattern associated with the un-entangled state.

Figure 4 shows the relationship between controller gain and the correlation coefficient and reveals the presence of two distinct zones (unstable and stable regions) separated by a critical gain value. It's noteworthy to see that a single regression model appears to be adequate for describing the general dependency for all the chaotic oscillators studied here.

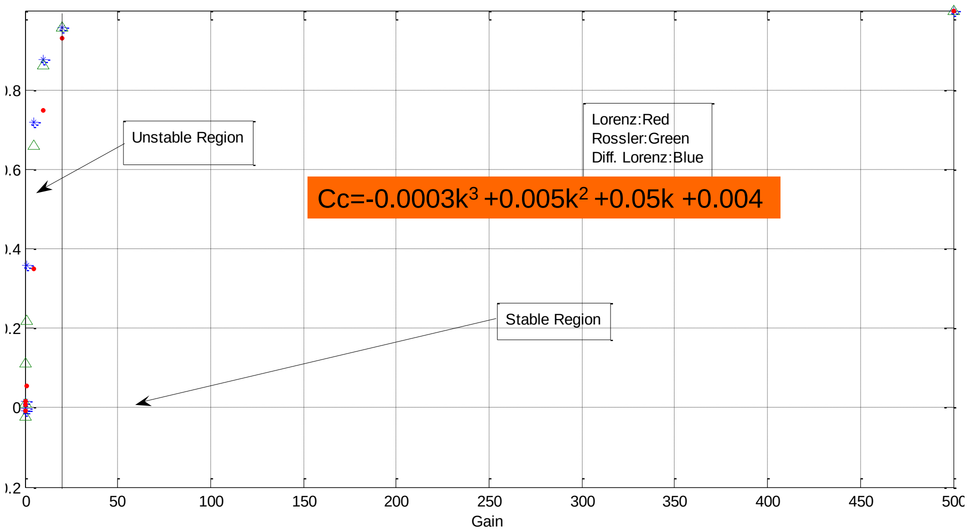


Figure 4: Correlation coefficient versus Controller Gain, k

3. Conclusions

Two methods were devised for determining the stable controller gains from descriptive statistics of the unsynchronized response (chaotic) oscillators. A simple regression model was found to adequately predict the relationship between synchronization fidelity and controller gain. Both the ellipsoidal volume and Lyapunov sum (trace) were excellent metrics for characterizing the synchronization transition process. The size of the instability region was found to be bounded by the length of the maximum fuzzy interval while the minimum fuzzy interval enclosed the un-entangled (critical) point.

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