Nonparametric Control Charts for Monitoring Location based on the Exceedance Statistic

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Abstract

Nonparametric control charts are useful when there is lack of knowledge about the underlying distribution. Two nonparametric control charts, based on the exceedance statistics, are considered for detecting a shift in the location parameter of a continuous distribution; the one being a cumulative sum (CUSUM)-type chart and the other being an exponentially weighted moving average (EWMA)-type chart. Advantages of the nonparametric charts include robustness to the violation of distributional assumptions and resistance to outliers. The fact that the exceedance statistics can save testing time and resources, as they can be applied as soon as a certain order statistic of the reference sample is available, may be a plus. A comparison with a number of existing control charts, comprising of the traditional (normal theory) CUSUM and EWMA charts for subgroup averages and the nonparametric CUSUM and EWMA charts based on the Wilcoxon rank-sum statistics, is made. It is seen that the proposed charts perform well in many cases and thus can be a useful alternative chart in practice.

Key Words: Exceedance, Nonparametric, Quality control, Robust, Run-length, Simulation

1. Introduction

The exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) control charts have enjoyed widespread popularity in practice with data analysts. These time sequential charts are particularly effective in detecting relatively small and persistent changes (step shifts) in the process (see e.g. Montgomery, 2009 pages 400 and 419). In typical applications of the traditional EWMA and CUSUM charts for subgroup averages it is usually assumed that the underlying process distribution is normal, or, at least, approximately so. However, in certain situations in practice, the normality assumption may not be tenable or justifiable for lack of information or data. Thus development and application of distribution-free (or nonparametric) charts seem desirable as they do not depend on a particular distributional assumption and their in-control (IC) performance is the same for all continuous distributions. For a thorough account of the nonparametric control charts literature see Chakraborti et al. (2001), Chakraborti and Graham (2007) and Chakraborti et al. (2011).

Amin et al. (1995) and Bakir and Reynolds (1979) considered nonparametric CUSUM charts based on the sign and the signed-rank test statistics, respectively, and, more recently, Graham et al. (2011a,b) considered nonparametric EWMA charts based on the sign and the signed-rank test statistics, respectively, and in all these cases the IC process median was specified or known. In many practical situations, the process median may not be known and would have to be estimated. It is well-known that ignoring the effects of estimation of parameters can be costly as the run-length properties of the chart are greatly impacted and this can lead to, for example, many more false alarms than are nominally expected. In this paper we consider a nonparametric EWMA chart and a nonparametric CUSUM chart for monitoring the unknown median using a reference sample.

2. Statistical Background: Precedence/Exceedance Statistic

The precedence test is a nonparametric test based on the number of observations from one of the samples that precede a specified (say the r^{th}) order statistic of the other sample. The precedence statistic is linearly related to the exceedance statistic, which is the number of observations from one of the samples that exceed the r^{th} order statistic of the other sample, so that precedence and exceedance tests are equivalent. Precedence/exceedance tests have been found to be useful in a number of applications including quality control and reliability studies with lifetime data. The reader is referred to Balakrishnan and Ng (2006) for the vast literature on precedence/exceedance tests. In particular, they note that (page 51) "Wilcoxon's rank-sum test performs better than the precedence tests if the underlying distributions are close to symmetry, such as the normal distribution, gamma distribution with large values of shape parameter, and lognormal distribution with small values of shape parameter. However, under some right-skewed distributions such as the exponential distribution, gamma distribution with shape parameter 2.0, and lognormal distribution with shape parameter 0.5, the precedence tests have higher power values than the Wilcoxon's rank-sum test for small values of r. It is evident that the more right-skewed the underlying distribution is, the more powerful the precedence test is." Motivated by these observations, Chakraborti et al. (2004) studied a class of nonparametric Phase II Shewhart-type charts based on the precedence statistics, called the Shewhart-type precedence charts. This paper has been the starting point for a number of follow-up papers in this area. In this paper two nonparametric Phase II control charts based on the exceedance statistics (denoted EWMA-EX and CUSUM-EX, respectively) is considered for detecting a shift in the location parameter of a continuous distribution.

3. Construction of the Proposed Control Charts

Assume that a Phase I reference sample $X_1, X_2, \dots, X_m$ is available from an IC process with a cdf $F(x)$. Let $Y_{j1}, Y_{j2}, \dots, Y_{jn}$, $j = 1,2, \dots$, denote the j^{th} test Phase II sample of size n from a cdf $G(y)$. Both F and G are unknown continuous distribution functions and the process is IC when $F = G$. For detecting a change in the location, we use the location model $G_Y(x) = F(x - \theta)$ where $\theta \epsilon [0, \infty)$ is the location parameter. Let $U_{j,r}$ denote the number of exceedances, that is, the number of Y observations in the j^{th} Phase II sample that exceeds $X_{(r)}$, the r^{th} ordered observation in the reference sample. The statistic $U_{j,r}$ is called an exceedance statistic and the probability $p_r = P[Y > X_{(r)} \,|\, X_{(r)}]$ is called an exceedance probability.

Note that the EWMA-EX and CUSUM-EX statistics defined in Sections 3.1 and 3.2, respectively, actually gives a class of control charts for various choice of r. From a practical point of view, and as used in Chakraborti et al. (2004), we take θ to be the median and $X_{(r)}$ to be the median of the reference sample. The reasons for focusing on the median are clear; it is robust and a better representative of the central reference value. However, in general, the precedence chart can be used to monitor other parameters, for example, the 1$^{\text{st}}$ quartile or the 70$^{\text{th}}$ percentile.

The proposed EWMA and CUSUM charts can be constructed in a straightforward manner. Since for a given value of the order statistic $X_{(r)} = x_{(r)}$, the variable $U_{j,r}$

follows a binomial(n, p_r) distribution, conditionally on $X_{(r)}$, we can construct a binomial-type EWMA chart and a binomial-type CUSUM chart using the $U_{j,r}$'s to monitor the process location. The reader should note that unconditionally these charts will not behave like binomial charts.

