Positive Trait Item Response Models

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Abstract
A new item response model is proposed for which the trait is positive. Three such models, the log-logistic, the log-normal, and the Weibull, are presented along with their item information curves. The data of seven addiction items from the DSM-IV from a study on alcohol addiction is analyzed by these three models using Bayesian Markov chain Monte Carlo methods. The item characteristic curves and item information curves are presented for all three models. The person scores for four item response patterns are presented for the log-logistic model.

Key Words: item response theory; positive latent trait; log-logistic; log-normal; Weibull; Bayesian inference; item characteristic curve; item information curve; person score;

1. Introduction

There has been increased interest in applying item response theory (IRT) models to measuring levels of addiction, including alcohol addiction (Beseler, Taylor, & Leeman, 2010; Gelhorn et al., 2008; Saha, Chou, & Grant, 2006; Wu et al., 2009), marijuana addiction (Wu et al., 2009), and gambling addiction (Sharp et al., 2012; Strong, Breen, & Lejuez, 2004; Strong, Breen, Lesieur, & Lejuez, 2003; Strong, Daughters, Lejuez, & Breen, 2004; Strong & Kahler, 2007; Strong, Lesieur, Breen, Stinchfield, & Lejuez, 2004).

Most current Bernoulli IRT models, including all of the models used above, assume each latent trait follows a standard normal density (Embretson & Reise, 2000; Fox, 2010). Although there is recent work that weakens the assumption of normality, especially symmetry (Azevedo, Bolfarine, & Andrade, 2011; Bazán, Branco, & Bolfarinez, 2006; Bolfarine & Bazán, 2010; Molenaar, Dolan, & Boeck, 2012; Woods & Thissen, 2006), the support of the trait is still assumed to be the entire real line. While this standard assumption may be appropriate for traits such as ability or attitude, it creates both conceptual and technical problems traits such as addiction.

Traits such as addiction have a positive probability for the absence of the trait. The standard assumption forces the trait for a non-addict to be located at negative infinity with probability zero, so that non-addicts are effectively excluded from the addiction continuum. Potentially dependent persons who endorse no items cannot be distinguished from non-addicts who would also endorse no items but can be independently identified by independent criteria. A more realistic assumption is that the trait for addiction follows a distribution with positive support with non-addicts located at zero.

IRT models with positive traits are not new. The original Rasch model posited a positive trait (Rasch, 1966), and similar models have been repeatedly proposed (Cressie & Holland, 1983; Ramsay, 1989). Recently, van der Maas, Molenaar, Maris, Kievit, and Borsboom (2011) proposed a "positive ability model" derived from information processing principles.

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2. Positive Trait Item Response Models

Let $Y_1, \ldots, Y_K$ be a set of Bernoulli random variables denoting items, such that $Y_k = 1$ if a person i endorses the item and $Y_k = 0$ if not. Let $Z_i, i = 1, \ldots, I$ be continuous random variables denoting i-th person's level of addiction such that $Z_i = 0$ if i is not addicted and $Z_i > 0$ otherwise. Let F be an absolutely continuous distribution function with positive support. The *positive trait item response model* (PTIRM) posits that the probability that person i endorses item k is

$$\begin{aligned}\pi_k(z_i) &= \mathrm{pr}\,(Y_k = 1 \mid Z_i = z_i, \alpha_k, \beta_k) \\ &= \begin{cases} 0 & \text{if} \quad z_i = 0; \\ F\left(\dfrac{z_i^{\beta_k}}{\alpha_k}\right) & \text{if} \quad z_i > 0. \end{cases}\end{aligned}$$

The parameter $\alpha_k > 0$ is interpreted as the severity of the addiction as revealed by the k-th item, with increasing α_k denoting increasing severity. The parameter $\beta_k > 0$ represents how well the k-th item can discriminate between levels of severity, with increasing β_k denoting finer discriminability.

