Prediction Intervals for *ARIMA* Processes: Sieve Bootstrap Approach

Maduka Rupasinghe* V.A. Samaranayake†

Abstract
The sieve bootstrap is a model-free re-sampling method that approximates an invertible linear process with a finite autoregressive model whose order increases with sample size. Prediction intervals based on this approach have been successfully implemented for stationary invertible ARMA processes. The coverage probabilities of sieve bootstrap intervals developed for $ARMA$ models, however, are well below the nominal level in the presence of a unit root in the autoregressive polynomial. An approach that overcomes this drawback is proposed and the asymptotic properties of the proposed method are derived. Monte Carlo simulation results indicate that the proposed method provides near nominal coverage at moderate sample sizes.

Key Words: Unit root processes; Forecast intervals; Integrated process; Nonstationarity

1. Introduction

Many financial and economic time series are non-stationary, and *Autoregressive Integrated Moving Average* (ARIMA) processes are often used to model such empirical process. For the practitioner, one of the main goals of empirical time series modeling is to obtain forecasts based on its past values. Standard parametric point and interval forecasts are quite accurate under normally distributed innovations. As noted by Stine (1987) and Thombs and Schucany (1990), parametric prediction intervals perform poorly when the normal assumption is violated. Nonparametric bootstrap based prediction intervals, on the other hand offer a potentially superior alternative to parametric methods. While nonparametric approaches have been proposed for stationary processes, a method that provides prediction intervals for the class of $ARIMA$ models with unknown orders p, q is not available. In the following sections, a nonparmetric bootstrap approach to obtain prediction intervals for $ARIMA$ processes with unknown orders is presented.

One drawback of the original bootstrap methods developed for time series is the requirement of the knowledge of the orders associated with the underlying process. For instance, the bootstrap approach proposed by Stine (1987) assumes that the order, p, of the $AR(p)$ process is known. The same is true for methods introduced by Thombs and Schucany (1990), Cao *et al.* (1997) and Pascual *et al.* (2004).

The approach proposed in this paper, however, does not require any knowledge of the orders associated with autoregressive and moving average polynomials. Our framework is identical to the Sieve Bootstrap prediction intervals implemented by Alonso, Pena and Romo (2002, 2003 and 2004), which resamples residuals obtained by a sequence of $AR(p)$ models with order $p = p(n)$ which increases with the sample size n. The foundation of this sieve bootstrap approach was laid by Kreiss (1988) and (1992), for time series that can be represented by an infinite autoregressive process. Bühlmann (1997), who introduced the term *sieve bootstrap*, extended this approach to more general class of time series that can be written as an infinite order moving average process. Alonso et *al*. (2002, 2003) formalized this sieve bootstrap concept and applied it to obtain prediction intervals for linear processes. The same authors made further refinements in 2004 by introducing model uncertainty in computing prediction intervals. Alonso's method was modified by Mukhopadhyay and Samaranayake (2010) to improve the coverages of the prediction intervals. They achieved

*Ashland Univeristy, Ashland OH 44805
†Missouri University of Science & Technology, Rolla MO 65401

this by introducing a variance inflation factor for bootstrap residuals. These preceding bootstrap methods are, however, limited to stationary linear processes such as $ARMA$ models. Rupasinghe and Samaranayake (2012) extended Alonso's 2003 sieve bootstrap procedure to compute prediction intervals for *long memory* processes ($FARIMA$). In this paper, we extend Alonso's 2003 sieve bootstrap procedure to obtain prediction intervals for $ARIMA$ processes.

1.1 *ARIMA* Processes

A real-valued process $\{x_t\}_{t\in\mathbb{Z}}$ is said to be a *Autoregressive Integrated Moving Average* (*ARIMA*(*p,d,q*)) process if it is stationary and satisfies

$$\alpha(B) \bigtriangledown^d (x_t - \mu) = \theta(B)\epsilon_t, t \in \mathbb{Z}, \tag{1}$$

where $\alpha(z) = 1 - \alpha_1 z - ... - \alpha_p z^p$ and $\theta(z) = 1 + \theta_1 z + ... + \theta(q) z^q$ represent autoregressive and moving average polynomials of degrees *p* and *q* respectively. The mean of the process is $\mu = E[x_t]$ for all t. It is assumed that $\alpha(.)$ and $\theta(.)$ do not share common zeros. The error terms, $\{\epsilon_t\}$, are assumed to be zero-mean white noise with finite variance σ^2. Note that $\bigtriangledown = 1 - B$, where B is the back-shift operator defined by $B^k x_t = x_{t-k}$ for $k \in \mathbb{N}$. The difference parameter, d, can take any non-negative integer, but we assume that $d = 1$ or 0, where represents the most common type of $ARIMA$ processes used in empirical modeling.

The literature on methods for obtaining prediction intervals for $ARIMA$ processes is very limited. Kim (2001) extended the forward and backward bootstrap procedure of Thombs and Schucany (1990) to obtain prediction intervals for $AR(p)$ models with unit roots by incorporating a bias correction on the bootstrap estimates of the forward and backward AR coefficients. The backward AR representation is obtained by reversing the forward (usual) $AR(p)$ model. This bias correction was adopted from Kilian (1998a) and utilized to improve the coverage probabilities in the presence of unit roots. Their method, however, assumes that the process is $AR(p)$ and the order, p, is known, which could be a weakness in situations where the order is unknown. They also assumed normal errors in establishing the asymptotic validity of the method.