3.1. The EWMA-EX control chart

The plotting statistic of the EWMA-EX chart is given by

$$Z_j = \lambda\, U_{j,r} + (1-\lambda) Z_{j-1} \quad \text{for} \quad j = 1,2,3,\ldots \tag{1}$$

where the starting value is taken as $Z_0 = E\left(U_{j-k,r} | X_{(r)}\right) = np_r$ and $0 < \lambda \leq 1$ is the smoothing constant. Note that we get the Shewhart-type precedence chart of Chakraborti et al. (2004) when $\lambda = 1$. To calculate the control limits of the proposed chart the IC mean and standard deviation of Z_j are necessary. It can be shown that the unconditional IC mean and standard deviation of Z_j are given by

$$E\left(Z_j\right) = n(1-a)\left(1-(1-\lambda)^j\right)$$
and
$$STDEV\left(Z_j\right) = \sqrt{\left(\frac{na(1-a)}{m+2}\right)\left\{n\left(1-(1-\lambda)^j\right)^2 + \frac{\lambda(m+1)}{2-\lambda}\left(1-(1-\lambda)^{2j}\right)\right\}} \tag{2}$$

respectively, where $a = r/(m+1)$ (see Appendix A of Graham et al. (2012) for the derivation of these formulae). Hence, the proposed nonparametric EWMA-EX chart has a plotting statistic Z_j given in (1) with $Z_0 = n(1-a)$ and the exact time varying upper control limit (UCL), lower control limit (LCL) and centreline (CL) of the chart are given by $CL = E\left(Z_j\right)$ and $UCL/LCL = E\left(Z_j\right) \pm L \times STDEV\left(Z_j\right)$ where the mean and the standard deviation are given in (2). The corresponding unconditional "steady-state" control limits are given by

$$CL = n(1-a)$$
and
$$UCL/LCL = n(1-a) \pm L\sqrt{\left(\frac{na(1-a)}{m+2}\right)\left\{n + \frac{\lambda(m+1)}{2-\lambda}\right\}} \tag{3}$$

These limits are typically used when the EWMA-EX chart has been running for several time periods and are obtained from (2) as $j \to \infty$ so that $\left(1-(1-\lambda)^j\right)$ and $\left(1-(1-\lambda)^{2j}\right)$ approach unity, respectively. Note that λ and L are the two design parameters of the chart which are chosen such that a desired nominal in-control ARL (denoted ARL_0) is attained. The smoothing parameter $0 < \lambda \leq 1$ is typically selected first (which depends on the magnitude of the shift to be detected) and then the constant $L > 0$ is selected (which determines the width of the control limits i.e. the larger the value of L, the wider the control limits and vice versa). The first step is to choose λ. If small shifts (roughly 0.5 standard deviations or less) are of primary concern the typical recommendation is to choose a small λ, say equal to 0.01, 0.025 or 0.05; if moderate shifts (roughly between 0.5 and 1.5 standard deviations) are of greater concern choose λ = 0.10, whereas if larger shifts (roughly 1.5 standard deviations or more) are of concern choose λ = 0.20 (see e.g. Montgomery (2009), page 423). Next we choose L, in conjunction with the chosen λ, so that a desired nominal ARL_0 is attained.

3.2. The CUSUM-EX control chart

As mentioned previously, given $X_{(r)} = x_{(r)}$, the variable $U_{j,r}$ follows a binomial distribution with parameters (n, p_r) and thus, conditionally on $X_{(r)}$, we can use a binomial-type CUSUM chart based on the $U_{j,r}$'s to monitor the process location (via the exceedance probabilities). Noting that $E\left(U_{j,r}\middle|X_{(r)}\right) = np_r$ and the conditional probability p_r is unknown, we may replace it by its unconditional IC value $d = \frac{m-r+1}{m+1}$ (the reader is referred to Result A.4 in the Appendix of Mukherjee et al. (2012) for the derivation of d). Hence the two-sided CUSUM-EX chart has plotting statistics

$$C_j^+ = \max[0, C_{j-1}^+ + \left(U_{j,r} - nd\right) - k] \quad \text{and} \quad C_j^- = \min[0, C_{j-1}^- + \left(U_{j,r} - nd\right) + k] \tag{4}$$

for $j = 1,2,3, \ldots$ with starting values $C_0^+ = C_0^- = 0$ and $k \geq 0$ is the so-called reference value. The chart signals a possible OOC situation for the first j at which either $C_j^- < -H$ or $C_j^+ > H,$ where $H > 0$ is called a decision constant. Otherwise, the process is considered IC and process monitoring continues without interruption.

The design parameters (k, H) are chosen so that the chart has a desired nominal ARL_0. The first step is to choose k. In Mukherjee et al. (2012) an extensive study was done to investigate the choice of k and they recommended using using $k = 0$ (or letting δ tend to 0 where δ represents the shift in the mean). As mentioned previously, we take θ to be the median and $X_{(r)}$ to be the median of the reference sample. In this case, d is taken to be equal to 0.5. Hence the exceedance CUSUM median chart based on the reference sample median, is given by the plotting statistics

$$C_j^+ = \max[0, C_{j-1}^+ + \left(U_{j,r} - n/2\right) - k] \quad \text{and} \quad C_j^- = \min[0, C_{j-1}^- + \left(U_{j,r} - n/2\right) + k] \tag{5}$$

for $j = 1,2,3, \ldots$ with starting values $C_0^+ = C_0^- = 0$. The next step is to choose H, in conjunction with the chosen k, so that a desired nominal ARL_0 is attained.

4. Performance Comparison

We compare the EWMA-EX chart to (i) the parametric EWMA-$\overline{X}$ chart and (ii) the nonparametric EWMA chart based on the Wilcoxon rank-sum statistic (denoted EWMA-Rank) proposed by Li et al. (2010). The CUSUM-EX chart is compared to (i) the parametric CUSUM-$\overline{X}$ chart and (ii) the nonparametric CUSUM chart based on the Wilcoxon rank-sum statistic (denoted CUSUM-Rank) proposed by Li et al. (2010). The distributions considered in the study are: (a) the standard normal distribution, N(0,1), (b) the exponential distribution with mean 1, EXP(1), which is positively skewed and (c) the Laplace (or double exponential distribution DE(0,1)) distribution with mean 0 and variance 2 which is standard normal like, but has heavier tails.