Three specific PTIRMs are readily available. First is the *log-logistic*:

$$\pi_k(z_i) = \frac{z_i^{\beta_k}}{\alpha_k + z_i^{\beta_k}}.$$

Second is the *log-normal*:

$$\pi_k(z_i) = \Phi\left[\log\left(\frac{z_i^{\beta_k}}{\alpha_k}\right)\right].$$

And third is the Weibull:

$$\pi_k(z_i) = 1 - \exp\left(-\frac{z_i^{\beta_k}}{\alpha_k}\right).$$

As previously mentioned, the log-logistic with $\beta_k = \beta$ is Rasch's (1966) original item response model . The log-normal is Steven's psychophysical stimulus-response function (Stevens, 1957; Thomas, 1983). The Weibull model, although frequently used in biostatistics, is, I believe, new as a psychometric model. Other distributions are possible, e.g., log-Cauchy, generalized gamma.

These three models can be expressed as a log-linear extension of generalized linear item response models (Mellenbergh, 1994), namely as $h\,[\pi(z)] = \beta \log(z) - \log(\alpha)$, where h is the logit, probit, or complementary log-log link function.

The item information function provides an index of item precision as a function of the latent trait. The log-logistic item information function is

$$I(z) = \frac{\alpha\beta^2 z^{\beta-2}}{(\alpha + z^\beta)^2} = \left(\frac{\beta}{z}\right)^2 \pi(z)\,[1 - \pi(z)]\,.$$

The log-normal item information function is

$$I(z) = \frac{\left\{\frac{\beta}{z}\phi\left[\log\left(\frac{z^\beta}{\alpha}\right)\right]\right\}^2}{\Phi\left[\log\left(\dfrac{z^\beta}{\alpha}\right)\right]\Phi\left[\log\left(\dfrac{\alpha}{z^\beta}\right)\right]}.$$

The Weibull item information function is

$$I(z) = \left[\frac{\beta z^{\beta-1}}{\alpha}\right]^2 \frac{\exp\left(-\frac{z^\beta}{\alpha}\right)}{1-\exp\left(-\frac{z^\beta}{\alpha}\right)} = \left[\frac{\beta z^{\beta-1}}{\alpha}\right]^2 \frac{1-\pi(z)}{\pi(z)}.$$

3. Inference

Bayesian inference was used to obtain parameter estimates. Let $\mathbf{Y}$ be the $I \times K$ matrix of binary outcomes with entries y_{ik} denoting the ith person's response to item k. Under the standard IRT assumptions of independence among subjects and local independence among items along with no missing data and prior independence among parameters, the posterior density of the model parameters is

$$\text{pr}\left(\{\alpha_k\}, \{\beta_k\}, \{z_i\} \,|\mathbf{Y}\right) \propto \prod_{i=1}^{I} \text{pr}(z_i) \prod_{k=1}^{K} \pi_k(z_i)^{y_{ik}} \left[1-\pi_k(z_i)\right]^{1-y_{ik}} \text{pr}(\alpha_k)\,\text{pr}(\beta_k).$$

Markov chain Monte Carlo (MCMC) methods were used to obtain the $2K + I$ marginal parameter distributions (Fox, 2010; Patz & Junker, 1999). The the mutually independent prior densities were $\alpha_k \sim \text{gamma}(.1, .1)$, $\beta_k \sim \text{gamma}(.1, .1)$, and $z_i \sim \text{log-normal}(0, 1)$. From the priors, $\Pr(0 < \alpha_k < 6) = .95$ and $\Pr(0 < \beta_k < 6) = .95$ for all k.

The Bayesian approach allows the responses of all respondents to be used, including those who endorse no items and those who endorse all items.

4. Data Set

The data sources were two public-use files from the Clinical Trials Network for the methadone and non-methadone maintenance trials for abstinence-based contingency management (Peirce et al., 2006; Petry et al., 2005) which had previously been analyzed using a standard IRT model (Wu et al., 2009). The data comprised 854 subjects responding to the 7 alcohol dependency items of the DSM-IV at baseline, prior to any intervention. Of the 854, 167 (19.6%) reported they had never used alcohol in the past nor were currently using alcohol. These subjects were given a trait score of $z = 0$. The remaining 687 were assumed to be potentially addicted to alcohol and assumed to have a trait score $z > 0$. The DSM-IV items were (1) *toler* — increasing tolerance of alcohol, (2) *wdraw* — experience withdrawal symptoms, (3) *amount* — using larger amounts, (4) *unable* — unable to control use, (5) *time* — large amount of time spent in acquiring alcohol, (6) *giveup* — giving up important activities, and (7) *contin* — continued use despite accompanying problems.