In their recent articles, Panichkitkosolkul and Niwitpong (2011, 2012) introduced parametric prediction intervals for Gaussian $AR(p)$ models that may include unit root processes. The prediction intervals are computed following preliminary unit root tests and two different formulations for prediction intervals were used based on the outcome of the initial tests. They used well known Dickey-Fuller (DF) (Dickey and Fuller (1979)), Augmented Dickey-Fuller (ADF) (Said and Dickey (1984)), and SSL (Shin, Sarkar and Lee (1996)) unit root tests. The random walk model is used to obtain point forecasts in case the preliminary test did not reject the null hypothesis that the process has an autoregressive root equal to unity. There are concerns on the use of unit root tests prior to computing prediction intervals, as the power of these tests is small under many situations. See Psaradakis (2000, 2001), Chang and Park (2003) and Palm, Smeekes and Urbain (2008). Specifically, the coverage probabilities that Panichkitkosolkul and Niwitpong (2011, 2012) report based on a simulation study are conditional on the results of the unit root test. Unconditional coverage probabilities can be lower due to the low power of the tests and the presence of Type I error.

Our method, however, do not alter the procedure of computing prediction intervals based on results of a unit root test. If the observed series $\{x_t\}$ satisfies $\alpha(B)(1 - B)x_t = \theta(B)\epsilon_t$, $\{\epsilon_t\} \sim WN(0, \sigma^2)$, observe that the differenced series $y_t = x_t - x_{t-1}$ is stationary. One can first compute the bootstrap distribution of the future observations, y_{n+h}, of the differenced series and then use it to obtain that of x_{n+h}. This implementation is simple if the underlying process of the original observations is $ARIMA(p, 1, q)$ because $\{y_t\}$ is then both stationary and invertible, but poses a problem if the underlying process is $ARMA(p,q)$. In the latter case, the differenced series is non-invertible since $\alpha(B)y_t = (1-B)\theta(B)\epsilon_t$. Alonso *et al.* (2003) and Bühlman (1997) both required that the moving average polynomial has no roots on or inside the unit circle so that the process is invertible. This was a key assumption in their sieve bootstrap procedure and was needed in order to approximate the time series by a sequence of AR polynomials.

Poskitt (2006, 2008) discussed ways of relaxing this condition while maintaining the statistical viability of finite order autoregressive approximations to non-invertible processes. Results in Poskitt (2006, 2008) provide a theoretical foundation on which sieve bootstrap based prediction intervals can be derived, irrespective of whether the differenced series is invertible or not. In the following sections we show how the method proposed by Alonso et *al*. (2003) can be modified, based on insights from Poskitt (2006, 2008), to obtain sieve bootstrap prediction intervals for a non-invertible process. Mukhopadhyay and Samaranayake (2010) provide additional refinements to the original method. We employ the 2003 paper by Alonso et *al*. which set the fundamental theoretical framework for the application of the sieve bootstrap for invertible processes, as the platform for our proposed modifications even though Alonso et *al*. (2004) and Mukhopadhyay and Samaranayake (2010) give further refinements to the original method. As the Monte Carlo simulation results in Section 4 show, the proposed method provide good finite sample coverage even without additional refinements adopted in the above two papers. Thus, the proposed method can be taken as good initial step in adopting the sieve bootstrap to obtain prediction intervals for *ARIMA* processes.

The rest of this paper is organized as follows. Section 2 introduces the sieve bootstrap procedure for obtaining prediction intervals and Section 3 establishes asymptotic validity of the proposed method. The simulation study along with an application is presented in Sections 4 and 5.

2. The proposed sieve bootstrap procedure

The main difference between the sieve bootstrap procedure given below and the procedure introduced by Alonso et *al*. (2002, 2003) is the criterion used in selecting the order of the autoregressive approximation. This change in the order, together with Poskitt's AR approximation to non-invertible processes, are sufficient to establish the convergence results. Unlike Panichkitkosolkul and Niwitpong (2011, 2012), the proposed procedure does not depend on a unit root test. We introduce a differencing step at the beginning of the procedure in order to accommodate *ARIMA* processes.

Assume that a realization $\{x_t\}_{t=1}^n$ is obtained from $ARIMA(p,d,q)$ process given in Equation (1) with $d = 1$ or 0. Define the differenced series, $\{y_t\}$, using $y_t = x_t - x_{t-1}$.

1. Select the order $p = p(n)$ of the autoregressive approximation from among models with $p \in \{1, 2, ..., M_n\}$ with $M_n = o\{[log(n)/n]^{1/2}\}$ by the AIC criterion. Alonso et *al*. (2003) preferred AICC over AIC and used $M_n = o\{[log(n)/n]^{1/4}\}$. This change in order is required to satisfy condition needed to utilize results of Poskitt (2006, 2008).

2. Estimate the autoregressive coefficients, $\hat{\phi}_{1,p,n}, ..., \hat{\phi}_{p,p,n}$, of the $AR(p)$ approximation, $\sum_{j=0}^{p} \phi_{j,p} y_{t-j} = \epsilon_{t,p}$, by the Yule-Walker method.

3. Obtain the $(n-p)$ residuals: $\hat{\epsilon}_{t,n} = \sum_{j=0}^{p} \hat{\phi}_{j,p,n}(y_{t-j} - \bar{y})$, $t = p+1, ..., n$ and define the empirical distribution function of the centered residuals, $\tilde{\epsilon}_t = \hat{\epsilon}_{t,n} - \hat{\epsilon}^{(\cdot)}$, where $\hat{\epsilon}^{(\cdot)} = (n-p)^{-1} \sum_{t=p+1}^{n} \hat{\epsilon}_{t,n}$, by $\hat{F}_{\tilde{\epsilon},n}(x) = (n-p)^{-1} \sum_{t=p+1}^{n} I_{[\tilde{\epsilon}_t \leq x]}$.

4. Draw a resample $\epsilon^*_{t,n}$, $t = p+1, ..., n$ of i.i.d. observations from $\hat{F}_{\tilde{\epsilon},n}$.

5. Obtain y_t^* by the recursion: $\sum_{j=0}^{p} \hat{\phi}_{j,p,n}(y_{t-j}^* - \bar{y}) = \epsilon_{t,n}^*$ for $t = p+1, ..., n$ and set $y_t^* = \bar{y}$ for $t = 1, ..., p$.