Because the EWMA-EX and CUSUM-EX charts are nonparametric, the IC run-length distribution and the associated characteristics remain the same for all continuous distributions. In other words, the IC run-length distribution is robust by definition and thus all IC characteristics such as the false alarm rate (*FAR*) and the *ARL* would all remain the same for all continuous distributions. For the OOC chart performance comparison it is customary to ensure that the ARL_0 values of the competing charts are fixed at (or very close to) an acceptably high value, such as 500 in this case, and then compare their out-of-control *ARL*'s i.e. their ARL_δ values, for specific values of the shift δ; the chart with the smaller ARL_δ value is generally preferred.

4.1. Comparison of the EWMA-EX chart to the parametric EWMA and the EWMA-Rank charts

Tables 1 to 3 show the OOC performance characteristics of the run-length distribution for various distributions and shifts $\delta = \gamma \frac{\sigma}{\sqrt{n}}$, where σ denotes the process standard deviation, $\gamma = 0.00(0.05)0.25$, 0.50, 0.75, 1.00, 1.50 and 2.00, represents the shift in the median, for $m = 100$ and $n = 5$, for the EWMA-EX, EWMA-$\bar{X}$ and EWMA-Rank charts, respectively. Note that although shifts as large as $\gamma = 3.00$ were considered in this study, the largest magnitude reported in the paper is $\gamma = 2.00$, since, for larger shifts, the run-length characteristics of the charts tend to converge. The first row of each cell in Tables 1 to 6 shows the *ARL* followed by the corresponding *SDRL* in parentheses, whereas the second row shows the values of the 5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th} percentiles (in this order).

Table 1. Control chart performance comparison under the $N(0,1)$ distribution for nominal $ARL_0 = 500$, $m = 100$, $n = 5$ and $\lambda = 0.05$

	EWMA-EX	**EWMA-$\bar{X}$**	**EWMA-Rank**
Shift (γ) / Control limits	1.991; 3.058 with $L = 1.75$	$\pm$ 0.462 with $L = 2.855$	234.2; 295.8
0.05	507.91 (795.12) 24, 72, 201, 589, 2048	499.63 (998.64) 22, 62, 172, 515, 2034	490.07 (863.99) 23, 64, 181, 535, 2000
0.10	495.49 (778.53) 23, 68, 193, 573, 2015	467.07 (1043.25) 21, 56, 153, 471, 1910	462.83 (828.64) 21, 59, 161, 489, 1930
0.15	468.66 (765.37) 22, 62, 176, 531, 1938	435.25 (941.97) 19, 50, 133, 415, 1810	427.94 (803.86) 20, 52, 140, 439, 1815
0.20	438.12 (738.95) 21, 56, 151, 477, 1853	371.67 (822.70) 17, 41, 105, 337, 1608	375.46 (748.86) 18, 44, 113, 360, 1614
0.25	398.98 (687.24) 20, 49, 130, 417, 1738	312.72 (695.72) 15, 35, 82, 262, 1367	326.08 (705.73) 16, 37, 88, 287, 1453
0.50	185.97 (456.32) 14, 26, 49, 129, 860	93.36 (305.26) 10, 18, 31, 62, 307	98.77 (306.09) 11, 19, 32, 66, 348
0.75	62.22 (189.51) 10, 17, 27, 48, 176	26.76 (53.78) 8, 12, 18, 27, 64	30.18 (84.12) 9, 13, 19, 29, 70
1.00	24.76 (34.77) 9, 13, 18, 26, 59	14.81 (11.84) 6, 9, 12, 17, 30	15.90 (14.64) 7, 10, 13, 18, 32
1.50	12.73 (6.13) 7, 9, 11, 15, 23	8.40 (3.11) 5, 6, 8, 10, 14	9.05 (3.16) 5, 7, 8, 10, 15
2.00	9.20 (2.64) 6, 7, 9, 10, 14	6.00 (1.71) 4, 5, 6, 7, 9	6.60 (1.68) 4, 5, 6, 7, 10

Table 2. Control chart performance comparison under the $EXP(1)$ distribution for nominal $ARL_0 = 500,\ m = 100, n = 5$ and $\lambda = 0.05$

	EWMA-EX	**EWMA-$\bar{X}$**	**EWMA-Rank**
Shift (γ) / Control limits	1.991; 3.058 with $L = 1.75$	$\pm$ 0.444 with $L = 2.745$	234.2; 295.8
0.05	493.73 (784.30) 24, 69, 190, 572, 2003	594.59 (2362.45) 18, 55, 154, 486, 2224	539.44 (1006.62) 22, 62, 175, 558, 2309
0.10	465.72 (757.07) 22, 61, 171, 525, 1958	626.89 (2342.87) 16, 49, 142, 484, 2445	535.78 (1134.54) 19, 50, 139, 495, 2408
0.15	424.85 (732.02) 20, 52, 141, 459, 1821	647.13 (2488.89) 15, 42, 122, 448, 2587	466.16 (1177.47) 16, 38, 94, 350, 2172
0.20	384.42 (702.73) 18, 44, 115, 387, 1700	672.79 (3214.85) 13, 36, 100, 376, 2558	385.49 (1209.37) 14, 30, 64, 218, 1809
0.25	317.06 (641.13) 16, 36, 86, 292, 1405	588.82 (2131.56) 12, 30, 76, 292, 2441	285.71 (1263.33) 13, 24, 44, 117, 1153
0.50	109.14 (320.47) 10, 17, 28, 64, 434	240.42 (1440.65) 8, 16, 29, 67, 639	48.25 (410.08) 9, 12, 18, 28, 81
0.75	29.89 (105.11) 7, 11, 15, 23, 65	58.22 (578.88) 7, 11, 17, 28, 90	14.81 (80.81) 7, 9, 11, 15, 26
1.00	12.74 (19.09) 6, 8, 10, 13, 26	18.10 (141.41) 5, 8, 12, 18, 36	9.36 (3.69) 6, 7, 9, 11, 15
1.50	6.06 (1.94) 5, 5, 5, 6, 9	8.36 (4.22) 4, 6, 7, 10, 15	6.37 (1.38) 5, 5, 6, 7, 9
2.00	5.02 (0.23) 5, 5, 5, 5, 5	5.86 (2.08) 3, 4, 5, 7, 10	5.14 (0.82) 4, 5, 5, 6, 7

