When should a tree be harvested?

Anders Muszta*

Abstract
We propose a simple mathematical model for the growth of annual rings of a single generic tree, subject to random competition for resources. The model rests on two premises: The ring diameters are bounded and are increasing with time. A simple way of satisfying these premises is to model the diameter as the running maximum of a bounded stochastic differential equation, driven by a Wiener process. To obtain an explicit solution to the equation, we require it to be reducible to an Ornstein-Uhlenbeck process. This allows us to obtain a law of the iterated logarithm for the ring diameter, which can assist in deciding when the tree should be harvested.

Key Words: Tree ring diameter, growth, competition, stochastic differential equation, maximum process, Ornstein-Uhlenbeck process, Langevin process, Law of the iterated logarithm.

1. Introduction

An essential part in managing a forest is knowing when to harvest. One tool in assisting making such a decision is simulation of forest growth. A forest is made up of trees, so if we can make simulation of tree growth then we have taken one step towards simulating the forest. There are mathematical models that focus on a single tree e.g., the models studied in 1980 by Bailey [1] and those investigated 21 years later by Fox, Ades and Bi [2], and these models often result in analytically tractable results. This mathematical elegance comes at a price, namely one disregards the interactions that exist between trees; nearby trees compete for light and water. To model competition, systems of interacting single-tree models are constructed as for example in 1979 by Garcia [3] and those constructed 29 later by Qiming, Scheider and Pitchford [4]. The advantage is more realistic models, but the disadvantage is the inability of obtaining analytically tractable solutions and often having to rely on numerical approximations of the system; this carries with it questions of whether the approximations really approximate the right things and at what rate the approximations converge to the true tree growth.

This presentation describes a simple model for the growth of a single tree, where effects of environmental disturbances and interactions with other trees are a natural part of the model, and do not have to be introduced as systems of interacting trees.

2. Methods

The model we propose studies the annual rings of a single tree and rests on two premises: the ring diameter is bounded and increases. A simple way of satisfying these premises is to model the ring diameter as the running maximum of the solution to a bounded stochastic differential equation.

$$D_t = \max_{s\in[0,t]} X_s \ , \quad t > 0 \ .$$

If the solution, X, has continuous trajectories, then the corresponding trajectories of the running maximum-process will be continuous and increasing as well. If we require the

*Centre of Biostochastics, Swedish University of Agricultural Sciences, Umeå, Sweden

stochastic differential equation to be reducible, then we obtain a bijective map of the diameter dynamics to the dynamics of the running maximum of a one-dimensional Ornstein-Uhlenbeck process. This strictly monotone map allows for a law of the iterated logarithm to be obtained, which can assist in deciding when the tree should be harvested.

2.1 Stochastic differential equation

Our stochastic differential equation describes the availability of sunlight and water, incorporating a deterministic growth model (μ) in the absence of competition, and a non-deterministic model of disturbances, conceivably caused by environmental effects and competing trees.

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t \; .$$

The connection between sunlight and water and their effect on diameter growth is a very complicated one – among other things it is governed by the familiar process of photosynthesis – which we disregard in this case, simply assuming it to be a proportionality. Another thing we assume is that the tree is insensitive to random disturbances when its diameter is very small and when it is very large; this is reflected in choosing the diffusion coefficient function to have a parabolic shape.

$$\sigma(x) = \sigma_0 x(K - x) \; , \quad x \in (0, K) \; .$$

The positive constant K denotes the largest possible diameter the tree rings can obtain; it is determined by the tree species, among other things. And together with the positive constant σ_0 the product $\sigma_0 K$ is related to the rate of growth of the tree volume. The random disturbances, W, are continuous and mostly of a small magnitude; hence we model them by a Wiener process. We do not take into account sudden major disturbances, such as storms or fires, the effects of which might be modeled by a Lévy process.

To obtain an analytically tractable (strong) solution, we require the stochastic differential equation to be reducible to an Ornstein-Uhlenbeck process, Y; this effectively determines the drift coefficient function to be

$$\mu(x) = \sigma(x) \cdot \Big\{ \frac{1}{2}\sigma'(x) - \alpha c_0 + \alpha \cdot \int^x \frac{1}{\sigma(y)}\,\mathrm{d}y \Big\} \; , \quad x \in (0, K) \; ,$$

where $\alpha > 0$ and c_0 are arbitrary constants. A stochastic differential equation with these drift- and diffusion coefficient functions has a unique strong solution equal to the stochastic process

$$X_t = f(Y_t) \; , \quad t \geq 0 \; ,$$

with the bounded and strictly increasing function

$$f(y) = \frac{K}{1 + e^{-(\sigma_0 K)\cdot(y - c_0)}} \; , \quad y \in (-\infty, \infty) \; ,$$

and the stochastic process Y equal to an Ornstein-Uhlenbeck process

$$Y_t = Y_0 e^{-\alpha t} + \sigma_0 e^{-\alpha t} \int_0^t e^{\alpha s}\,\mathrm{d}W_s \; .$$

Thus we have constructed a model in which the availability of sunlight and water (X) change in a smooth[1], but random, fashion and does not exceed the upper bound K. Consequently, the tree ring diameter – being the running maximum of the sunlight-and-water process – also increases in a smooth random fashion, and does not exceed the upper bound K.