6. Compute the estimates $\hat{\phi}_{1,p,n}^*, ..., \hat{\phi}_{p,p,n}^*$ as in Step 2, using $\{y_t^*\}_{t=1}^n$.

7. For $h > 0$, compute the future bootstrap observations of the differenced series by the recursion: $y_{n+h}^* - \bar{y} = \sum_{j=1}^{p} \hat{\phi}_{j,p,n}^*(y_{n+h-j}^* - \bar{y}) + \epsilon_{n+h,n}^*$ where, $y_t^* = y_t, t \leq n$.

   Up to this point we have followed Alonso *et al.* (2003) sieve bootstrap procedure but the next step is crucial to obtaining bootstrap future observation of the original time series $\{x_t\}$.

8. Compute the future bootstrap observations of the original series by the recursion: $x_{n+h}^* = x_{n+h-1}^* + y_{n+h}^*$ where, $x_t^* = x_t, t \leq n$, $h > 0$.

9. Obtain a Monte Carlo estimate of the bootstrapped distribution function of x_{n+h}^* by repeating steps 4-8 *B* times.

10. Use the bootstrapped distribution to approximate the unknown distribution of x_{n+h} given the observed sample.

11. The $100(1-\alpha)\%$ prediction interval for x_{n+h} is given by $\{Q^*(\frac{\alpha}{2}), Q^*(1-\frac{\alpha}{2})\}$ where, $Q^*(.)$ are the quantiles of the estimated bootstrap distribution.

3. Asymptotic results

Note that if the original process $\{x_t\}$ is indeed an $ARIMA(p, 1, q)$ process, then the differenced process $\{y_t\}$ is $ARMA(p, q)$ and the results of Alonso et *al*. (2003) applies directly to the bootstrap distribution of y_{n+h}^*. It then follows by simple arguments that the bootstrap distribution of x_{n+h}^* converges to that of x_{n+h}. On the other hand, complications arise if $\{x_t\}$ has no unit root. Then $\{y_t\}$ would not be invertible and hence the results of Alonso et *al*. (2003) do not apply. This is where the new order for M_n (Step 1) and results of Poskitt (2006, 2008) come into play. This approach avoids the need to pre-test for unit roots and then select the prediction interval procedure based on the outcome of the test.

In order to establish the asymptotic validity of the sieve bootstrap intervals, Alonso (2003) first established the convergence of $\hat{\boldsymbol{\phi}}_{p,n}^*$ to $\hat{\boldsymbol{\phi}}_{p,n}$. We follow the same approach, but modify the proofs to accommodate the changes arising out of the possibility that the differenced series is non-invertible. We first establish asymptotic properties of the differenced series $\{y_t\}$ and then move onto proving results for $\{x_t\}$.

Rupasinghe and Samaranayake (2012) extended some of the results in Bühlmann (1995, 1997) and Alonso et *al*. (2003) to *regular processes*, a general class of linear processes that includes both *FARIMA* and non-invertible time series. As stated in Poskitt (2006), the process $\{y_t\}_{t\in\mathbb{Z}}$ is said to be linearly regular if $\{y_t\}_{t\in\mathbb{Z}}$ is covariance stationary with,

$$y_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \tag{2}$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$, is a zero mean white noise process with finite variance σ^2 and the impulse response coefficients $\{\psi_j\}_{j=0}^{\infty}$ satisfy the condition $\psi_0 = 1$ and $\sum_{j\geq 0} \psi_j^2 < \infty$.

In the following derivations, we will use the $AR(p)$ approximation suggested by Poskitt (2006) for such time series.

Definition 1. *Let $\{y_t\}$ satisfy equation (2) and define, for $p < n$, $\{\epsilon_{t,p}\}$ such that $\sum_{j=0}^{p} \phi_{j,p} y_{t-j} = \epsilon_{t,p}$, where the AR coefficients vector $\boldsymbol{\phi}_p = (\phi_{1,p}, ..., \phi_{p,p})'$ is obtained using the Yule-Walker equations, $\boldsymbol{\Gamma}_p \boldsymbol{\phi}_p = -\boldsymbol{\gamma}_p$, $\boldsymbol{\gamma}_p = (\gamma(1), ..., \gamma(p))'$, $\boldsymbol{\Gamma}_p = [\gamma(i-j)]_{i,j=1}^{p}$, with $\gamma(k) = E[y_t y_{t+k}]$ for $k \in \mathbb{N}_0$.*

This definition provides us the AR approximation to $\{y_t\}$ even in the case where the series is not invertible and thus cannot be written as an infinite autoregressive process. It should be noted that Lemma 1 of Poskitt (2006) establishes that $\epsilon_{t,p} \to \epsilon_t$ in mean square as $p \to \infty$.

The following sets of assumptions are required in order to prove our asymptotic results.

A1: Let ξ_t denote the σ-algebra of events determined by ϵ_s, $s \leq t$. Also, assume ϵ_t is $i.i.d.$ and that

$$E[\epsilon_t|\xi_{t-1}] = 0 \qquad and \qquad E[\epsilon_t^2|\xi_{t-1}] = \sigma^2, t \in \mathbb{Z}.$$

Furthermore, assume $E[\epsilon_t^4] < \infty$ for $t \in \mathbb{Z}$.

A2: The series y_t is a linearly regular covariance-stationary process with Wold representation $y_t = \sum_{j\geq 0} \psi_j \epsilon_{t-j}$ with $\sum_{j\geq 0} |\psi_j^2| < \infty$.

B: Let $p(n) = o\{[n/log(n)]^{1/2}\}$ and $\hat{\boldsymbol{\phi}}_{p,n} = (\hat{\phi}_{1,p,n}, ..., \hat{\phi}_{p,p,n})'$ satisfy the empirical Yule-Walker equations $\hat{\boldsymbol{\Gamma}}_{p,n} \hat{\boldsymbol{\phi}}_{p,n} = -\hat{\boldsymbol{\gamma}}_{p,n}$, where
$\hat{\boldsymbol{\Gamma}}_{p,n} = [\hat{R}(i-j)]_{i,j=1}^{p}, \hat{\boldsymbol{\gamma}}_{p,n} = (\hat{R}(1), ..., \hat{R}(p))'$, and $\hat{R}(j) = n^{-1} \sum_{t=1}^{n-|j|} (y_t - \bar{y})(y_{t+|j|} - \bar{y})$ for $|j| < n$.