Table 3. Control chart performance comparison under the $DE(0,1)$ distribution for nominal $ARL_0 = 500, m = 100, n = 5$ and $\lambda = 0.05$

	EWMA-EX	**EWMA-$\bar{X}$**	**EWMA-Rank**
Shift (γ) / Control limits	1.991; 3.058 with $L = 1.75$	$\pm$ 0.449 with $L = 2.774$	234.2; 295.8
0.05	498.83 (787.98) 24, 68, 194, 572, 2033	490.13 (1475.95) 20, 56, 147, 427, 1890	486.51 (855.26) 22, 64, 177, 530, 1987
0.10	452.16 (748.64) 22, 61, 166, 502, 1867	461.25 (1291.09) 19, 51, 135, 403, 1801	446.37 (815.47) 20, 56, 152, 473, 1852
0.15	392.14 (688.25) 20, 51, 130, 408, 1699	425.17 (1354.14) 17, 45, 115, 357, 1711	391.53 (765.37) 18, 45, 120, 387, 1688
0.20	310.62 (612.46) 18, 40, 94, 281, 1396	374.96 (1220.86) 16, 38, 93, 289, 1505	330.57 (727.18) 16, 37, 88, 282, 1482
0.25	236.07 (500.18) 16, 34, 70, 192, 1083	324.82 (1009.33) 15, 33, 75, 235, 1353	257.97 (620.30) 14, 31, 66, 197, 1159
0.50	50.72 (155.31) 11, 17, 25, 43, 126	96.56 (466.12) 10, 17, 29, 58, 290	56.48 (209.38) 10, 16, 24, 42, 147
0.75	20.06 (40.58) 9, 12, 16, 22, 39	29.81 (139.13) 7, 12, 17, 26, 61	18.68 (49.11) 7, 11, 14, 20, 39
1.00	13.57 (8.86) 7, 10, 12, 16, 24	14.26 (11.63) 6, 9, 12, 16, 29	11.75 (6.62) 6, 8, 10, 14, 21
1.50	9.34 (2.41) 6, 8, 9, 11, 14	8.01 (3.11) 4, 6, 7, 9, 14	7.39 (2.15) 5, 6, 7, 8, 11
2.00	7.66 (1.48) 6, 7, 7, 8, 10	5.81 (1.74) 4, 5, 6, 7, 9	5.70 (1.27) 4, 5, 5, 6, 8

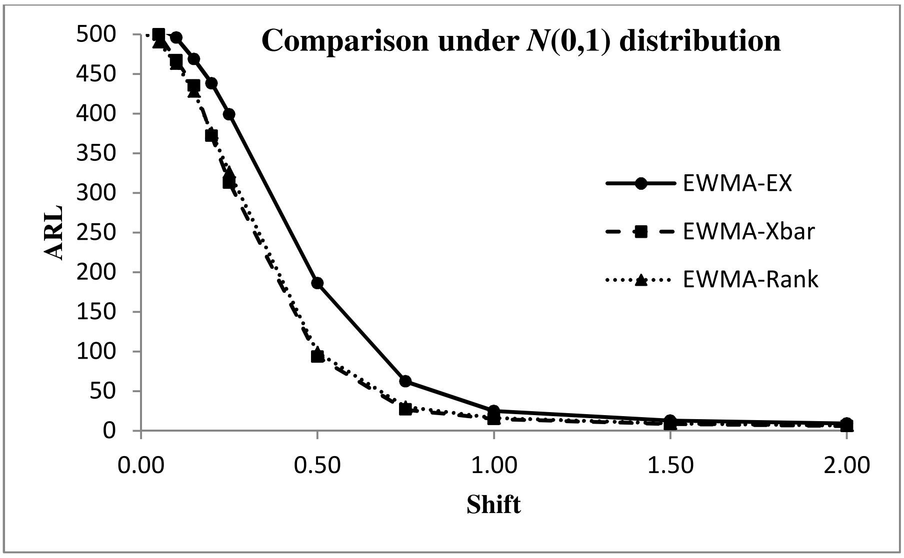

Figure 1. *ARL* performance comparison of the competing charts under the $N(0,1)$ distribution for $m = 100$, $n = 5$ and $\lambda = 0.05$.

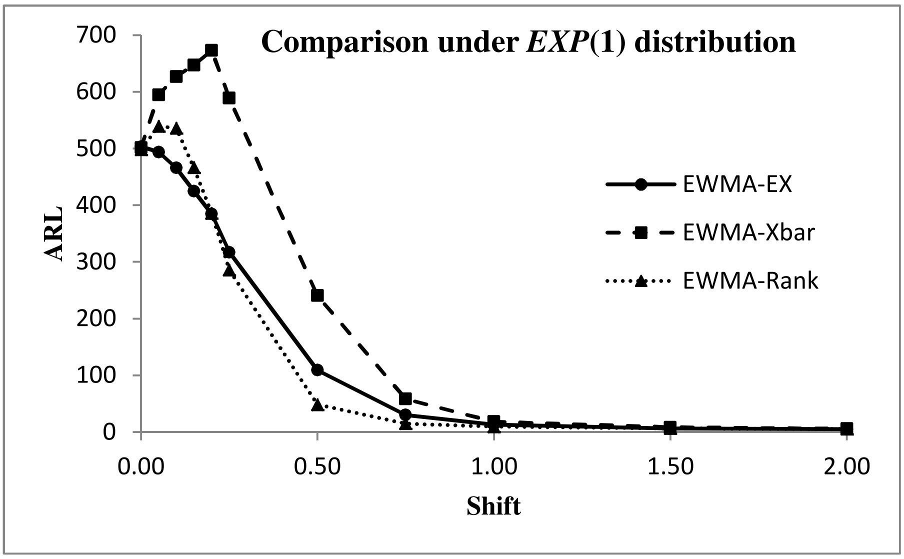

Figure 2. *ARL* performance comparison of the competing charts under the $EXP(1)$ distribution for $m = 100$, $n = 5$ and $\lambda = 0.05$.

