Conditional Confidence Intervals of the Process Capability Index C_{pk}

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Abstract
The process capability index C_{pk} involves both of the two parameters of a process, the process mean μ and the process standard deviation σ. When μ and σ are unknown and we have uncertain prior information about their values, we may check the uncertain prior information by preliminary tests. Then we can construct a conditional confidence interval of the process capability index C_{pk} following rejection of the preliminary tests. In this paper, we adopt two tests for testing μ and σ separately and sequentially, instead of testing μ and σ jointly. Conditional confidence intervals of C_{pk} following different results of the two tests are provided. For the case that both of the two null hypotheses are rejected, we construct the confidence interval of C_{pk} for which the two parameters μ and σ are all unknown. An extension of the general method for finding a confidence interval of an unknown quantity that is a function of two parameters is also discussed.

Key Words: the process capability index C_{pk}; conditional confidence interval; preliminary test; sequential tests.

1. Introduction

The conditional confidence intervals of the process capability index C_p is discussed by Zhang J.C. and Han C.-P. (2011). When the process mean μ, which is also the expected value of the measurement (X) of a product, is equal to the midpoint of the specified interval (LSL,USL), where LSL and USL are the lower and upper specification limits of measurements respectively, then the expected proportion of non-conforming (NC) product is equal to $2\Phi(-3C_p)$. But if the expected value of X is not equal to the midpoint of the specified interval, i.e., $\mu \neq (1/2)(LSL + USL)$, then the expected proportion of NC product will be bigger than $2\Phi(-3C_p)$. In this case, the process capability index C_p is no longer the best index to measure the quality of a product. Thus, we introduce another process capability index C_{pk}.

If we consider the effects of the value of the process mean μ, then the process capability index C_{pk} is defined as

$$C_{pk} = \frac{\min(USL-\mu, \mu-LSL)}{3\sigma}$$

Since $\min(a, b) = \frac{1}{2}(|a + b| - |a - b|)$ for any $a \geq 0$ and $b \geq 0$, therefore, we also have

$$C_{pk} = \frac{d - \left|\mu - \frac{1}{2}(LSL + USL)\right|}{3\sigma}$$

or simply

$$C_{pk} = \frac{d - |\mu - m|}{3\sigma}$$

where $m = \frac{1}{2}(LSL + USL)$ is the midpoint of the specified interval.

In the above discussion, we assume that LSL ≤ μ ≤ USL. If μ is outside of the specified interval, then by the initial definition of C_{pk}, the value of C_{pk} would be negative, and the process would clearly be inadequate for controlling the quality of the product.

The process capability index C_{pk} involves both the process mean and the process variance. When we have a random sample $X_1, X_2, \ldots, X_n$ from a process which follows a $N(\mu, \sigma^2)$ distribution, then an estimator of μ is the sample mean $\overline{X}$, and an estimator of σ is the sample standard deviation S . Therefore, a point estimator of C_{pk} is given by

$$\hat{C}_{pk} = \frac{d - |\overline{X} - m|}{3S}$$

Since $\overline{X}$ and S are mutually independently distributed, it is possible for us to calculate the mean and the variance of $\hat{C}_{pk}$ by first carrying out its rth moment about the origin. But this procedure is much more complicated than what we did for the process capability index C_p (Kotz & Lovelace (1998), page 55), and it involves another type of distribution which is so called “folded” distribution.

If we consider both the mean μ and the variance σ^2 as unknown parameters, then the construction of (unconditional) confidence intervals of C_{pk} is difficult due to the fact that the distribution of $\hat{C}_{pk}$ involves the joint distribution of two non-central t-distributed random variables. No single technique is considered best in practice at this time (Kotz & Lovelace (1998), page 57). Although the explicit expression of such a confidence interval is almost impossible, but theoretically, this confidence interval is still possible to be determined for particular problems. The idea is to extend the general method for finding a confidence interval of an unknown parameter to the two parameters case. That is, first try to find the joint confidence region of the two parameters μ and σ, and then use this joint confidence region to obtain a confidence interval of C_{pk}, this method is discussed in Section 4.

In the case that we have some uncertain prior information about the values of μ and σ, we will use preliminary tests for testing the values of the two parameters. We will adopt two tests for testing μ and σ separately and sequentially, instead of testing μ and σ jointly.

The conditional confidence interval (CCI) of C_{pk} will be considered following rejection of any of the tests.