Assumptions in A1 imposes a Martingale difference structure on the innovations. In Proposition 1 we assume $i.i.d.$ innovations for the underlying processes because the sieve bootstrap scheme draws resamples independently and identically, it is unable to capture the correlation structure of the innovations if they are correlated.

The order of p in Assumption B is slightly different from that of Rupasinghe and Samaranayake (2012). They assumed that $p(n) = o\{[n/log(n)]^{1/2-d}\}$, where d is the difference parameter taking fractional values from -0.5 to 0.5. The value for d is set to zero throughout since we are only interested in $ARIMA(p, d, q)$ models with $d = 0$ or 1 and differencing removes the unit root, if present.

Next we present asymptotic properties of the sieve bootstrap method given in Section 2 by adopting some results form Rupasinghe and Samaranayake (2012).

The following results follows from the same arguments use in Lemmas 1, 2 and 3 of Rupasinghe and Samaranayake (2012). Therefore, they are stated without proof.

Lemma 1. *Assume that A1, A2 and B hold. Then,*

$$\textstyle\sum_{j=0}^{p} (\hat{\phi}_{j,p,n} - \phi_{j,p})^2 = o_{a.s.}\{[log(n)/n]^{1/2}\},$$

where $\phi_{j,p}$, $j = 1, 2, ..., p$, $p < n$ are the coefficients given in Definition 1.

Lemma 2. *Assume that A1, A2 and B hold. Then, for any fixed $t \in \mathbb{Z}$,*

$$E^*(\epsilon_{t,n}^{*2}) = E(\epsilon_t^2) + o_p(1).$$

The next Lemma states asymptotic convergence of bootstrap innovations to theoretical innovations, and is similar to Lemma 5.4 in Buhlmann (1997).

Lemma 3. *Assume that assumptions given in A1, A2 and B hold. Then, for each fixed* $t \in \mathbb{N}$*,*

$$\epsilon^*_{t,n} \xrightarrow{d^*} \epsilon_t, \textit{ in probability.}$$

The following proposition is analogous to Proposition 1 of Alonso et *al*. (2003) and shows that the bootstrap autoregressive coefficients obtained in Step 6 converge to the autoregressive coefficients of the fitted model obtained in Step 2.

Proposition 1. *Assume A1, A2 and B hold. Then,*

$$\max_{1 \le j \le p(n)} |\hat{\phi}^*_{j,p,n} - \hat{\phi}_{j,p,n}| \xrightarrow{P^*} 0, \textit{ in probability}. \tag{3}$$

The $AR(p)$ approximation described in Definition 1 is used to establish the convergence of future bootstrap values of the differenced series. It is, therefore, essential to show the convergence of bootstrap innovations, $\epsilon^*_{t,n}$, to the approximated errors, $\epsilon_{t,p}$. This is a strategic feature proposed to overcome issues raised in generalizing Alonso *et al.* (2003) results for regular processes.

Lemma 4. *Assume that assumptions given in A1, A2 and B hold. Then, for each fixed* $t \in \mathbb{N}$*,*

$$\epsilon^*_{t,n} \xrightarrow{d^*} \epsilon_{t,p}, \textit{ in probability.}$$

Proof. Let $F_{\epsilon,n}(x) = (n-p)^{-1} \sum_{t=p+1}^{n} 1_{[\epsilon_{t,p} \le x]}$, $F_{\epsilon,p}(x) = \mathbb{P}[\epsilon_{t,p} \le x]$ for $x \in \mathbb{R}$, and denote the Mallows metric by $d_2(.,.)$. Then, from standard results it follows that $d_2(F_{\epsilon,n}, F_{\epsilon,p}) = o_{a.s.}(1)$. Thus we need to only show that $d_2(\hat{F}_{\bar{\epsilon},n}, F_{\epsilon,n}) = o_p(1)$. Let S be uniformly distributed on $\{p+1, ..., n\}$ and let $Z_1 = \epsilon_S$, $Z_2 = \bar{\epsilon}_S$, where $\bar{\epsilon}_{t,n} = \hat{\epsilon}_{t,n} - \hat{\epsilon}^{(.)}_n$. Then, $d_2(\hat{F}_{\bar{\epsilon},n}, F_{\epsilon,n})^2 \le E|Z_1 - Z_2|^2 = (n-p)^{-1} \sum_{t=p+1}^{n} (\bar{\epsilon}_{t,n} - \epsilon_{t,p})^2 = (n-p)^{-1} \sum_{t=p+1}^{n} (\hat{\epsilon}_{t,n} - \hat{\epsilon}^{(.)}_n - \epsilon_{t,p})^2$. From the proof of Lemma 2 in Rupasinghe and Samaranayake (2012), $\hat{\epsilon}^{(.)}_n = o_p(1)$ and $(n-p)^{-1} \sum_{t=p+1}^{n} |\hat{\epsilon}_{t,n} - \epsilon_{t,p}| = o_p(1)$. Hence $d_2(\hat{F}_{\epsilon,n}, F_{\epsilon,n}) = o_p(1)$. □

Now we establish the convergence of the bootstrap differenced series.