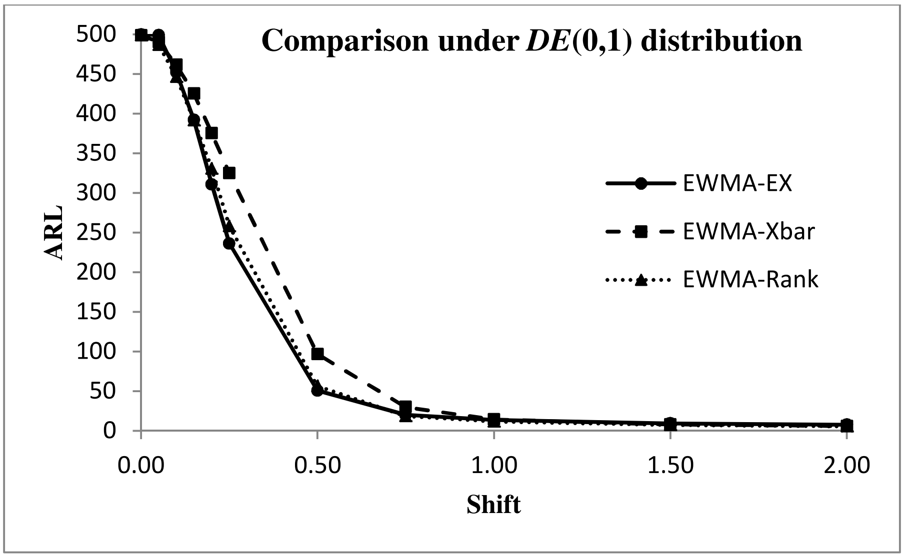

Figure 3. *ARL* performance comparison of the competing charts under the *DE*(0,1) distribution for $m = 100$, $n = 5$ and $\lambda = 0.05$.

The results from Tables 1 to 3 and Figures 1 to 3 are summarized below.

N(0,1)	The EWMA-$\overline{X}$ and EWMA-Rank charts perform similarly for all shifts under consideration and both charts outperform the proposed chart.
EXP(1)	• The proposed chart performs the best for small shifts of $\gamma < 0.20$. • The performances of the two nonparametric charts are very similar for shifts of γ = 0.20 and 0.25. • The EWMA-Rank chart outperforms the proposed chart for shifts of γ = 0.50, 0.75 and 1.00. • The EWMA-$\overline{X}$ chart performs the worst except for large shifts ($\gamma \geq 1.50$) where the three charts perform very similarly.
DE(0,1)	• The proposed chart outperforms the EWMA-$\overline{X}$ chart for all shifts under consideration, except for γ = 0.05 where the performance is very similar and for $\gamma \geq 1.50$ where the EWMA-$\overline{X}$ performs the best. • The EWMA-Rank chart outperforms the EWMA-$\overline{X}$ chart for all shifts under consideration, except for γ = 0.05 and 2.00, where the performance is very similar. • The EWMA-Rank chart and the proposed chart perform similarly for shifts of sizes $\gamma \leq 0.15$ and $\gamma \geq 0.75$, but the proposed chart detects shifts faster for γ = 0.20, 0.25 and 0.50.

4.2. Comparison of the CUSUM-EX chart to the parametric CUSUM and the CUSUM-Rank charts

In this section we compare the performance of the CUSUM-EX chart to the CUSUM-$\overline{X}$ and CUSUM-Rank charts, respectively. The results are shown in Tables 4 to 6 and also graphically represented in Figures 4 to 6 for m = 10 and n = 5. The reference value, k, is taken as follows for each CUSUM chart:

Chart	Reference value	Motivation
CUSUM-EX	$k = n(d^* - d)$ where $d^* = 0.5\sqrt{\frac{n(m+n+1)}{4(m+2)}}$	Refer to Mukherjee et al. (2012) for motivation
CUSUM-$\overline{X}$	$k = 0.5\sigma/\sqrt{n}$	As done by Li et al. (2010)
CUSUM-Rank	$k = 0.5\sqrt{mn(m+n+1)/12}$	As done by Li et al. (2010). This is 0.5 times the standard deviation of the Wilcoxon rank-sum statistic

Table 4. Control chart performance comparison under the $N(0,1)$ distribution for nominal $ARL_0 = 500$, $m = 100$ and $n = 5$

	CUSUM-EX	**CUSUM-$\overline{X}$**	**CUSUM-Rank**
Shift (γ)	$H = 9.675$	$H = 10.6$	$H = 353$
0.05	496.56 (734.10) 24, 73, 210, 604, 1911	489.90 (971.02) 25, 65, 169, 503, 1990	507.97 (814.79) 18, 76, 224, 600, 1930
0.10	484.87 (733.29) 23, 69, 195, 590, 1979	452.08 (935.92) 23, 59, 151, 457, 1821	476.49 (771.58) 17, 70, 209, 567, 1807
0.15	463.80 (723.14) 22, 62, 184, 547, 1854	422.34 (910.41) 22, 52, 128, 396, 1753	441.46 (740.21) 15, 60, 181, 511, 1724
0.20	420.86 (691.41) 20, 55, 153, 471, 1714	368.57 (882.13) 20, 45, 105, 323, 1554	397.85 (723.33) 14, 50, 147, 438, 1573
0.25	380.01 (636.35) 18, 45, 124, 408, 1668	309.10 (737.37) 18, 38, 83, 255, 1328	350.65 (653.51) 12, 41, 122, 371, 1446
0.50	178.37 (438.32) 13, 25, 49, 129, 765	90.59 (289.13) 12, 21, 33, 62, 296	130.67 (363.49) 8, 17, 36, 98, 534
0.75	58.94 (180.04) 11, 16, 25, 45, 163	28.43 (64.72) 9, 14, 20, 29, 65	35.02 (87.57) 6, 10, 17, 32, 105
1.00	24.25 (37.82) 8, 12, 18, 26, 56	16.47 (9.85) 8, 11, 14, 19, 33	15.43 (21.62) 5, 7, 11, 17, 38
1.50	12.30 (6.38) 6, 9, 11, 14, 22	9.54 (3.19) 6, 7, 9, 11, 15	7.09 (3.41) 4, 5, 6, 8, 13
2.00	8.75 (2.61) 5, 7, 8, 10, 13	6.87 (1.83) 4, 6, 7, 8, 10	4.87 (1.63) 3, 4, 5, 6, 8