We will discuss the conditional confidence intervals of C_{pk} for the following three different cases:

(1) The mean μ is known, the variance σ^2 is unknown
(2) The mean μ is unknown, the variance σ^2 is known
(3) Both the mean μ and the variance σ^2 are unknown

2. CCIs of C_{pk} When μ Is Known and σ^2 Is Unknown

If the process mean μ is known, then for the process capability index C_{pk}, there is only one unknown parameter σ. This situation is similar to the one for finding a conditional confidence interval of the process capability index C_p. In this case, the preliminary test should be constructed as H_o: $C_{pk} \le C_o$ vs. H_1: $C_{pk} > C_o$, or, simply use the parameter σ: H_o: $\sigma \ge \sigma_o$ vs. H_1: $\sigma < \sigma_o$, where the value of σ_o can be determined by the formula

$$C_o = \frac{d - |\mu - m|}{3\sigma_o}$$

[Result 2.1] If a process has a known mean μ and an unknown variance σ^2, then a 100(1- α_1 - α_2)% conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $C_{pk} \le C_o$ (or H_o: $\sigma \ge \sigma_o$) can be determined by the following interval

$$\left(\frac{d - |\mu - m|}{3\sigma_U} \; , \; \frac{d - |\mu - m|}{3\sigma_L} \right)$$

where (σ_L^2, σ_U^2) is a 100(1- α_1 - α_2)% conditional confidence interval of σ^2 following rejection of the preliminary test for testing H_o: $\sigma \ge \sigma_o$ vs. H_1: $\sigma < \sigma_o$. The value σ_U^2 is a 100(1- α_1)% conditional upper confidence limit of σ^2, and the value of σ_L^2 is a 100(1- α_2)% conditional lower confidence limit of σ^2. These two values can be obtained by using the method given by Zhang J.C. & Han C.-P. (2011).

3. CCIs of C_{pk} When μ Is Unknown and σ^2 Is Known

In some situations, if we have enough information about the variance of a process, i.e. the variance σ^2 of the process can be regarded as known. Then for the process capability index C_{pk} , there is only one unknown parameter, the process mean μ. If the measurement of a process follows a normal distribution, then we can use the sample mean $\overline{X}$ as a point estimator of μ. Therefore, a point estimator of the process capability index C_{pk} becomes

$$\hat{C}_{pk} = \frac{d - |\overline{X} - m|}{3\sigma}$$

$$= \begin{cases} \dfrac{d-(\overline{X}-m)}{3\sigma}, & if\ \overline{X} \geq m \\ \\ \dfrac{d+(\overline{X}-m)}{3\sigma}, & if\ \overline{X} < m \end{cases}$$

In this case, finding a conditional confidence interval of the process capability index C_{pk} is really a matter of finding a conditional confidence interval of the process mean μ.

For the process mean μ, if a random sample X_1, X_2, …… X_n drawn from the process follows a normal distribution N(μ, σ^2), then a 100(1-α)% unconditional confidence interval of μ is given by the interval

$$(\overline{X} - z_{1-\alpha/2}\sigma/\sqrt{n},\quad \overline{X} + z_{1-\alpha/2}\sigma/\sqrt{n})$$

After we obtain an unconditional confidence interval of μ, then an unconditional confidence interval of C_{pk} can be easily determined by using the formula C_{pk} = $\dfrac{d-|\mu-m|}{3\sigma}$, since the only unknown parameter in this formula is μ.

The test hypothesis for the parameter C_{pk} for this case (μ is unknown, σ^2 is known) can be constructed as H_o: $C_{pk} = C_o$ vs. H_1: $C_{pk} \neq C_o$, or equivalent to the hypothesis H_o: μ = μ_o vs. H_1: μ ≠ μ_o, where $\mu_o = m + d - 3\sigma C_o$ if $\mu_o \geq m$, and $\mu_o = m - d + 3\sigma C_o$ if $\mu_o < m$. For the same value of C_o, whether we choose the value μ_o by using the condition $\mu_o \geq m$ or $\mu_o < m$ depend on prior information. For example, if we allow more deviation from the lower side of the mean, then we need to use the condition $\mu_o \geq m$. That is, we choose $\mu_o = m + d - 3\sigma C_o$. Otherwise, we use the condition $\mu_o < m$ and choose $\mu_o = m - d + 3\sigma C_o$.