All data management, analyses, and graphical displays were conducted in **R** (R Development Core Team, 2012) with **Rstudio** (RStudio, Inc, 2012). The $2K + I = 14 + 687$ marginal parameter distributions were obtained by MCMC using **JAGS** (Plummer, 2003, 2011) and the **R2jags** package (Su & Yajima, 2012). All MCMC convergence diagnostics were satisfactory; in particular, the Brooks-Gelman-Rubin potential scale reduction statistic was less than 1.1 for all parameters (Gelman, Carlin, Stern, & Rubin, 2004). The graphics were produced with the **lattice** package (Sarkar, 2008).

5. Results

Figure 1 presents the item characteristic curves for the three models. Although the curves show, as expected, slightly different forms, the ordering of the curves along the latent trait axis (Addiction Score) is the same for all three models. For each model, all the characteristic curves show roughly the same item severity (α_k) and discriminability (β_k), except for *wdraw*, which has larger severity and smaller discriminability.

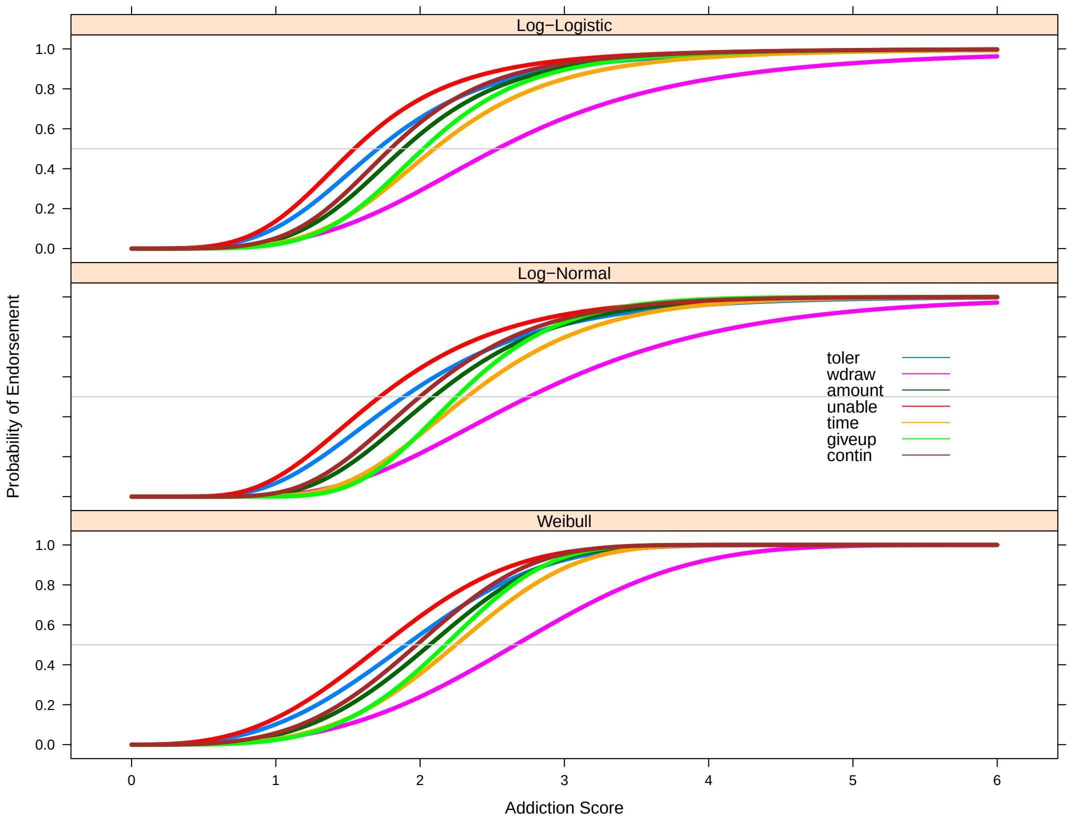

Figure 1: Item Character Curves for the Log-Logistic, Log-Normal, and Weibull Models

