A Robust Tobit Regression Model when Errors are from the Epsilon Skew Exponential Power Family

Ahmad Flaih * Hassan Elsalloukh † Jose Guardiola ‡

Abstract

In this paper, we generalize the Epsilon Skew Normal (ESN) Tobit regression model, proposed by Mashtare Jr. and Huston (2011), to the Epsilon Skew Exponential Power (ESEP) Tobit regression, which was proposed by Elsalloukh (2004). Tobit model assumes the normality of residuals term. Elsalloukh et al.(2005) proposed the Epsilon Skew Exponential Power (ESEP) family of distributions which includes the ESN distribution and many others distributions as special cases. This flexible family of distributions can accommodate both heavy tails and skewness behaviors. Therefore, ESEP can be considered as a ”robust model” to cope with the deviation from normality. We propose the use of the Epsilon Skew Exponential Power family of distribution as an alternative model to make inference on estimating the interested parameters of the Tobit regression model. In the process, we develop the basic properties of the ESEP Tobit model, such as the structural equation, the expected value of the censored variable, and the loglikelihood functions based on the piecewise nature of the ESEP density.

KEY WORDS: Epsilon Skew Exponential Power (ESEP) family, Tobit regression, maximum likelihood estimation.

1. Introduction

Tobit model was first studied by Tobin (1958) who discussed the estimation of the parameters of the truncated normal regression model which represents the relationship of durable goods expenditure to age and liquid asset in a random sample of households. Holden (2004) tested the normality assumption in the Tobit model. Xianbo (2007) considered the semiparametric and nonparametric estimation of Tobit models such as the truncated and censored regression models. Jeong and Jeong (2010) proposed a new test on normality in the censored regression (Tobit) model. Arabmazar and Schmidt (1982) explained that the maximum likelihood estimation of the Tobit parameters becomes inconsistent when the normality assumptions of the error random variable are untenable. Amemiya (1973) proved the consistency and the asymptotic normality of the maximum likelihood estimators of the Tobit parameters when the dependent variable is normal but truncated to the left of zero. Han and Kronmal (2004) considered the Box-Cox transformation to address the problem of departure from normality assumptions in the Tobit model. Amemiya and Boskin (1974) considered the estimation of the censored regression model when the errors are distributed according to log-normal. Mashtar Jr. and Huston (2011) considered the estimation of a censored regression (Tobit) model when the errors are distributed as Epsilon-Skew Normal

*UALR, Department of Applied Science, 2801 South University Avenue, Little Rock, AR 72204

†UALR, Department of Mathematics and Statistics, 2801 South University Avenue, Little Rock, AR 72204

‡Department of Mathematics and Statistics, Texas AM University-Corpus Christi 6300 Ocean Drive, Corpus Christi, Texas 78412

distribution with application in the biostatistics field. Hence, instead of using the robust approach or the Cox-Box transformations to estimates the Tobit regression parameter when the assumptions of normality are untenable, we adopt an alternative approach based on using the ESEP distribution to develop a Tobit model as a robust model.

2. The Epsilon Skew Exponential Power (ESEP) distribution

Elsalloukh (2004 and 2005) introduced the Epsilon-Skew Exponential Power (ESEP) distribution that can accommodate heavy-tailed (Leptokurtic) and skewed data. The ESEP density is denoted by *ESEP*$(\theta, \sigma, \alpha, \epsilon)$ and defined by

$$f(y) = \frac{\alpha}{2\sigma\sqrt{2}\Gamma(\frac{1}{\alpha})} \begin{cases} exp[-(\frac{y-\theta}{\sqrt{2}\sigma(1-\epsilon)})^{\alpha}]; & y \geq \theta \\ exp[-(\frac{\theta-y}{\sqrt{2}\sigma(1+\epsilon)})^{\alpha}]; & y < \theta, \end{cases} \tag{1}$$

where $-1 < \epsilon < 1$ is the skewness parameter, $\theta \in \Re$ is the location parameter, $\sigma > 0$ is the scale parameter, and $\alpha \in \Re$ is the shape parameter. Moreover, the density function (1) is known as the Epsilon Skew Exponential Power of order α.