A common rule of how to use the above preliminary test is that, if the null hypothesis is not rejected, then we use μ_0 as an estimate of μ to give the estimate of C_{pk}, there is no need to construct a conditional confidence interval of C_{pk} in this case. But if the null hypothesis is rejected, we should use $\overline{x}$ as an estimate of μ to give the estimate of C_{pk} , and then we need to find a conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $C_{pk} = C_o$, or equivalently H_o: μ = μ_o.

As we already know, when the process variance σ^2 is known, the process capability index C_{pk} contains only one unknown parameter, the process mean μ. Therefore, in order to find a conditional confidence interval of C_{pk}, we need to find a conditional confidence interval of the mean μ.

Arabatzis, Gregoire and Reynolds (1989) investigated the conditional confidence interval of the normal mean following rejection of a two-sided test when σ is known, although the main conclusion they have reached is still discussible, but some partial results are useful.

Next, we'll follow the general method to find a conditional confidence interval of μ following rejection of the null hypothesis H_o: $\mu = \mu_o$.

If a random sample $X_1, X_2, \dots, X_n$ is taken from a normal distribution $N(\mu, \sigma^2)$, where μ is unknown and σ^2 is known. Then a level α test for testing H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ has the critical region

$$K = \left\{\bar{X} : \left|\bar{X} - \mu_o\right| > z_{1-\alpha/2}(\sigma/\sqrt{n})\right\}$$

where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution. The null hypothesis is rejected if $\bar{X} \in K$, and a conditional confidence interval of μ is computed only after we rejected the null hypothesis.

The conditional pdf of $\bar{X}$ can be expressed as

$$f_c(\bar{x}) = \begin{cases} f(\bar{x})/D, & if \left|\bar{x} - \mu_o\right| > z_{1-\alpha/2}(\sigma/\sqrt{n}) \\ 0, & otherwise \end{cases}$$

where $f(\bar{x})$ is the unconditional pdf of $\bar{X}$, and D is the power of the test which is given by

$$\begin{aligned} D &= P(\left|\bar{x} - \mu_o\right| > z_{1-\alpha/2}(\sigma/\sqrt{n}) | \mu) \\ &= 1 - \Phi\left\{z_{1-\alpha/2} - \sqrt{n}(\mu - \mu_o)/\sigma\right\} + \Phi\left\{- z_{1-\alpha/2} - \sqrt{n}(\mu - \mu_o)/\sigma\right\} \\ &= 1 - \Phi\left\{z_{1-\alpha/2} - \gamma\right\} + \Phi\left\{- z_{1-\alpha/2} - \gamma\right\} \end{aligned}$$

where $\gamma = \sqrt{n}(\mu - \mu_o)/\sigma$, and $\Phi(\cdot)$ is the CDF of the standard normal distribution. Under H_o, D = α. When $\gamma \to \infty$, D approaches 1.

The conditional CDF of $\bar{X}$ can be expressed as

$$F_c(\bar{x}) = \begin{cases} \dfrac{\Phi\left\{\sqrt{n}(\bar{x} - \mu)/\sigma\right\}}{1 - \Phi\left\{z_{1-\alpha/2} - \gamma\right\} + \Phi\left\{- z_{1-\alpha/2} - \gamma\right\}}, & if\ \bar{x} < \mu_o - z_{1-\alpha/2}\sigma/\sqrt{n} \\ \dfrac{\Phi\left\{\sqrt{n}(\bar{x} - \mu)/\sigma\right\} - \Phi\left\{z_{1-\alpha/2} - \gamma\right\} + \Phi\left\{- z_{1-\alpha/2} - \gamma\right\}}{1 - \Phi\left\{z_{1-\alpha/2} - \gamma\right\} + \Phi\left\{- z_{1-\alpha/2} - \gamma\right\}}, & if\ \bar{x} > \mu_o + z_{1-\alpha/2}\sigma/\sqrt{n} \end{cases}$$