Theorem 1. *Assume that A1, A2 and B hold. Then, in probability, as* $n \to \infty$*,*

$$y^*_{n+h} \xrightarrow{d^*} y_{n+h}, \textit{ for fixed } h \in \mathbb{N} \tag{4}$$

Proof.

$$\text{Observe that, } y_{n+h} = -\sum_{j=1}^{p} \phi_{j,p} y_{n+h-j} + \epsilon_{n+h,p} \tag{5}$$

$$\text{and } y^*_{n+h} = -\sum_{j=1}^{p} \hat{\phi}_{j,p,n} y^*_{n+h-j} + \epsilon^*_{n+h,n}, \tag{6}$$

where $y^*_t = y_t$ for $t \leq n$. For beverity, we prove the theorem for $h = 1$.
From Lemma 4, $\epsilon^*_{n+1,n} \xrightarrow{d^*} \epsilon_{n+1,p}$ and thus we need only to show that the difference of the first terms on the right hand side of (5) and (6) converges to zero in probability. Therefore consider,

$$-\sum_{j=1}^{p}(\hat{\phi}_{j,p,n} - \phi_{j,p})y_{n+1-j} \leq \left(\sum_{j=1}^{p}(\hat{\phi}_{j,p,n} - \phi_{j,p})^2\right)^{1/2} \left(\sum_{j=1}^{p(n)} y^2_{n+1-j}\right)^{1/2}$$
$$= \{o_{a.s.}[[log(n)/n]^{\frac{1/2}{2}}]\}\{O_p[p^{1/2}]\} = o_p(1).$$

Thus, $y^*_{n+1} \xrightarrow{d^*} y_{n+1}$, in probability. □

Finally, we establish the large sample validity of sieve bootstrap prediction intervals for $ARIMA(p, d, q)$ processes with $d = 0$ or 1 by proving the convergence of the future bootstrap values of the original time series, obtained in Step 8.

Theorem 2. *Assume that A1, A2 and B hold. Then, in probability, as $n \to \infty$,*

$$x^*_{n+h} \xrightarrow{d^*} x_{n+h},\ \textit{for}\ h = 0, 1, ... \tag{7}$$

Proof. The future values of the originally observed time series, $\{x_{n+h}\}$ can be written as $x_{n+h} = x_{n+h-1} + y_{n+h}$. Then the bootstrap one-step ahead value exhibit the following property: $x^*_{n+1} = x_n + y^*_{n+1} \xrightarrow{d^*} x_n + y_{n+1} = x_{n+1}$. For $h > 1$, the result can be proven using the mathematical induction.

□

4. Simulation Study

In order to investigate the finite sample performances of the method proposed in this paper, a Monte-Carlo simulation study, using a series of models given in Table 1, was carried out with three different error distributions and sample sizes 100 and 200. The coverage, bootstrap length, and the length of the interval theoretically achievable under known order and parameter values were computed for 95% and 99% prediction intervals to asses the performance of the proposed method. Results are reported in Tables 2 through 7.

Table 1: Models considered in the simulation study

Nomenclature	Model	AR roots	MA roots
$M1$	$(1 - 0.75B + 0.5B^2)X_t = \epsilon_t$	1.414, 1.414	-
$IM1$	$(1 - 0.75B + 0.5B^2)(1 - B)X_t = \epsilon_t$	1, 1.414, 1.414	-
$M2$	$X_t = (1 - 0.9B)\epsilon_t$	-	$1.\bar{1}$
$IM2$	$(1 - B)X_t = (1 - 0.9B)\epsilon_t$	1	$1.\bar{1}$
$M3$	$X_t = (1 - 0.3B + 0.7B^2)\epsilon_t$	-	1.195, 1.195
$IM3$	$X_t = (1 - 0.3B + 0.7B^2)\epsilon_t$	1	1.195, 1.195
$M4$	$(1 - 0.7B)X_t = (1 - 0.3B)\epsilon_t$	1.428	$3.\bar{3}$
$IM4$	$(1 - 0.7B)(1 - B)X_t = (1 - 0.3B)\epsilon_t$	1, 1.428	$3.\bar{3}$
$M5$	$(1 - 0.95B)X_t = (1 - 0.3B)\epsilon_t$	1.05	$3.\bar{3}$
$IM5$	$(1 - 0.7B)(1 - B)X_t = (1 - 0.3B)\epsilon_t$	1, 1.05	$3.\bar{3}$

Note that the models employed in the study are the same $ARMA$ models studied by Mukhopadhyay and Samaranakaye (2010) and Alonso *et al.* (2004). We also considered corresponding $ARIMA$ models (with labels beginning with I) since we are interested in unit root processes. The standard normal distribution, t-distribution with 3 degrees of freedom, and exponential (1) distribution centered at zero, were considered for error distributions. Prediction intervals for leads $h = 1, 2, 3$ were computed. The Matlab (Version 2011a) software was used for these simulations.

For each combination of model, sample size, nominal coverage and error distribution, $N = 1,000$ independent series were generated and for each of these simulated series, steps 1 to 15 were implemented. To compute the coverage probabilities for each of this N simulations, $R = 1,000$ future observations (x_{n+h}) were generated using the original model. The proportion of those falling in between the lower and upper bounds of the bootstrap prediction interval was then defined to be the coverage. Thus, the coverage at the i^{th} simulation run is given by $C(i) = R^{-1}\sum_{r=1}^{R} I_A[x_{n+h}^r(i)]$ where $A = [Q^*(\alpha/2), Q^*(1-\alpha/2)]$, $I_A(.)$ is the indicator function of the set A and $x_{n+h}^r(i)$, $r = 1, 2, ...1,000$ are the R future values generated at the i^{th} simulation run.