The probability density and the cumulative distribution functions of the standard form of *ESEP*$(0, 1, \alpha, \epsilon)$ are, respectively

$$f_0(y) = \frac{\alpha}{2\sqrt{2}\Gamma(\frac{1}{\alpha})} \begin{cases} exp[-(\frac{y}{\sqrt{2}(1-\epsilon)})^{\alpha}]; & y \geq 0 \\ exp[-(\frac{-y}{\sqrt{2}(1+\epsilon)})^{\alpha}]; & y < 0, \end{cases} \tag{2}$$

and

$$F_0(y) = \begin{cases} 1 - \frac{(1-\epsilon)}{2\Gamma(\frac{1}{\alpha})}\Gamma(\frac{1}{\alpha}, g(y)); & y \geq 0 \\ \frac{(1+\epsilon)}{2\Gamma(\frac{1}{\alpha})}\Gamma(\frac{1}{\alpha}, h(y)); & y < 0, \end{cases} \tag{3}$$

where

$$\Gamma(\alpha, x) = \int_{x}^{\infty} t^{\alpha-1} exp(-t)dt, \alpha, x > 0, \tag{4}$$

is the incomplete gamma function,

$$g(y) = (\frac{y}{\sqrt{2}(1-\epsilon)})^{\alpha}, \tag{5}$$

and

$$h(y) = (\frac{-y}{\sqrt{2}(1+\epsilon)})^{\alpha}, \tag{6}$$

The mean of the ESEP is defined as

$$E(Y) = \theta - \frac{4\epsilon\sigma\Gamma(\frac{2}{\alpha})}{\sqrt{2}\Gamma(\frac{1}{\alpha})}.$$

The general form of the density function (1) is $\frac{1}{\sigma}f_0(\frac{|y-\theta|}{\sigma})$, f_0 is as defined by (2), the general form of the c.d.f. is $F_0(\frac{|y-\theta|}{\sigma})$, and F_0 is defined by (3).

By varying the value of the shape and the skewness parameters, the ESEP fits data with heavy-tailed, light-tailed, and skewed, that is, the ESEP reduces the effect of the outliers and increases the robustness of the data analysis; therefore the ESEP is a *"robust family"*.

3. Tobit Regression Model

In this section, we look at the setup of Tobit (censored) regression model and the parameters estimation of Tobit regression. The left censored measurment function, Maddala (1983) and Long (1997), is defined by

$$y_i = \begin{cases} y^*; & y^* > c \\ c; & y^* \leq c, \end{cases} \tag{7}$$

where y_i are the *censored variables*, y^* is the *latent variable*, and c is the censoring point. Let the dependent variable y^* be

$$y^* = \theta + u_i. \tag{8}$$

By substituting (8) in (7), where θ is the mean of y^* and $u_i \sim N(0, \sigma^2)$, the Tobit model measurment function (7) becomes

$$y_i = \begin{cases} y^*; & \text{if } y^* = \theta + u_i > 0 \\ 0; & \text{if } y^* = \theta + u_i \leq 0. \end{cases} \tag{9}$$

The expected value of Tobit model, Long (1997), is defined by

$$\begin{aligned} E(y) = E[E(y/\eta)] &= P(\text{uncensored})E(y/\eta = 1) + P(\text{censored})E(y/\eta = 0) \\ &= \theta\Phi(\frac{\theta}{\sigma}) + \sigma\phi(\frac{\theta}{\sigma}), \end{aligned}$$

where $E[E(y/\eta)]$ is the law of iterated expectation of a bivariate random variables, y and η, and

$$\eta = \begin{cases} 1; y > 0 \\ 0; y = 0 \end{cases} \tag{10}$$

The maximum likelihood estimation of the Tobit model can be defined by the following general likelihood function of the censored regression, Maddala (1983),

$$f(y_1, ..., y_n) = \prod_{i=1}^{n} [F(y^*)]^{d_i} [F(c)]^{1-d_i}.$$

Traditionally, in Tobit model, θ is parameterized as $x_i'\beta$, thus, we can define the Likelihood function as

$$L = \prod_{i=1}^{n} [\frac{1}{\sigma}\phi(\frac{y_i - x_i'\beta}{\sigma})]^{d_i} [1 - \Phi(\frac{x_i'\beta}{\sigma})]^{1-d_i},$$

where d is an indicator defined as

$$d = \begin{cases} 1; y_i > 0 \\ 0; y_i = 0. \end{cases}$$

Amemiya (1973) considered the estimation of the parameters of the regression model β_i and σ^2, when the response variable is normal and left truncated.