The above formula implies that if the null hypothesis is rejected by a small observation of $\bar{X}$, i.e., if $\bar{x} < \mu_o - z_{1-\alpha/2}\sigma/\sqrt{n}$, then the conditional CDF of $\bar{X}$ can be expressed as

$$F_c(\bar{x}) = \frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}}$$

$$= \frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}} \quad (3.1)$$

If the null hypothesis is rejected by a large observation of $\bar{X}$, i.e., if $\bar{x} > \mu_o + z_{1-\alpha/2}\sigma/\sqrt{n}$, then the conditional CDF of $\bar{X}$ can be expressed as

$$F_c(\bar{x}) = \frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}}$$

$$= \frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}} \quad (3.2)$$

It's quite obvious from equations (3.1) and (3.2) that the conditional CDF of $\bar{X}$ depends only on the parameter μ, but not on any other nuisance parameters. It also can be verified numerically that the two functions $h_1(\mu)$ and $h_2(\mu)$ constructed by the following equations

$$F_c(h_1(\mu);\mu) = \alpha_1$$

and

$$F_c(h_2(\mu);\mu) = 1-\alpha_2$$

are increasing functions. So we can use the general method for finding a confidence interval of an unknown parameter to find a conditional confidence interval of μ. Thus, following the general method, we get

[Result 3.1] Suppose the random sample X_1, X_2, …… X_n is taken from a normal distribution $N(\mu, \sigma^2)$, where μ is unknown and σ^2 is known. Let $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ with $\alpha_1 + \alpha_2 < 1$, and $\bar{x}$ be an observed value of $\bar{X}$. Let $\Phi(\cdot)$ denote the CDF of the standard normal distribution. If the observed value $\bar{x}$ results in rejecting the null hypothesis H_o: $\mu = \mu_o$ at level α by the condition $\bar{x} < \mu_o - z_{1-\alpha/2}\sigma/\sqrt{n}$, then the solutions of

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_u^c)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}} = \alpha_1 \quad (3.3)$$

And

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_l^c)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}} = 1 - \alpha_2 \qquad (3.4)$$

construct a 100(1- α_1 - α_2)% conditional confidence interval (μ_l^c, μ_u^c) of μ. Otherwise, if the observed value $\bar{x}$ results in rejecting the null hypothesis H_o: μ = μ_o at level α by the condition $\bar{x} > \mu_o + z_{1-\alpha/2}\sigma/\sqrt{n}$, then the solutions of

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_u^c)/\sigma\right\}-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}}=\alpha_1 \qquad (3.5)$$

and

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_l^c)/\sigma\right\}-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}}=1-\alpha_2 \qquad (3.6)$$

construct a 100(1- α_1 - α_2)% conditional confidence interval (μ_l^c, μ_u^c) of μ.

The above equations look like complicated, but if we use IMSL numerical library, we can solve the equations for the conditional lower and upper confidence limits of μ easily.

Once we obtain the conditional confidence interval of μ as (μ_l^c, μ_u^c), to obtain a conditional confidence interval of C_{pk} just follows some simple calculations.

The relationship between the conditional confidence interval of C_{pk} and the unconditional confidence interval of C_{pk} for the case that μ is unknown and σ is known can be similarly obtained following the analysis by Meeks, S. L. & D'Agostino, R. B. (1983). Except in this case, the procedure is much more complicated. We will not discuss in detail at this time.

In some special cases, we still need to test a one-sided hypothesis for the process capability C_{pk}, this include the following two different situations, H_o: $C_{pk} \le C_o$ vs. H_1: $C_{pk} > C_o$ or H_o: $C_{pk} \ge C_o$ vs. H_1: $C_{pk} < C_o$. To find a conditional confidence interval of C_{pk} following rejection of any of the above null hypotheses follows a similar procedure discussed in this section. That is, first we need to find a conditional confidence interval of the process mean μ following rejection of the preliminary test, and then we use the relationship $C_{pk} = \dfrac{d-|\mu-m|}{3\sigma}$ to obtain a conditional confidence interval of C_{pk}.

4. CCls of C_{pk} When both μ and σ^2 Are Unknown

So far, we discussed the conditional confidence intervals of the process capability index C_{pk} for either μ is known and σ^2 is unknown or μ is unknown and σ^2 is known. But in most situations, the true values of the two parameters μ and σ^2 are all unknown. So next, we'll discuss the conditional confidence intervals of C_{pk} when both μ and σ^2 are unknown.

The testing hypotheses we need to consider for this situation depends on how much prior information we have. If we have prior information for both parameters μ and σ^2, then we need to construct testing hypotheses for the two parameters μ and σ^2. But in some cases, we may only have information for one of the two parameters, so we can construct only one testing hypothesis. Next, we will discuss case by case.