Figure 2 presents the item information curves for the three models. Unexpectedly, the item information curves are different from each other. The log-logistic model shows greatest precision for *unable*, followed by the precisions for *contin* and *giveup*. In contrast, the log-normal model shows greatest precision for *giveup* followed by *unable* and next *contin*. In further contrast, the Weibull model shows greatest precision for *giveup* followed by *contin* then by *time* and *amount*. Also, the location of the score of greatest precision is greater for the Weibull than it is for either the log-logistic or log-normal for all items.

Figure 3 presents the person scores for four item response patterns under the log-logistic model. The upper left panel presents the results for a potential addict endorsing none (0000000) of the 7 items. The mean addiction score is 0.58. The black line displays the prior standard log-normal density for the person score. The red line presents the posterior density of the score for that pattern. The blue line is the log-normal density with the observed mean and variance as parameters. The red and blue lines have similar location but do not coincide. The red posterior density has the same location but less variance than the black prior.

The upper right panel displays the results of a person with the pattern 0001011 and an observed mean score of 1.79. The lines have the same interpretation as in the previous

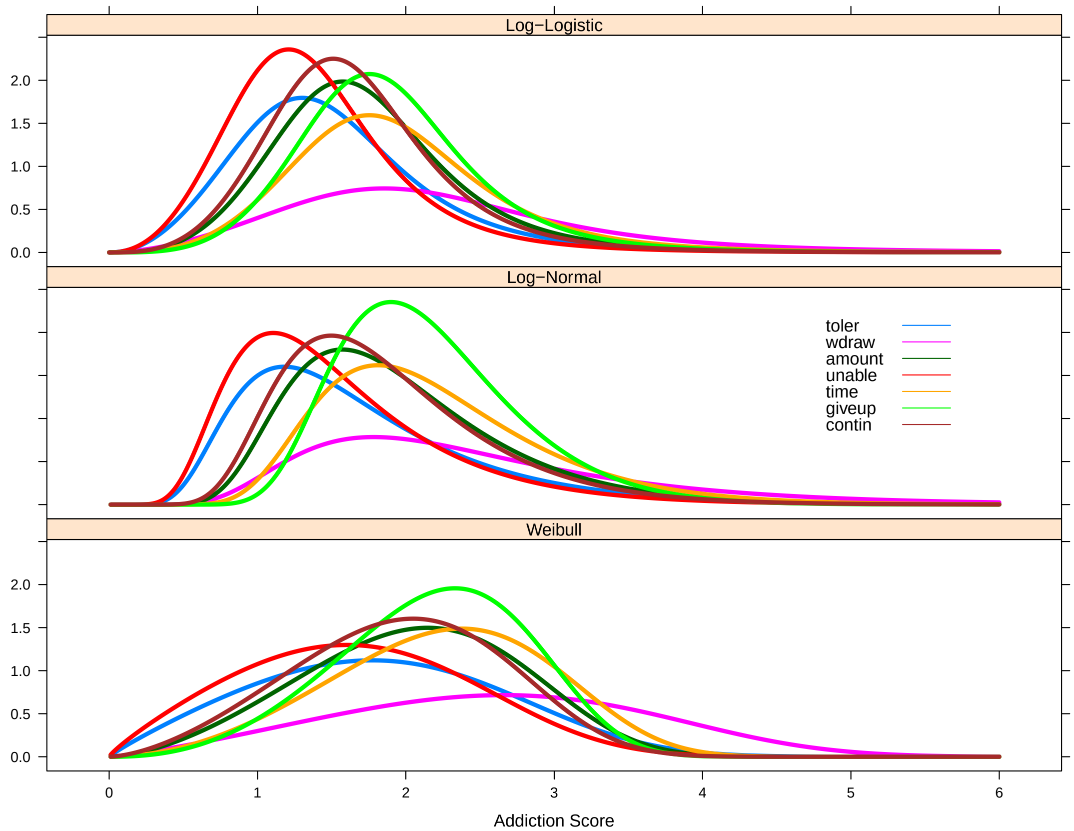

Figure 2: Item Information Curves for the Log-Logistic, Log-Normal, and Weibull Models

panel. The blue observed density coincides with the the red posterior density.