Table 5. Control chart performance comparison under the $EXP(1)$ distribution for nominal $ARL_0 = 500$, $m = 100$ and $n = 5$

	CUSUM-EX	**CUSUM-$\overline{X}$**	**CUSUM-Rank**
Shift (γ)	$H = 9.675$	$H = 9.9$	$H = 353$
0.05	479.55 (717.07) 24, 71, 207, 574, 1885	553.30 (2673.83) 21, 59, 152, 444, 1950	597.99 (1010.16) 19, 81, 246, 681, 2334
0.10	457.13 (696.04) 21, 62, 179, 541, 1858	574.48 (2303.95) 19, 51, 142, 449, 2232	613.28 (1252.61) 16, 66, 208, 651, 2559
0.15	410.06 (682.05) 20, 52, 149, 459, 1676	970.65 (28775.14) 17, 45, 123, 423, 2449	627.22 (1892.39) 13, 46, 154, 559, 2575
0.20	354.36 (613.76) 17, 41, 112, 384, 1548	715.95 (10164.77) 15, 39, 102, 367, 2354	551.89 (2335.34) 11, 32, 99, 405, 2410
0.25	301.34 (569.42) 15, 33, 84, 298, 1338	678.40 (4737.46) 14, 33, 82, 302, 2409	461.33 (1557.74) 10, 24, 66, 269, 2040
0.50	106.24 (308.62) 10, 17, 27, 60, 440	348.59 (6365.84) 10, 18, 32, 71, 611	97.47 (1645.64) 6, 11, 17, 34, 183
0.75	26.67 (77.76) 7, 11, 14, 22, 64	247.32 (14456.46) 8, 13, 19, 31, 90	15.55 (80.45) 5, 7, 9, 14, 31
1.00	11.90 (17.63) 5, 7, 9, 13, 24	20.40 (226.26) 6, 10, 13, 19, 38	7.73 (5.19) 4, 5, 7, 9, 15
1.50	5.77 (1.87) 5, 5, 5, 6, 9	9.26 (4.27) 5, 7, 8, 11, 17	4.70 (1.34) 3, 4, 4, 5, 7
2.00	5.02 (0.20) 5, 5, 5, 5, 5	6.59 (2.25) 4, 5, 6, 8, 11	3.65 (0.74) 3, 3, 4, 4, 5

Table 6. Control chart performance comparison under the $DE(0,1)$ distribution for nominal $ARL_0 = 500$, $m = 100$ and $n = 5$

	CUSUM-EX	CUSUM-$\overline{X}$	CUSUM-Rank
Shift (γ)	$H = 9.675$	$H = 10.05$	$H = 353$
0.05	485.99 (710.01) 24, 72, 212, 592, 1917	506.30 (1805.17) 22, 58, 141, 410, 1827	509.43 (820.05) 18, 77, 222, 595, 1927
0.10	438.43 (692.36) 21, 59, 168, 504, 1821	454.84 (1400.71) 21, 54, 133, 386, 1720	470.23 (785.96) 17, 68, 204, 550, 1775
0.15	365.36 (613.08) 19, 48, 126, 400, 1544	405.98 (1575.96) 20, 46, 110, 323, 1546	411.42 (725.92) 14, 51, 159, 469, 1642
0.20	287.07 (551.27) 17, 38, 89, 271, 1221	379.28 (1384.18) 19, 41, 95, 281, 1439	350.48 (641.76) 12, 41, 124, 372, 1466
0.25	219.30 (479.72) 15, 32, 65, 183, 955	325.22 (1126.52) 16, 34, 75, 225, 1251	297.15 (635.51) 11, 33, 91, 286, 1263
0.50	46.82 (125.16) 11, 17, 25, 41, 119	106.23 (627.57) 11, 19, 30, 57, 288	73.78 (252.11) 7, 13, 24, 52, 253
0.75	18.73 (15.30) 8, 12, 16, 21, 38	32.04 (217.60) 9, 13, 18, 28, 62	19.06 (35.12) 5, 8, 12, 20, 49
1.00	12.93 (5.26) 7, 10, 12, 15, 22	15.63 (15.25) 7, 10, 13, 18, 31	10.11 (9.50) 4, 6, 8, 12, 22
1.50	8.96 (2.51) 6, 7, 9, 10, 13	10.07 (3.28) 5, 7, 8, 11, 15	5.59 (2.17) 3, 4, 5, 7, 10
2.00	7.24 (1.64) 5, 6, 7, 8, 10	6.50 (1.85) 4, 5, 6, 7, 10	4.15 (1.21) 3, 3, 4, 5, 6

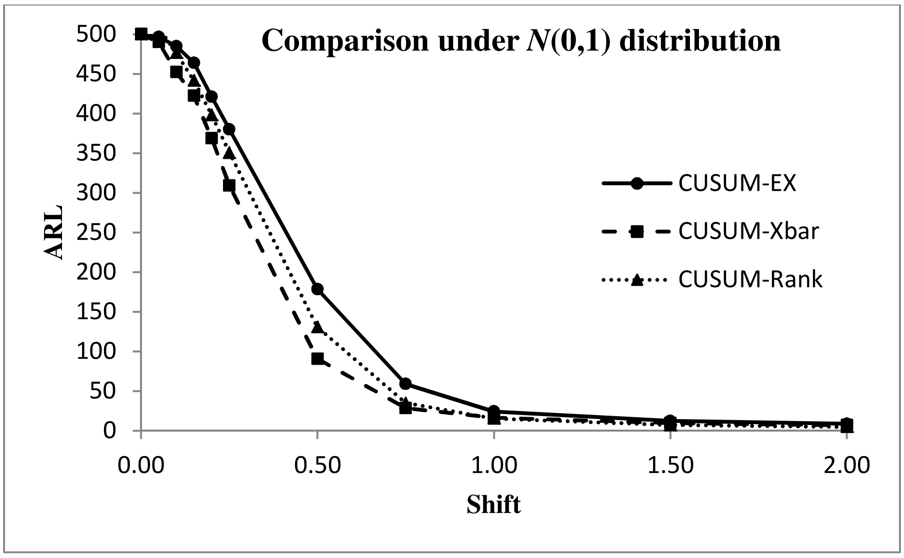

Figure 4. ARL performance comparison of the competing charts under the $N(0,1)$ distribution for $m = 100$ and $n = 5$