4.1 Testing for both Parameters

As we mentioned at the beginning, if both the mean μ and the variance σ^2 of a process are unknown, and we have uncertain prior information for both of them, then we will test the parameters μ and σ^2 separately using two sequential tests. The conditional confidence interval of C_{pk} will be considered following rejection of any of the two tests. The procedure is given as the following. First, test the hypothesis H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$, if the null hypothesis is not rejected, we regard σ as given ($\sigma = \sigma_o$), and then test H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ for the parameter μ, this test is a normal test since σ is given. If the null hypothesis H_o: $\mu = \mu_o$ is also not rejected, then we use μ_o and σ_o as two estimates of μ and σ to give the estimate of C_{pk}, no conditional confidence interval of C_{pk} is needed. But if the null hypothesis H_o: $\mu = \mu_o$ is rejected, we use $\bar{x}$ and σ_o as two estimates to give the estimate of C_{pk}. And then we will find a conditional confidence interval of μ following rejection of the null hypothesis H_o: $\mu = \mu_o$. Finally, we use the above conditional confidence interval of μ together with the value of σ_o (since $\sigma = \sigma_o$ is regarded as known in this case) to obtain a conditional confidence interval of C_{pk}. This procedure is similar to the one we discussed in Section3.

If the null hypothesis of the first test for testing H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$ has been rejected, in this case, we need to use the sample standard deviation *s* as an estimate of σ, and then regard σ as unknown to construct the second hypothesis H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ for testing the process mean μ. This time the test is a t-test since σ is unknown. If the null hypothesis of the second test is not rejected, we need to use μ_o and *s* as two estimates of μ and σ to give the point estimate of C_{pk}, and then find a conditional confidence interval of σ following rejection of the null hypothesis H_o: $\sigma = \sigma_o$. The conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ of the two sequential tests can be obtained as following. We regard μ as known ($\mu = \mu_o$) and σ as unknown and use the conditional confidence interval of σ together with the known value of μ ($\mu = \mu_o$) to construct a conditional confidence interval of C_{pk}.

If the null hypothesis of the second test for testing H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ is also rejected, then both μ and σ need to be considered as unknown, and the conditional confidence interval of C_{pk} should be considered following rejection of the two preliminary tests. In order to find a conditional confidence interval of C_{pk} in this case, we should first consider a joint confidence region of μ and σ. Next, we'll give some basic analyses for how to find a conditional confidence interval of C_{pk} for this situation.

Following the general method, to find such a conditional joint confidence region of μ and σ, we need first to find the conditional joint CDF of $\overline{X}$ and S. We start with finding the unconditional joint pdf of $\overline{X}$ and S.

If a random sample $X_1, X_2, \dots, X_n$ is taken from a normal distribution $N(\mu, \sigma^2)$, then $\overline{X} \sim N(\mu, \sigma^2/n)$, $S^2 \sim \dfrac{\sigma^2}{n-1}\chi^2_{n-1}$, and $\overline{X}$ and S are independent. Follows Arabatzis, Gregoire and Reynolds (1989), the unconditional joint pdf of $\overline{X}$ and S can be expressed as

$$f(\bar{x}, s) = \frac{2s^{n-2}\sqrt{n}((n-1)/2)^{((n-1)/2}}{\sqrt{2\pi}\sigma^n\Gamma((n-1)/2)} e^{(-[n(\bar{x}-\mu)^2+(n-1)s^2]/2\sigma^2}$$

for $-\infty < \bar{x} < \infty, 0 < s < \infty$, where $\Gamma(\cdot)$ is the Gamma function.

The conditional joint pdf of $\overline{X}$ and S following rejection of the two tests for testing H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$ and H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ can be expressed as

$$f_c(\bar{x}, s) = \begin{cases} f(\bar{x}, s)/D, & if\ (\bar{x}, s) \in K \\ \\ 0, & otherwise \end{cases}$$

where K is the critical region of the two tests determined by the intersection of $|\bar{x} - \mu_o| > t_{1-\alpha/2}(s/\sqrt{n})$ and $\dfrac{(n-1)s^2}{\sigma_o^2} < \chi^2_{n-1;\alpha/2}$ or $\dfrac{(n-1)s^2}{\sigma_o^2} > \chi^2_{n-1;1-\alpha/2}$, which is also the total shaded open regions of I, II, III and IV shown in figure 4.1; D is the total unconditional probability of $(\bar{x}, s)$ falling into the above critical region, which is determined by the following double integral

$$D = \iint_K f(\bar{x}, s)d\bar{x}ds$$

The conditional joint CDF of $\overline{X}$ and S following rejection of the two tests for testing H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$ and H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ can be expressed as

$$F_c(\bar{x}, s) = \frac{\iint f(\bar{x}, s)d\bar{x}ds}{D}, \quad for\ (\bar{x}, s) \in K \tag{4.1}$$