3.1 The Epsilon-Skew Exponential Power Tobit Regression Model

Let $y_1, y_2, ..., y_n$ denote a left censored as defined in (7) with $c = 0$. Moreover, suppose $y^* \sim ESEP(\theta, \sigma, \alpha, \epsilon)$ and y^* as defined in (8) with $u_i \sim ESEP(\theta, \sigma, \alpha, \epsilon)$. Thus, the ESEP Tobit model is

$$y_i = \begin{cases} y^*; & y^* = \theta + u_i > 0 \\ 0; & y^* = \theta + u_i \leq 0, \end{cases} \tag{11}$$

where y is the observed value of the dependent variable, and θ is the mode of y^*.

proposition 1. *The expected value of a left censored variable Y with latent ESEP variable y^* is*

$$E(Y) = \begin{cases} \theta[1 - F_0(\frac{-\theta}{\sigma})] + \frac{\sigma}{\sqrt{2}\Gamma(\frac{1}{\alpha})}[(1+\epsilon)^2\Gamma(\frac{2}{\alpha}, g(-\theta) \\ ) + (1-\epsilon)^2\Gamma(\frac{1}{\alpha})]; & \theta > 0 \\ \theta[1 - F_0(\frac{-\theta}{\sigma})] + \frac{(1-\epsilon)^2\sigma}{\sqrt{2}\Gamma(\frac{1}{\alpha})}\Gamma(\frac{2}{\alpha}, h(-\theta)); & \theta \leq 0, \end{cases} \tag{12}$$

where $g(-\theta)$, $h(-\theta)$, and $\Gamma(\cdot, \cdot)$ are as defined in (4),(5), and (6), respectively.

Proof.

$$\begin{aligned} E(y) &= E[E(y/\eta)] \\ &= P(\text{uncensored})E(y/\eta = 1) + P(\text{censored})E(y/\eta = 0), \end{aligned}$$

where $E[E(y/\eta)]$ is the law of iterated expectation of a bivariate random variables, y and η as defined in (10). Then

$$\begin{aligned} P(uncensored) &= p(y^* > 0) \\ &= p(u_i > -\theta) \\ &= 1 - F_0(\frac{-\theta}{\sigma}), \end{aligned} \tag{13}$$

where $F(\cdot)$ is the general form of the cdf of the ESEP density. Let $u_i^\star$ be the a random variable with density $g(\cdot)$ given by:

$$\begin{aligned} g(\lambda/\lambda > 0) &= \frac{g(\lambda)}{p(\lambda > 0)} \\ &= \frac{\frac{1}{\sigma}f_0(\frac{\lambda}{\sigma})}{1 - F(\frac{-\theta}{\sigma})}; \quad -\theta < \lambda < \infty, \end{aligned} \tag{14}$$

where $\frac{1}{\sigma}f_0(\frac{\lambda}{\sigma})$ is the general form of the ESEP distribution. See Amemiya (1973) for details. Thus,

$$\begin{aligned} E(y|\eta = 1) &= E(\theta + \lambda) \\ &= \theta + E(\lambda). \end{aligned} \tag{15}$$

Given this, we have

$$\begin{aligned} E(y) &= 1 - F_0(\frac{-\theta}{\sigma})[\theta + E(\lambda)] \\ &= \theta[1 - F_0(\frac{-\theta}{\sigma})] + \int_{-\theta}^{\infty} \frac{\lambda}{\sigma} f_0(\frac{\lambda}{\sigma}) d\lambda. \end{aligned} \tag{16}$$

If $\theta > 0$ the integral in (16) is

$$\int_{-\theta}^{0} \frac{\alpha\lambda}{2\sqrt{2}\sigma\Gamma(\frac{1}{\alpha})} exp[-(\frac{-\lambda}{2^{\frac{1}{2}}(1+\epsilon)\sigma})^{\alpha})]d\lambda + \int_{0}^{\infty} \frac{\alpha\lambda}{2\sqrt{2}\sigma\Gamma(\frac{1}{\alpha})} exp[-(\frac{\lambda}{2^{\frac{1}{2}}(1-\epsilon)\sigma})^{\alpha})]d\lambda$$
$$= \frac{\sigma(1+\epsilon)^2}{2\Gamma(\frac{1}{\alpha})}\Gamma(\frac{2}{\alpha}, g(-\theta)) + \frac{\sigma(1-\epsilon)^2}{2\Gamma(\frac{1}{\alpha})}\Gamma(\frac{2}{\alpha}). \quad (17)$$

If $\theta < 0$, the integral in (16) is

$$\int_{-\theta}^{\infty} \frac{\alpha\lambda}{2\sqrt{2}\sigma\Gamma(\frac{1}{\alpha})} exp[-(\frac{\lambda}{2^{\frac{1}{2}}(1-\epsilon)\sigma})^{\alpha})]d\lambda$$
$$= \frac{\sigma(1-\epsilon)^2}{2\Gamma(\frac{1}{\alpha})}\Gamma(\frac{2}{\alpha}, g(-\theta)) \quad (18)$$