The bootstrap length and theoretical length for the i^{th} simulation run are given by $L_B(i) = Q^*(1-\alpha/2) - Q^*(\alpha/2)$ and $L_T(i) = x_{n+h}^r(1-\alpha/2) - x_{n+h}^r(\alpha/2)$ respectively. The theoretical length $L_T(i)$ is the difference between the $100(1-\alpha/2)^{th}$ and $100(\alpha/2)^{th}$ percentile points the empirical distribution of the 1,000 future observations that were generated using the underlying time series model with known order and the true values of the coefficients. Using these statistics, the mean coverage, mean length of bootstrap prediction intervals, mean length of theoretical intervals, and their standard errors were computed as:

Mean Coverage $\bar{C} = N^{-1}\sum_{i=1}^{N} C(i)$

Standard Error of Mean Coverage $SE_{\bar{C}} = \{[N(N-1)]^{-1}\sum_{i=1}^{N}[C(i)-\bar{C}]^2\}^{1/2}$

Mean Length (bootstrap) $\bar{L}_B = N^{-1}\sum_{i=1}^{N} L_B(i)$

Standard Error of Mean Length $SE_{\bar{L}_B} = \{[N(N-1)]^{-1}\sum_{i=1}^{N}[L_B(i)-\bar{L}_B]^2\}^{1/2}$

Mean theoretical Length $\bar{L}_B = N^{-1}\sum_{i=1}^{N} L_B(i)$

In total 120 different combinations of model type, sample size, nominal coverage probability, and error distributions were investigated in this simulation study. However, due to space limitations, we report only a representative sample of results for 95% intervals, in Table 2 through 7. These tables report the mean coverage, mean interval length, and mean theoretical length, standard error of mean coverage and standard error of mean interval length. The complete results of the simulation study are available upon request from the corresponding author.

To investigate the behaviour of the intervals for each of the 120 combinations, the minimum value, percentiles (25th, 50th, and 75th), and the maximum value of (a) the coverage probabilities, (b) the bootstrap interval bounds (upper and lower), and (c) the theoretical interval bounds (upper and lower), were further computed, based on the 1,000 values generated through simulation, and these statistics are also available upon request.

From Tables 2-7, we can see that the mean coverages of the proposed sieve bootstrap method are very close to the nominal coverage for all the leads regardless of presence or absence of a unit root and of the nature their error distribution. Also, it is seen that the mean bootstrap interval lengths are close to the theoretical lengths.

It is interesting to note (Tables 4,6 and 7) how the proposed sieve bootstrap procedure performs for models $M5$ and $IM5$ in which the AR root is close to unity. In practice, many parametric and nonparametric prediction intervals produce very liberal coverages when the AR polynomial has a root close to unity (see Alonso *et al.* (2002, 2004)). However, from Tables 4, 6 and 7, we can see that our proposed method is capable of producing accurate prediction intervals for time series with an AR root close to one.

Table 2: Coverage of 95% intervals for Models $M1$ & $IM1$ with normal errors

Leads	Size	Model $M1$			Model $IM1$		
		Theo. Length	Coverage Mean (SE)	Length Mean (SE)	Theo. Length	Coverage Mean (SE)	Length Mean (SE)
1	100	3.9040	0.9548 (0.0026)	4.2994 (0.0444)	3.9339	0.9561 (0.0029)	4.4168 (0.0492)
	200	3.9178	0.9503 (0.0024)	4.2013 (0.0298)	3.9153	0.9598 (0.0019)	4.3081 (0.0403)
2	100	6.7753	0.9500 (0.0031)	7.5408 (0.0901)	10.3208	0.9464 (0.0034)	11.1515 (0.1263)
	200	6.7719	0.9484 (0.0031)	7.4664 (0.0619)	10.2422	0.9539 (0.0023)	10.9454 (0.1049)
3	100	8.9874	0.9456 (0.0041)	10.1540 (0.1308)	18.5608	0.9365 (0.0040)	19.5478 (0.2382)
	200	8.9473	0.9476 (0.0036)	10.1857 (0.0961)	18.3911	0.9491 (0.0027)	19.3903 (0.1969)

Table 3: Coverage of 95% intervals for Models $M4$ & $IM4$ with normal errors

Leads	Size	Model $M4$			Model $IM4$		
		Theo. Length	Coverage Mean (SE)	Length Mean (SE)	Theo. Length	Coverage Mean (SE)	Length Mean (SE)
1	100	3.8980	0.9463 (0.0031)	4.2205 (0.0275)	3.9305	0.9545 (0.0020)	4.1908 (0.0380)
	200	3.9102	0.9438 (0.0037)	4.2945 (0.0462)	3.9267	0.9579 (0.0012)	4.1649 (0.0218)
2	100	4.2136	0.9425 (0.0045)	4.6845 (0.0329)	6.7384	0.9528 (0.0023)	7.2182 (0.0729)
	200	4.2094	0.9512 (0.0043)	4.9479 (0.0550)	6.7303	0.9573 (0.0015)	7.1610 (0.0472)
3	100	4.3746	0.9400 (0.0063)	5.0388 (0.0374)	9.4048	0.9492 (0.0028)	9.9920 (0.1182)
	200	4.3675	0.9523 (0.0044)	5.3092 (0.0704)	9.4416	0.9568 (0.0018)	10.0501 (0.0803)

Table 4: Coverage of 95% intervals for Models $M5$ & $IM5$ with normal errors

Leads	Size	Model $M5$			Model $IM5$		
		Theo. Length	Coverage Mean (SE)	Length Mean (SE)	Theo. Length	Coverage Mean (SE)	Length Mean (SE)
1	100	3.9134	0.9396 (0.0029)	3.9704 (0.0369)	3.9281	0.9439 (0.0026)	4.0540 (0.0368)
	200	3.9051	0.9466 (0.0021)	4.0106 (0.0313)	3.9068	0.9481 (0.0020)	3.9901 (0.0293)
2	100	4.6468	0.9431 (0.0029)	4.8376 (0.0469)	7.5649	0.9485 (0.0024)	8.1119 (0.0854)
	200	4.6862	0.9465 (0.0021)	4.8470 (0.0378)	7.5790	0.9525 (0.0019)	7.9298 (0.0573)
3	100	5.2358	0.9468 (0.0032)	5.6353 (0.0595)	11.6686	0.9480 (0.0031)	12.7837 (0.1745)
	200	5.2679	0.9488 (0.0025)	5.5933 (0.0537)	11.6951	0.9523 (0.0022)	12.3978 (0.1048)

Table 5: Coverage of 95% intervals for Models $M4$ & $IM4$ with exponential errors