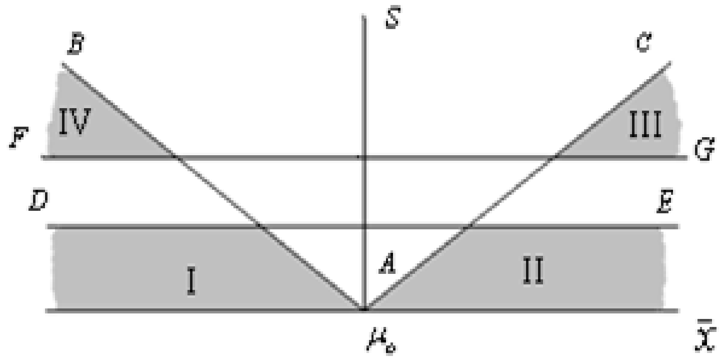

Figure 4.1 The joint domain of $(\bar{x}, s)$ for the conditional joint CDF of $\bar{X}$ and S following rejection of two tests. The lines AB and AC are determined by $|\bar{x}-\mu_o| = t_{1-\alpha/2}(s/\sqrt{n})$. The lines DE and FG are determined by $\frac{(n-1)s^2}{\sigma_o^2} = \chi^2_{n-1;\alpha/2}$ *and* $\frac{(n-1)s^2}{\sigma_o^2} = \chi^2_{n-1;1-\alpha/2}$

It should be noticed that the calculations of the double integral $\iint f(\bar{x}, s) d\bar{x} ds$ in equation (4.1) are quite different when the pair of observations $(\bar{x}, s)$ falls into different regions of I, II, III or IV. From equation (4.1), it's quite obvious that the conditional joint CDF of $\bar{X}$ and S only depends on the two unknown parameters μ and σ. With powerful computer programs, it's possible to calculate the cumulated probability for any observed value of $(\bar{x}, s)$ using equation (4.1), if the two parameters μ and σ^2 are given.

Now, let us focus on the conditional joint confidence region of μ and σ. Since the conditional joint CDF of $\bar{X}$ and S only depends on the two unknown parameters μ and σ but not on any other unknown nuisance parameters, if a conditional joint confidence region of μ and σ exists, it could be found by using the above information. Next, we'll try to extend the general method for finding a confidence interval of an unknown parameter to the two parameters case.

Suppose $\Omega \in K$ is one relatively small region of $\bar{X}$ and S such that $P[(\bar{X}, S) \in \Omega] = 1-\alpha$, if we regard $(\bar{x}, s)$ as random statistics and let (μ, σ) change jointly, then the statement $(\bar{x}, s) \in \Omega$ is equivalent to the statement $\alpha_1 < F_c(\bar{x}, s) < 1-\alpha_2$ for some α_1 and α_2 such that $\alpha_1 + \alpha_2 = \alpha$. Therefore, if the inequality $\alpha_1 < F_c(\bar{x}, s) < 1-\alpha_2$ has a solution for the region of (μ, σ), then this solution should construct a 100(1-α)% joint confidence region of μ and σ. In other words, if we plug any pair of (μ, σ) values into the above inequality and make the inequality a true statement for a pair of observed statistics $\bar{X}$ and *s*. then this pair of (μ, σ) value should be in a 100(1- α_1 - α_2)% conditional joint confidence region of μ and σ which is related to this observed pair of statistics $\bar{X}$ and *s*. In this way, we can extend the general method to the two parameters case, and obtain the following result.

[Result 4.1] Suppose a random sample X_1, X_2, …… X_n is taken from a normal distribution $N(\mu, \sigma^2)$, where μ and σ^2 are both unknown. Let $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ such that $0 < 1 - \alpha_1 - \alpha_2 < 1$. Let $\bar{x}$ and s be the observed values of $\bar{X}$ and S, and let $F_c(\bar{x}, s)$ denote the conditional joint CDF of $\bar{X}$ and S (equation (4.1)). If the observed values of $\bar{x}$ and s result in rejecting the two null hypotheses H_o: $\mu = \mu_o$ and H_o: $\sigma = \sigma_o$ at level α, then the solution of

$$\alpha_1 < F_c(\bar{x}, s) < 1 - \alpha_2 \tag{4.2}$$

for all pairs of (μ, σ) construct a $100(1- \alpha_1 - \alpha_2)$% conditional joint confidence region of μ and σ.