By substituting (17) and (18) in equation (16), the expected value of the censored variable Y is

$$E(Y) = \begin{cases} \theta[1 - F_0(\frac{-\theta}{\sigma})] + \frac{\sigma}{\sqrt{2}\Gamma(\frac{1}{\alpha})}[(1+\epsilon)^2\Gamma(\frac{2}{\alpha}, g(-\theta)) \\ +(1-\epsilon)^2\Gamma(\frac{2}{\alpha})]; & \theta > 0 \\ \theta[1 - F_0(\frac{-\theta}{\sigma})] + \frac{(1-\epsilon)^2\sigma}{\sqrt{2}\Gamma(\frac{1}{\alpha})}\Gamma(\frac{2}{\alpha}, h(-\theta)); & \theta \leq 0, \end{cases} \quad (19)$$

where $g(-\theta)$, $h(-\theta)$, and $\Gamma(\cdot,\cdot)$ are as defined in (4),(5), and (6), respectively. □

The first moment of some known distributions, which are special cases of ESEP Tobit regression $(\theta, \sigma, \alpha, \epsilon)$, can be derived from (19) as follows:

Case1: ESEP$(\theta, \sigma, 2, \epsilon)$

$$E(Y) = \begin{cases} \theta[1 - F_0(\frac{-\theta}{\sigma})] + \frac{\sigma^2(1+\epsilon)^2}{\sigma\sqrt{2\pi}} e^{-(\frac{-\theta}{\sqrt{2}(1+\epsilon)\sigma})^2} - \frac{4\sigma\epsilon}{\sigma\sqrt{2\pi}}; & \theta > 0 \\ \theta 1 - F_0(\frac{-\theta}{\sigma})] + \frac{\sigma^2(1-\epsilon)^2}{\sigma\sqrt{2\pi}} e^{-(\frac{-\theta}{\sqrt{2}(1-\epsilon)\sigma})^2}; & \theta \leq 0. \end{cases}$$

Case2: ESEP$(\theta, \sigma, 1, \epsilon)$

$$E(Y) = \begin{cases} \theta[1 - F_0(\frac{-\theta}{\sigma})] + \frac{2\sigma^2(1+\epsilon)^2}{2\sigma\sqrt{2}} e^{-(\frac{-\theta}{\sqrt{2}(1+\epsilon)\sigma})} - \frac{2\theta\sigma(1+\epsilon)}{2^{\frac{1}{2}}2\sqrt{2}\sigma} e^{-(\frac{-\theta}{\sqrt{2}(1+\epsilon)\sigma})} - \frac{4\sigma\epsilon}{\sqrt{2}}; & \theta > 0 \\ \theta[1 - F_0(\frac{-\theta}{\sigma})] + \frac{2\sigma^2(1-\epsilon)^2}{2\sigma\sqrt{2}} e^{-(\frac{-\theta}{\sqrt{2}(1-\epsilon)\sigma})} - \frac{2\theta\sigma(1-\epsilon)}{2^{\frac{1}{2}}2\sqrt{2}\sigma} e^{-(\frac{-\theta}{\sqrt{2}(1-\epsilon)\sigma})} - \frac{\sigma(1-\epsilon)^2}{\sqrt{2}}; & \theta \leq 0. \end{cases}$$

Case3: ESEP$(\theta, \sigma, 2, 0)$

$$E(Y) = \theta\Phi(\frac{\theta}{\sigma}) + \frac{1}{\sigma}\phi(\frac{\theta}{\sigma}).$$

4. Maximum Likelihood Estimation of the Parameters of ESEP Tobit Model

The Log Likelihood (LL) function per observation of the parameters of the ESEP Tobit model is

$$l(y_i; \theta, \sigma, \alpha, \epsilon) = d_i log[\frac{1}{\sigma} f_0(\frac{y_i - \theta}{\sigma})] + (1 - d_i) log[F_0(\frac{-\theta}{\sigma})], \quad (20)$$

where d is an indicator given as

$$d = \begin{cases} 1; y_i > 0 \\ 0; y_i \leq 0. \end{cases}$$

Since the pdf of the ESEP is a piecewise function, we consider the following three cases for the LL function