[1] As a more realistic model of the physical process of Brownian motion, the Ornstein-Uhlenbeck process is designed to have differentiable trajectories.

2.2 Comparison with data

How do trajectories of our model compare with growth curves of actual trees? Do they have qualitative features in common?

One of the most notable features of our model is that it displays periods of constancy, just like real data; this is due to taking the running maximum of a continuous process. Another feature of our model is that the rate of growth slows down when the tree becomes old; this is found in nature as well, where the cambial growth of trees slows down when the tree becomes old. Located in the tree trunk, the cambium is a layer of cells that continually differentiates into short-lived xylem cells and long-lived phloem cells. The xylem cells transport minerals and water from the roots upward in the tree, and when the xylem cells die they become the rings we are trying to describe with our model. The phloem cells transport sugars, created by photosynthesis in the leaves, downward in the tree.

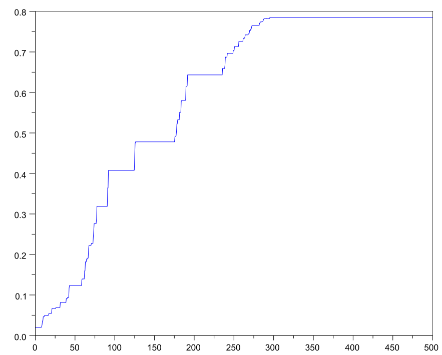

Figure 1: Simulation of the growth of a single tree, using our proposed model.Observe the periods of constancy, which can be interpreted as the tree experiencing harsh environmental conditions for growth, like competition with neighboring trees or enduring periods of drought. If the tree is competing for resources it does not grow in girth, but rather in height, until it reaches favorable circumstances for ring growth.

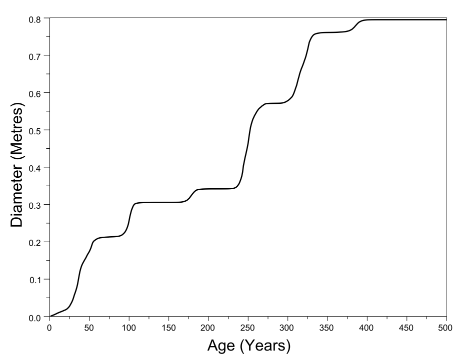

Figure 2: Part of tree ring data on a 1533 year old fig that grew by the river *Cisne* in the Los Alerces National Park in Argentina during the years 441-1974; the tree diameter became 0.8 meters.

2.3 Law of the iterated logarithm

From an economic point of view, a tree might be harvested when the relative increase of its diameter over e.g., a ten-year time period is small enough. When does this occur?

To answer this question we make use of the fact that we have a process (X) given by a strictly increasing function of the well-known Ornstein-Uhlenbeck process, and determine a Law of the iterated logarithm for the corresponding tree ring diameter D. To do this we make use of the fact that the Ornstein-Uhlenbeck process can be represented as a time-changed Wiener process, with an explicitly known strictly increasing time-change, τ.

$$Y_t = Y_0 e^{-\alpha t} + \sigma_0 e^{-\alpha t} W_{\tau(t)} \;,$$
$$\tau(t) = (2\alpha)^{-1}(e^{2\alpha t} - 1) \;, \quad t > 0 \;.$$

This will give us a deterministic growth curve around which the trajectories of D will eventually fluctuate.

The Wiener process obeys the Law of the iterated logarithm, i.e., almost all trajectories of the process have the following property: For any positive number ε, the trajectories of the Wiener process will eventually be found somewhere between the two curves

$$t \mapsto \pm(1+\varepsilon)\sqrt{2t \log\log t} \;,$$

and occasionally the trajectories will be above the curve $t \mapsto (1-\varepsilon)\sqrt{2t \log\log t}$ or below the curve $t \mapsto -(1-\varepsilon)\sqrt{2t \log\log t}$. Or, more succinctly $P\{\limsup_{t\to\infty} \frac{|W_t|}{L(t)} = 1\} = 1$, where we introduce the function

$$L(t) = \sqrt{2t \log\log t} \;, \quad t \in (3, \infty) \;.$$

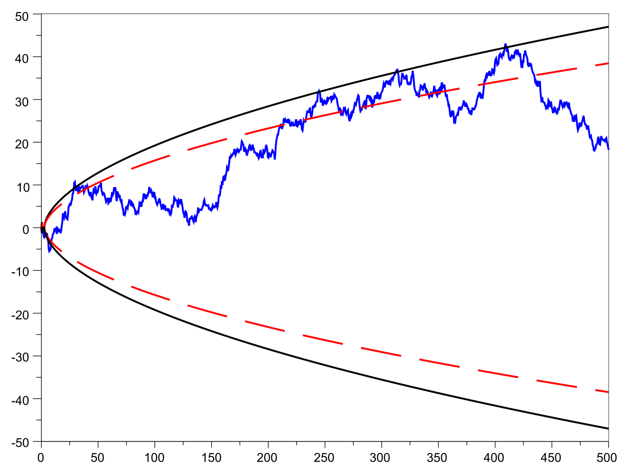

Figure 3: Almost all trajectories of a Wiener process will eventually be found somewhere between the solid curves, and occasionally those same trajectories will be found above the upper dashed curve or below the lower dashed curve.