Leads	Size	Model $M4$			Model $IM4$		
		Theo. Length	Coverage Mean (SE)	Length Mean (SE)	Theo. Length	Coverage Mean (SE)	Length Mean (SE)
1	100	3.6257	0.9522 (0.0081)	4.2604 (0.0626)	3.6652	0.9583 (0.0045)	3.8065 (0.0502)
	200	3.6535	0.9610 (0.0055)	4.2893 (0.0743)	3.6562	0.9694 (0.0020)	4.1766 (0.0766)
2	100	4.0041	0.9499 (0.0075)	4.7753 (0.0665)	6.4765	0.9520 (0.0032)	6.6051 (0.0841)
	200	4.0081	0.9617 (0.0038)	4.8360 (0.0813)	6.4841	0.9662 (0.0025)	7.1918 (0.1256)
3	100	4.2433	0.9498 (0.0080)	5.1285 (0.0688)	9.1766	0.9468 (0.0031)	9.2601 (0.1174)
	200	4.2318	0.9553 (0.0059)	5.1947 (0.0974)	9.1801	0.9592 (0.0032)	10.0295 (0.1871)

Table 6: Coverage of 95% intervals for Models $M5$ & $IM5$ with exponential errors

Leads	Size	Model $M5$			Model $IM5$		
		Theo. Length	Coverage Mean (SE)	Length Mean (SE)	Theo. Length	Coverage Mean (SE)	Length Mean (SE)
1	100	3.6596	0.9637 (0.0041)	4.0219 (0.0525)	3.6823	0.9520 (0.0048)	4.0145 (0.0725)
	200	3.6978	0.9613 (0.0036)	4.0387 (0.0723)	3.6831	0.9570 (0.0043)	3.8241 (0.0558)
2	100	4.4705	0.9481 (0.0051)	4.8064 (0.0528)	7.2723	0.9479 (0.0044)	8.0177 (0.1331)
	200	4.4917	0.9493 (0.0041)	4.8302 (0.0790)	7.2880	0.9510 (0.0034)	7.4932 (0.0941)
3	100	5.1322	0.9463 (0.0052)	5.5740 (0.0641)	11.3299	0.9462 (0.0046)	12.6384 (0.2140)
	200	5.1352	0.9503 (0.0044)	5.6875 (0.0920)	11.3583	0.9497 (0.0029)	11.7789 (0.1525)

Table 7: Coverage of 95% intervals for $M5$ & $IM5$ with t-dist errors

Leads	Size	$M5$			$IM5$		
		Theo. Length	Coverage Mean (SE)	Length Mean (SE)	Theo. Length	Coverage Mean (SE)	Length Mean (SE)
1	100	6.3676	0.9367 (0.0029)	6.3982 (0.1329)	6.3954	0.9442 (0.0026)	6.8128 (0.1451)
	200	6.3828	0.9428 (0.0022)	6.5108 (0.0989)	6.3827	0.9445 (0.0019)	6.4212 (0.0884)
2	100	7.7655	0.9379 (0.0031)	8.2689 (0.1965)	12.5904	0.9470 (0.0026)	13.9975 (0.3207)
	200	7.7979	0.9441 (0.0022)	8.1410 (0.1388)	12.5435	0.9446 (0.0020)	12.7702 (0.1609)
3	100	8.7704	0.9405 (0.0034)	9.8278 (0.3086)	19.5889	0.9450 (0.0032)	22.6252 (0.6393)
	200	8.8182	0.9443 (0.0026)	9.4723 (0.1757)	19.3835	0.9442 (0.0022)	20.1380 (0.3077)

5. Application to a real data set

The proposed sieve bootstrap method was applied to the daily highest Yahoo stock prices from February 4, 2009 to March 31, 2011; 544 observations in total. The data set can be found at `http://finance.yahoo.com/q/hp?s=YHOO`. The time series is displayed in Figure 1 and clearly exhibits a unit root behavior. The first 535 observations were used to compute 95% prediction intervals for the next consecutive 10 days using the proposed method. The dashed lines in Figure 2 show the upper and lower bounds of the computed prediction intervals. The sieve bootstrap method was able to capture the true future values of this empirical time series accurately confirming the results in the simulation study.