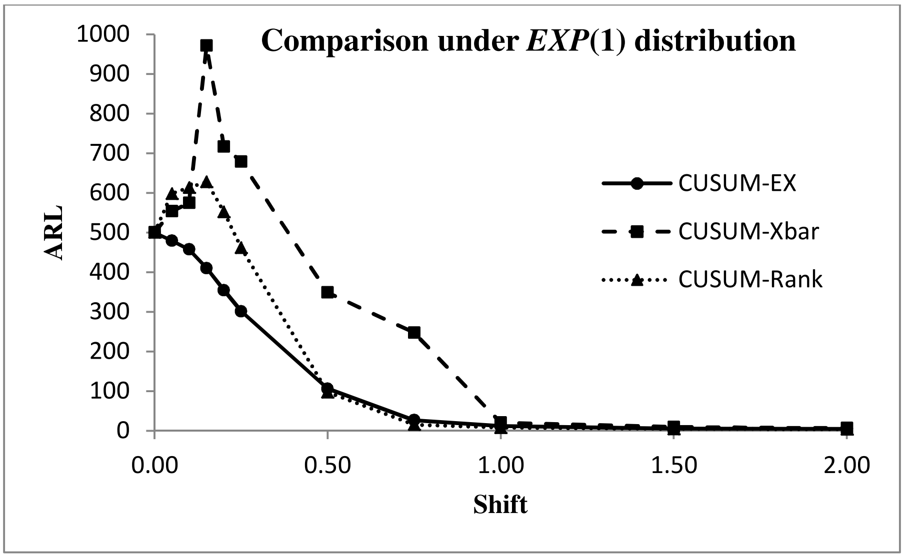

Figure 5. *ARL* performance comparison of the competing charts under the *EXP*(1) distribution for $m = 100$ and $n = 5$

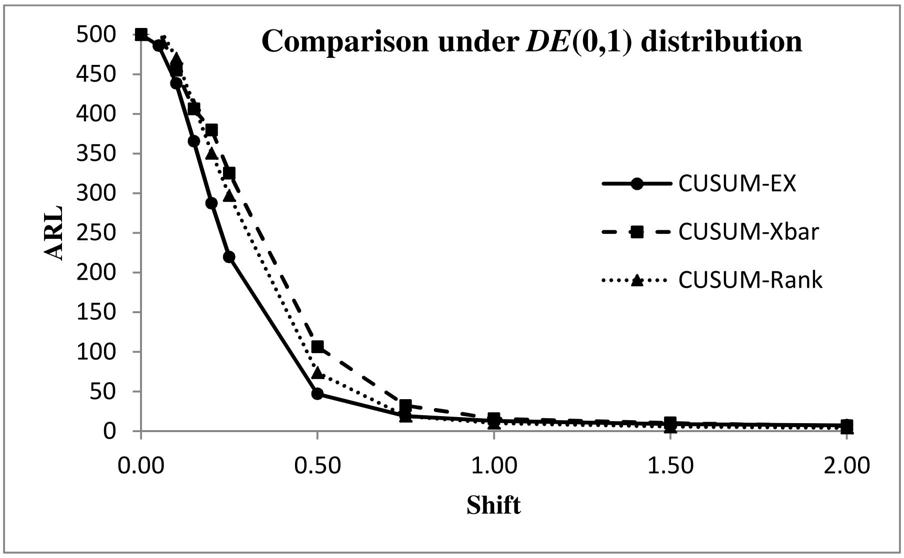

Figure 6. *ARL* performance comparison of the competing charts under the *DE*(0,1) distribution for $m = 100$ and $n = 5$

The results from Tables 4 to 6 and Figures 4 to 6 are summarized below.

$N(0,1)$	For shifts $\gamma \leq 0.75$ the CUSUM-$\overline{X}$ is performing best, whereas for $\gamma > 0.75$ the CUSUM-$\overline{X}$ and CUSUM-Rank charts perform similarly and both charts outperform the proposed chart. It isn't surprizing that the CUSUM-$\overline{X}$ is superior to the proposed chart in this case, since it is typical for parametric methods to outperform their nonparametric counterparts when all assumptions are met.
$EXP(1)$	• The proposed chart performs the best for small shifts of $\gamma < 0.50$. • The performances of the nonparametric charts are very similar for shifts of $\gamma \geq 0.50$. • For all shifts under consideration the CUSUM-$\overline{X}$ chart performs the worst, except for large shifts ($\gamma \geq 1.50$) where the performance of the three charts are very similar.
$DE(0,1)$	• The proposed chart performs the best for small shifts of $\gamma < 1.00$. • The performances of the nonparametric charts are very similar for shifts of $\gamma \geq 1.00$. • For all shifts under consideration the CUSUM-$\overline{X}$ chart performs the worst, except for large shifts ($\gamma \geq 1.50$) where the performance of the three charts are very similar.

4.3. Comparison of the CUSUM-EX and the EWMA-EX charts

By using the first columns of Tables 1 and 4 (for the $N(0,1)$ distribution), the first columns of Tables 2 and 5 (for the $EXP(1)$ distribution) and the first columns of Tables 3 and 6 (for the $DE(0,1)$ distribution) we can compare the CUSUM-EX chart to the EWMA-EX chart. The results are illustrated in Figures 7 to 9.