The solution of equation (4.2) for the joint confidence region is not easy to be formulated, but we may think in the following way to get a rough picture. In equation (4.2), if we fix one of the two unknown parameters, say σ, at one value σ_1, then the problem becomes to finding a conditional confidence interval of one single unknown parameter. By the general method, the solution should be a finite interval if the value of σ_1 is within the joint confidence region. If we change σ to another fixed value σ_2, then the solution of μ is another finite interval if σ_2 is also in the joint confidence region. Same situation happens when we fix μ at one value and try to find the solution of σ. So, we may conclude that the solution of equation (4.2) is just one connected region of μ and σ, and this region should contain the pair of observed value of $(\bar{x}, s)$.

Result 4.1 is numerically verified by examples, but hasn't been proved theoretically. After we obtained the conditional joint confidence region of μ and σ, the conditional confidence interval of C_{pk} following rejection of the two tests for testing H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$ and H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ can also be determined, but the computation is still very complicated.

4.2 Testing for One of the Two Parameters

In some situations, we may have uncertain prior information on one of the two unknown parameters. If this is the case, then we can construct only one preliminary test. We now consider the first case that we have some prior information about the process mean μ, and we test the hypothesis H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$. If the null hypothesis is not rejected, then we regard $\mu = \mu_o$ as known, no conditional confidence interval of C_{pk} is needed to construct. If the null hypothesis H_o: $\mu = \mu_o$ has been rejected, then we need to use $\bar{x}$ and s as two estimates of μ and σ to give a point estimate of C_{pk}. The conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\mu = \mu_o$ can be obtained by using a similar procedure discussed in this section (4.1).

The conditional joint pdf of $\bar{X}$ and S following rejection of the null hypothesis H_o: $\mu = \mu_o$ can be expressed as

$$f_c(\bar{x}, s) = \begin{cases} f(\bar{x}, s)/D, & if \ (\bar{x}, s) \in K \\ 0, & otherwise \end{cases}$$

where $f(\bar{x}, s)$ is the unconditional joint pdf of $\bar{X}$ and S; K is the critical region of the test which is determined by $|\bar{x} - \mu_o| > t_{1-\alpha/2}(s/\sqrt{n})$, i.e., the regions I and II shown in figure 4.2; D is the total unconditional probability of $(\bar{x}, s)$ falling into the above critical region, which is determined by the following double integral.

$$D = \iint_K f(\bar{x}, s) d\bar{x} ds$$

In this situation, D is also the power of the test for testing H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$, which can be calculated by using the non-central t-distribution, that is

$$D = P(|\bar{X} - \mu_o| > t_{1-\alpha/2}(S/\sqrt{n})|\mu)$$

$$= P(\left|\frac{\bar{X} - \mu + \mu - \mu_o}{S/\sqrt{n}}\right| > t_{1-\alpha/2}|\mu)$$

$$= 1 - H(t_{1-\alpha/2}) + H(-t_{1-\alpha/2})$$

where $H(\cdot)$ is the CDF of the non-central t-distribution with (n-1) degrees of freedom and with non-centrality parameter $\delta = \frac{\mu - \mu_0}{\sigma/\sqrt{n}}$. It's quite obvious that D involves the two unknown parameters μ and σ.

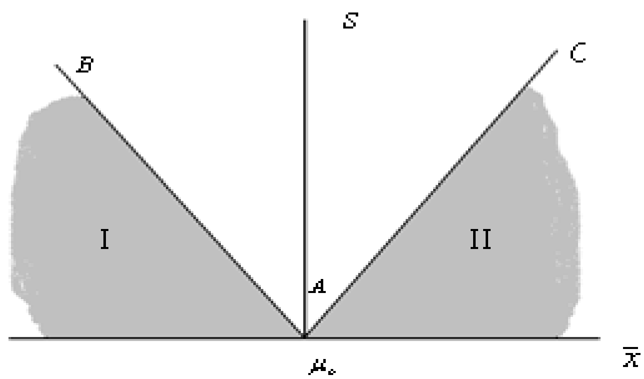

Figure 4.2 The joint domain of $(\bar{x}, s)$ for the conditional joint CDF of $\bar{X}$ and S following rejection of one test for the mean H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$. The lines AB and AC are determined by $|\bar{x} - \mu_o| = t_{1-\alpha/2}(s/\sqrt{n})$.