The time-changed Wiener process W_τ will obey the same law, since the time-change is non-random.

$$P\{\limsup_{t\to\infty} \frac{|W_{\tau(t)}|}{(L\circ\tau)(t)} = 1\} = 1 \;.$$

This implies that (for any positive number ε) almost all trajectories of the process $f(Y)$ will occasionally be found above the curve

$$t \mapsto f\big(Y_0 e^{-\alpha t} + \sigma_0 e^{-\alpha t}(1-\varepsilon)(L \circ \tau)(t)\big) ,$$

and this will also be true for the running maximum of this process. Thus, eventually the trajectories of the ring diameter process, D, will be above a certain curve, G, essentially determined by the diffusion coefficient function σ. To put it succinctly,

$$P\{\limsup_{t\to\infty} \frac{D_t}{G(t)} \geq 1\} = 1 ,$$

where we express the asymptotic growth curve G implicity by

$$c_0 - \frac{1}{\sigma_0 K} \log\big(\frac{K}{G(t)} - 1\big) = e^{-\alpha t}\{Y_0 + \frac{\sigma_0}{\alpha}(e^{2\alpha t} - 1) \log\log \frac{(e^{2\alpha t} - 1)}{2\alpha}\} .$$

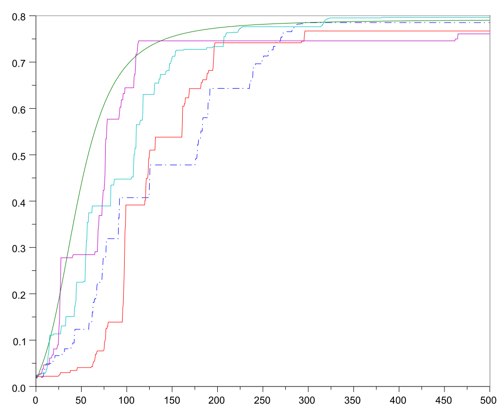

Figure 4: The asymptotic growth curve G as function of time, together with five simulated trajectories of tree ring growth, generated using the parameters $K = 0.8$, $\alpha = 0.01$, $\sigma_0 = 0.5$ and $c_0 = -3$.

When the relative increase in ring diameter is too small, then it might be a good time to harvest. We can use the asymptotic curve G to compute its corresponding relative increase over a time period, $h > 0$, and compare it with the corresponding relative increase in the data; to be of any practical use, the time period must not be too long; we choose $h = 10$ years.

$$G'_h(t) = \frac{G(t+h) - G(t)}{G(t)} , \quad t > 0 .$$

Should we declare a relative increase less than 0.3 to be too small, then the asymptotic curve in Figure 5 suggests harvest to occur after about 45 years; this agrees quite well with the data, as its relative increase goes below 0.3 for the first time after about 40 years.

3. Conclusion

Using a simple stochastic differential equation we have been able to derive a model whose qualitative behavior resembles that of annual tree ring growth. The asymptotic behavior

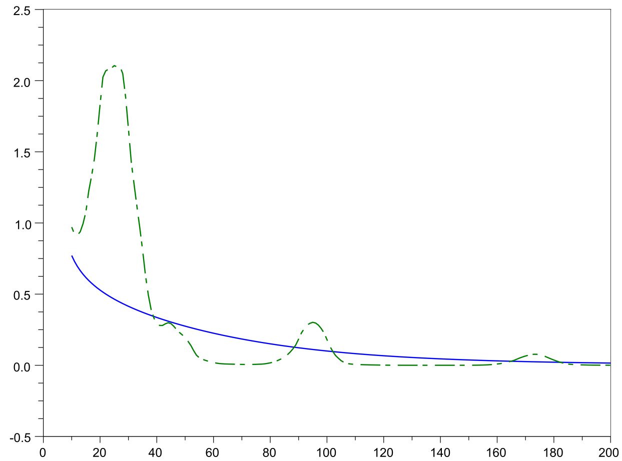

Figure 5: The solid curve represents the 10-year relative increase in ring diameter according to our model, when using the same parameters as in Figure 4. The dashed curve represents the 10-year relative increase in ring diameter of the Los Alerces fig depicted in Figure 2.

of the model was derived using the law of the iterated logarithm for a Wiener process, and using the asymptotic growth curve a corresponding relative increase in diameter growth was be computed. This relative increase can be used when deciding the best time to harvest.

References

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