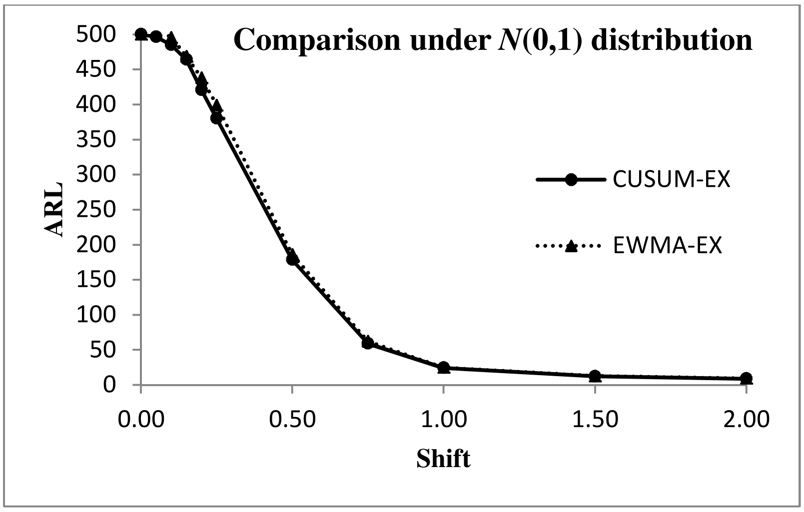

Figure 7. *ARL* performance comparison of the CUSUM-EX and EWMA-EX charts under the $N(0,1)$ distribution for $m = 100$ and $n = 5$

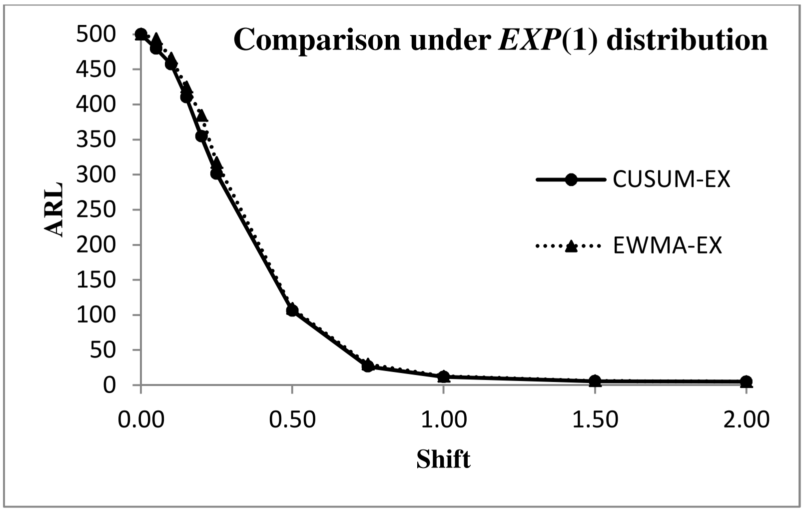

Figure 8. *ARL* performance comparison of the CUSUM-EX and EWMA-EX charts under the *EXP*(1) distribution for $m = 100$ and $n = 5$

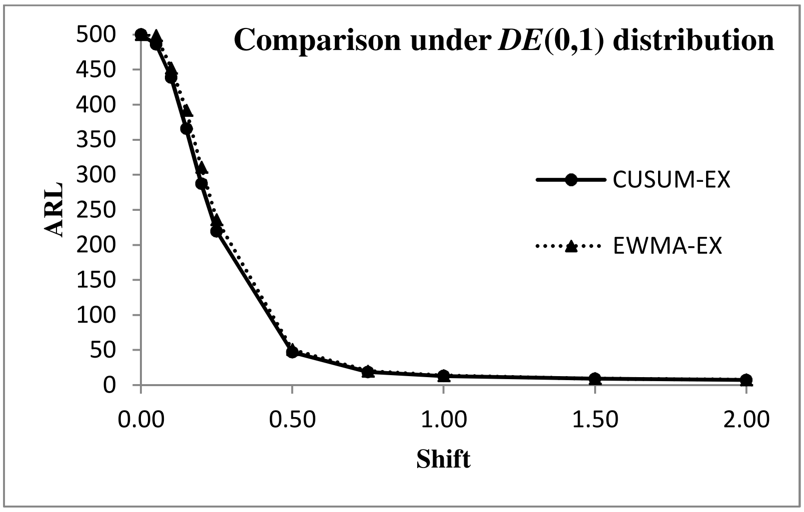

Figure 9. *ARL* performance comparison of the CUSUM-EX and EWMA-EX charts under the *DE*(0,1) distribution for $m = 100$ and $n = 5$

From Figures 7 to 9 we find that the two nonparametric exceedance charts are performing similarly for all shifts and distributions under consideration. However, the EWMA charts are preferred over the CUSUM charts by some in the industry. They are easier to implement and as Steiner and Jones (2010) put it, "The main advantage of an EWMA is that it provides an ongoing local estimate of the average score…Another minor advantage is the inherent two-sided nature of an EWMA." Thus, although the two nonparametric exceedance charts perform similarly, the EWMA-EX may be preferred by practitioners.

4.4. Some General comments regarding the *ARL* comparisons

It may be noted that there is some bias in the *ARL* (the ARL_δ is bigger than the ARL_0) of the charts for the exponential distribution when the shift is small. The bias is most prominent for the EWMA-$\overline{X}$ and CUSUM-$\overline{X}$ charts and it is also slightly present in the EWMA-Rank and CUSUM-Rank charts, whereas the two exceedance charts don't seem to have this problem. The bias could be due to many extreme long run-lengths observed in the simulation of the *ARL*, which can be a result of the right-skewedness of the exponential distribution coupled with the fact that the run-length distribution is itself highly right-skewed with a long right tail. Or it could also be a result of simulation error because the ARL_δ values are very close to the ARL_0 values. Some authors have considered *ARL*-unbiased parametric charts and this would be a topic of further research in the context of nonparametric charts. On the other hand, Steiner and Jones (2010), among others, have recommended examining the median run-length (*MRL*) instead "which is easier to simulate and gives arguably a better summary." It will be interesting to study *MRL*-unbiasedness.

5. Concluding Remarks

The traditional parametric EWMA-$\overline{X}$ and CUSUM-$\overline{X}$ charts can lack in-control robustness and as such the corresponding false alarm rates can be a practical concern. We propose two nonparametric Phase II charts based on the exceedance statistics (denoted EWMA-EX and CUSUM-EX, respectively) for detecting a shift in the unknown location parameter of a continuous distribution. A performance comparison of the EWMA-EX chart is done with its competitors: the traditional parametric CUSUM and EWMA charts for subgroup averages and some nonparametric charts i.e. CUSUM and EWMA charts based on the Wilcoxon rank-sum statistics, respectively. It is seen that the proposed charts perform as well as and, in many cases, better than its competitors, particularly for distributions that are heavier-tailed and more peaked than the normal. Thus the proposed chart can be a useful tool for the quality data analyst.

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