The conditional joint CDF of $\overline{X}$ and S following rejection of the null hypothesis H_o: $\mu = \mu_o$ can be expressed as

$$F_c(\bar{x}, s) = \frac{\iint f(\bar{x}, s) d\bar{x} ds}{D}, \quad for\ (\bar{x}, s) \in K \tag{4.3}$$

This conditional joint CDF of $\overline{X}$ and S depends only on the two unknown parameters μ and σ but not on any other nuisance parameters, so we can follow the same procedure discussed in this section (4.1) to find a conditional joint confidence region of μ and σ following rejection of the null hypothesis H_o: $\mu = \mu_o$. After we obtained the joint confidence region of μ and σ, we can use it to obtain a conditional confidence interval of C_{pk}.

In case we only have uncertain prior information about the process variance σ^2, we need to test the hypothesis H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$. If the null hypothesis is not rejected, then we regard $\sigma = \sigma_o$ as known. No conditional confidence interval of C_{pk} is needed for this case, since there is no test hypothesis has been rejected. If the null hypothesis H_o: $\sigma = \sigma_o$ has been rejected, then we need to use $\bar{x}$ and s as two estimates of μ and σ to give a point estimate of C_{pk}. The conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ can be constructed similarly to the previous case, except the rejection region is different.

The conditional joint pdf of $\overline{X}$ and S following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ can be expressed as

$$f_c(\bar{x}, s) = \begin{cases} f(\bar{x}, s)/D, & if\ (\bar{x}, s) \in K \\ 0, & otherwise \end{cases}$$

where $f(\bar{x}, s)$ is the unconditional joint pdf of $\overline{X}$ and S. K is the critical region of the test which is determined by $\frac{(n-1)s^2}{\sigma_o^2} < \chi^2_{n-1;\alpha/2}$ and $\frac{(n-1)s^2}{\sigma_o^2} > \chi^2_{n-1;1-\alpha/2}$, that is, the regions I and II shown in figure 4.3. D is the total unconditional probability of $(\bar{x}, s)$ falling into the above critical region, which is also determined by the double integral.

$$D = \iint_K f(\bar{x}, s) d\bar{x} ds$$

The conditional joint CDF of $\overline{X}$ and S following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ can be expressed as

$$F_c(\bar{x}, s) = \frac{\iint f(\bar{x}, s) d\bar{x} ds}{D}, \quad for\ (\bar{x}, s) \in K \tag{4.4}$$

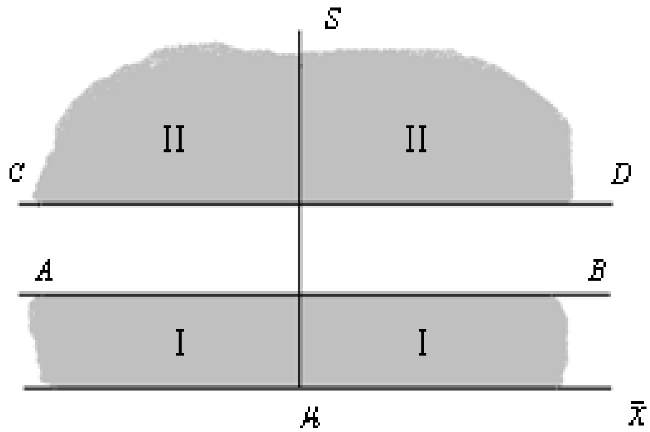

Figure 4.3 The joint domain of $(\bar{x}, s)$ for the conditional joint CDF of $\bar{X}$ and S following rejection of one test for σ H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$. The lines AB and CD are determined by $\frac{(n-1)s^2}{\sigma_o^2} = \chi^2_{n-1;\alpha/2}$ and $\frac{(n-1)s^2}{\sigma_o^2} = \chi^2_{n-1;1-\alpha/2}$

As we can check, this conditional joint CDF of $\bar{X}$ and S only depends on the two unknown parameters μ and σ. Therefore, we can use result 4.1 to find a conditional joint confidence region of μ and σ following rejection of the null hypothesis H_o: $\sigma = \sigma_o$. Once the conditional joint confidence region of μ and σ is obtained, a conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ can also be determined.

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