The lower left panel displays the results of person with the pattern 111110 and an observed mean score of 2.98. The lower right panel displays the results of person who endorses all items (111111) with an observed mean score of 5.81. In both cases the blue observed densities nearly coincide with the red posterior densities.

6. Summary

The PTIRM appears to be a viable alternative to the usual IRT models for positive traits. Interpretation of item parameters is roughly the same as that for standard IRT models. Bayesian inference via MCMC is a satisfactory method for obtaining parameter and person distributions. Different PTIRMs yield similar ICCs but different IICs. Different PTIRMS yield similar person scores.

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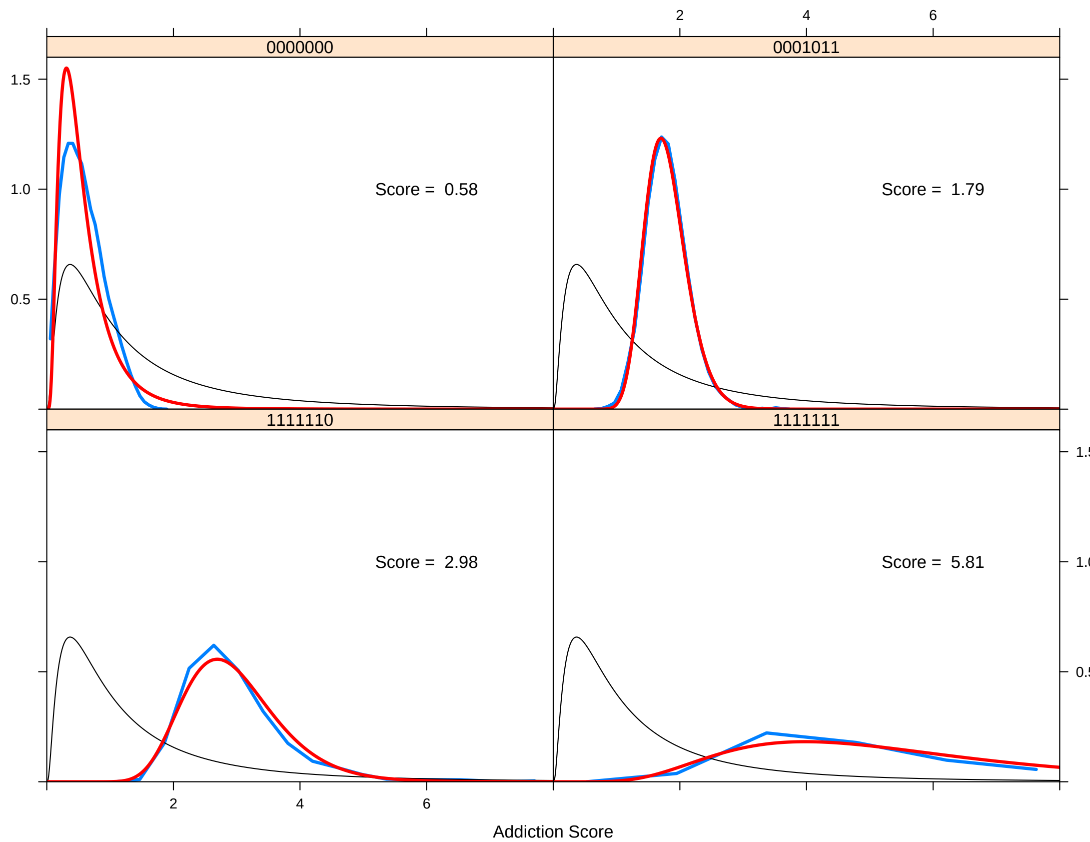

Figure 3: Person Scores for the Log-Logistic Model

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