Figure 1: Daily Highest Yahoo Stock Prices

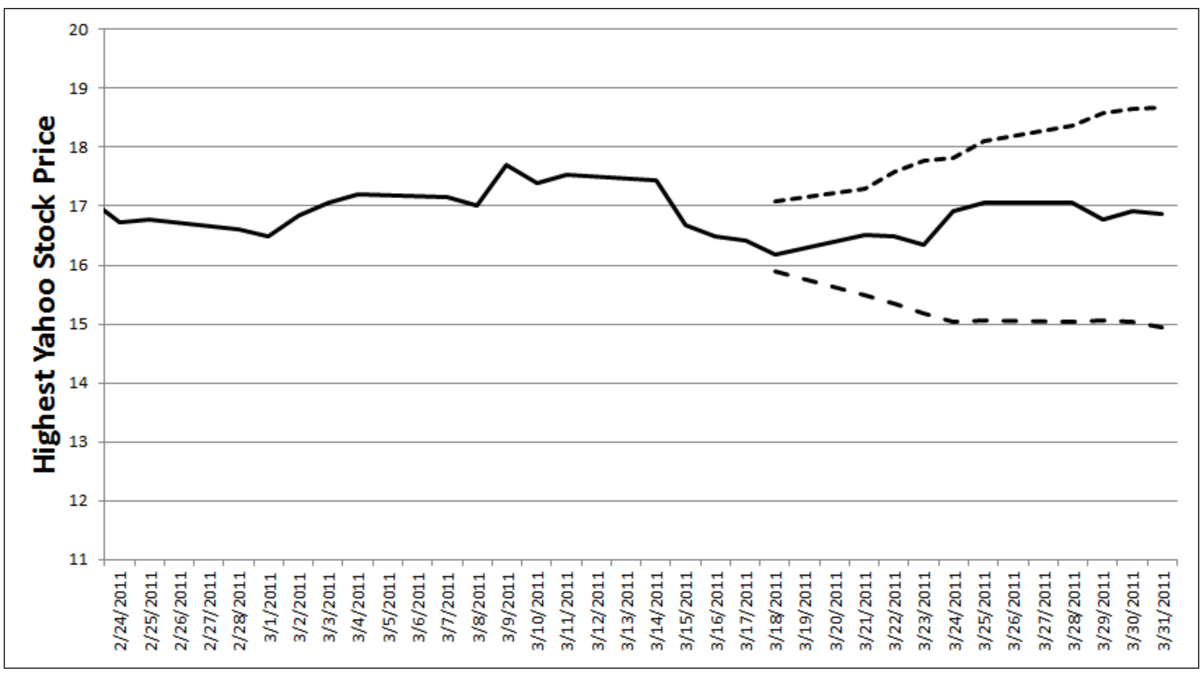

Figure 2: 95% SB Prediction Bands for Yahoo Stock Prices in dashed lines; Only a segment of Figure 1 is displayed

6. Conclusion

In this paper, we proposed a sieve bootstrap based prediction intervals for unit root ($ARIMA$) processes that provides proper coverage without altering the computational steps based on the results of a unit root test. Large sample properties are established for the proposed method and a Monte-Carlo simulation study was carried out determine finite sample properties. The simulation results indicates that the procedure works very well under normal, exponential and t distributed errors. Most importantly, the method is stable even when the AR polynomial of the underlying process has a root close to unity.

REFERENCES

Alonso A. M., Pena D., Romo J. (2002), "Forecasting time series with sieve bootstrap," *Journal of Statistical Planning and Inference*, 100, 1–11.

Alonso A. M., Pena D., Romo J. (2003), "On sieve bootstrap prediction intervals," *Statistics and Probability Letters*, 65, 13–20.

Alonso A. M., Pena D., Romo J. (2004), "Introducing model uncertainty in time series bootstrap," *Statistica Sinica*, 14, 155–174.

Bühlmann P. (1995), "Moving-average representation of autoregressive approximations," *Stochastic Process. Appl.*, 60, 331-342.

Bühlmann P. (1997), "Sieve bootstrap for time series," *Bernoulli*, 3, 123–148.

Cao, R., Febrero-Bande, M., Gonzalez-Manteiga, W., Prada-Sanchez, J.M., Garcia-Jurado, I. (1997), "Saving computer time in constructing consistent bootstrap prediction intervals for autoregressive processes," *Comm. Statist. Simulation Comput.*, 26, 961–978.

Chang Y.,Park Y. (2003), "A sieve bootstrap for the test of a unit root," *Journal of Time Series Analysis*, 24, 379–400.

Dickey D.A.,Fuller W.A. (1979), "Distribution of estimators for autoregressive time series with a unit root," *Journal of the American Statistical Association*, 74, 427–431.

Kilian L. (1998(a)), "Small-sample Condence Intervals for Impulse Response Functions," *Review of Economics and Statistics*, 80(2), 218–230.

Kim J. (2001), "Bootstrap-After-Bootstrap Prediction Intervals for Autoregressive Models," *Journal of Business and Economic Statistics*, 19, 117–128.

Kreiss J.P. (1988), "Asymptotic Statistical Inference for a Class of Stochastic Processes," *Habilitationsschrift, Universiti at Hamburg*.

Kreiss J.P. (1992), "Bootstrap procedures for AR(∞)-processes In: JEockel, K.H., Rothe, G., Sendler, W. (Eds.),Bootstrapping and Related Techniques," *Springer, Heidelberg*, pp. 107–113.

Mukhopadhyay P., Samaranayake V.A, (2010), "Prediction intervals for time series: A Modified Sieve Bootstrap Approach," *Communications in Statistics - Simulation and Computation*, 39, 517–538.

Palm F.C., Smeekes S., Urbain J. (2007), "Bootstrap unit-root tests: comparison and extensions," *Journal of Time Series Analysis*, 20, 371–401.

Panichkitkosolkul W., Niwitpong S. (2011), "On Multistep-Ahead Prediction Intervals Following Unit Root Tests for a Gaussian AR(1) Process with Additive Outliers," *Applied Mathematical Sciences*, 47, 2297–2316.

Panichkitkosolkul W., Niwitpong S. (2011), "Prediction intervals for the Gaussian autoregressive processes following the unit root tests," *Model Assisted Statistics and Applications*, 7, 1–15.

Pascual, L., Romo, J., Ruiz, E. (2004), "Bootstrap predictive inference for ARIMA processes," *Journal of Time Series Analysis*, 25 , 449–465.

Poskitt D. S. (2006), "Autoregressive approximation in nonstandard situations: the fractionally integrated and non-invertible cases," *Annals of the Institute of Statistical Mathematics*, 59, 697–725

Poskitt D. S. (2008), "Properties of the Sieve bootstrap for fractionally integrated and non-invertible processes," *Journal of Time Series Analysis*, 29, 224–250.

Psaradakis Z. (2000), "Bootstrap test for unit roots in seasonal autoregressive models," *Statistics and Probability Letters*, 22, 389–395.

Psaradakis Z. (2001), "Bootstrap test for an autoregressive unit root in the presence of weakly dependent errors," *Journal of Time Series Analysis*, 22, 577–594.

Rupasinghe M., Samaranayake V.A. (2012), "Asymptomatic Properties of Sieve Bootstrap Prediction Intervals for FARIMA Processes," *Statistics and Probability Letters*, 82, 2108–2114.

Said E., Dickey D. (1984), "Testing for Unit Roots in Autoregressive Moving Average Models of Unknown Order," *Biometrika*, 71, 599-607

Shin D., Sarkar S., Lee J. (1996), "Unit root tests for time series with outliers," *Statistics & Probability Letter*, 30, 189–197.

Stine, R.A. (1987), "Estimating properties of autoregressive forecasts," *J. Amer. Statist. Assoc.* 82, 1072–1078.