Case1: For $\theta \leq 0$, the Log likelihood function per observation is,

$$\begin{aligned} l(y_i; \theta, \sigma, \epsilon, \alpha) = d_i[log(\frac{\alpha}{2\sigma\sqrt{2}\Gamma(\frac{1}{\alpha})}) - (\frac{y_i - \theta}{\sqrt{2}\sigma(1-\epsilon)})^{\alpha}] \\ + (1 - d_i)log[1 - \frac{(1-\epsilon)}{2\Gamma(\frac{1}{\alpha})}\Gamma(\frac{1}{\alpha}, g(y))]. \end{aligned} \tag{21}$$

Case2: For $\theta > 0$ and $y_i < 0$, the Log likelihood function per observation is,

$$\begin{aligned} l(y_i; \theta, \sigma, \epsilon, \alpha) = d_i[log(\frac{\alpha}{2\sigma\sqrt{2}\Gamma(\frac{1}{\alpha})}) - (\frac{\theta - y_i}{\sqrt{2}\sigma(1+\epsilon)})^{\alpha})] \\ + (1 - d_i)log[\frac{(1+\epsilon)}{2\Gamma(\frac{1}{\alpha})}\Gamma(\frac{1}{\alpha}, h(y))]. \end{aligned} \tag{22}$$

Case3: For $0 < \theta \leq y_i$, the Loglikelihood function per observation is,

$$l(y_i; \theta, \sigma, \epsilon, \alpha) = log[\frac{\alpha}{2\sigma\sqrt{2}\Gamma(\frac{1}{\alpha})}] - (\frac{y_i - \theta}{\sqrt{2}\sigma(1-\epsilon)})^{\alpha}, \tag{23}$$

where $g(y) = (\frac{y-\theta}{\sqrt{2}(1-\epsilon)})^{\alpha}$, $h(y) = (\frac{\theta-y}{\sqrt{2}(1+\epsilon)})^{\alpha}$.

We can use the above three cases to find the maximum likelihood estimates of the parameters for the ESEP Tobit regression model. In the traditional tobit model, the location parameter θ is parametrized as $x_i'\beta$ in the tobit regression. Thus, the likelihood function of the ESEP Tobit regression is

$$l(y_i; \theta, \sigma, \alpha, \epsilon) = \prod_i^n [\frac{1}{\sigma} f_0(\frac{y_i - x_i'\beta}{\sigma})]^{d_i} [F_0(\frac{-x_i'\beta}{\sigma})]^{1-d_i}, \tag{24}$$

REFERENCES

Amemiya, T. (1973), "Regression analysis when the dependent variable is truncated normal," *Econometrica*, 41, 997-1016.

Amemiya, T. and Boskin, M. (1974, "Regression analysis when the dependent variable is truncated lognormal, with an application to the determinants of the duration of welfare dependency," *International Economic Review*, 15(2), 485-496.

Arabmazar, A. and Schmidt, P. (1982) "An investigation of the robustness of the Tobit Estimation to Non-Normality," *Econometrica*,50,1055-1063.

Elsalloukh,H. (2004), "The epsilon skew exponential power distribution," *PhD. thesis, Baylor university*, USA.

Elsalloukh, H., Guardiola, J., and Young, M.D. (2005), "The Epsilon-Skew Exponential Power Distribution Family," *Far East Journal of Theoretical Statistics,* 16, No. 1, 97-112.

Han, C. and Kronmal, R. (2004), ”Box-Cox transformation of left-censored data with application to the analysis of coronary artery calcification and pharmacokinetic data,” *Statistics in Medicine*, 23, 3671-3679.

Holden, D. (2004), ”Testing the Normality Assumption in the Tobit Model,” *Journal of Applied Statistics,* vol. 31,521-532.

Jeong, J. and Jeong, D. (2010), ”A Test on the Normality in the Tobit Model” *(in press)*.

Long, J.S. (1997), ”Regression Models for Categorical and Limited Dependent Variables,” *London: Sage Publications*.

Maddala, G.S. (1983), ”Limited-Dependent and Qualitative Variables in Econometrics,” *Econometric Society Monographs*.

Mashtare Jr., T. and Huston, A. (2011), ”Utilizing the Flexibility of the Epsilon-Skew-Normal Distribution for Tobit Regression Problems,” *Communications in Statistics*, Theory and Methods,Vol.40,Issue 3.

Tobin, J. (1958), ”Estimation of Relationships for Limited Dependent Variables,” *Econometrica,* 26, 24-36.

Xianbo, Z. (2007), ”Semiparametric and nonparameteric Estimation of Tobit Models,” *PhD. thesis, the Hong Kong University and technology, Hong Kong*.