A New Method for the Simultaneous Balancing of the Economic Accounting Systems at Current and Constant Prices

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Abstract

The balancing of a system of national accounts is usually carried out by balancing first the flows at current prices and then those at constant prices. However, this procedure in two distinct steps makes it impossible to control the consistency of the system of deflators. The most appropriate way to obtain a consistent system of flows at both current and constant prices is to balance them simultaneously. The main complexity in balancing accounting systems simultaneously at current and constant prices is the nonlinearity of the balancing systems. This work introduces a new simultaneous balancing method to balance huge sets of national accounting data simultaneously at current and constant prices through the definition of a procedure based on the predictor-corrector method. The latter is the distinctive element of the proposed balancing method. The main features of the balancing method introduced in this work are its very high flexibility compared to the other methods in literature, and its useful capability to allow the control of the consistency of the system of deflators. An application to the Italian economic data has been worked out, and it has yielded significant results.

Key Words: Simultaneous balancing method at current and constant prices, Statistical modelling, Very large national accounting balancing systems, National economic and social accounts.

1. Introduction

Almost all National Institutes of Statistics publish a complete set of national accounting data at current prices and only a small number of aggregates at constant prices. The reason behind this practice is the considerable complexity of obtaining balanced accounting systems simultaneously at both current and constant prices. The most relevant problem of obtaining a complete set of national accounts at constant prices is the impossibility of defining a complete and appropriate set of price indices. National accounts are, in fact, normally compiled and balanced first at current prices, and then converted to constant prices using a suitable system of deflators. It is known that, however, the accounting system at constant prices calculated in this way will be unbalanced and it is, therefore, necessary to balance it. The balancing procedure, moreover, determines modifications in the value of the deflators and the latter may not reflect the true economic dynamic of the country to which the accounts refer.

Some methods to solve the problem of balancing an accounting system simultaneously at current and constant prices are presented in the literature. They are essentially based on the balancing method originally proposed by Stone, Champernowne, and Meade (1942) and use an *ad hoc* transformation for variables, and the definition of a suitable matrix of variances and covariances to solve the nonlinearity of the accounting balancing system at both current and constant prices (Weale, 1988).

This work delineates a new method to balance an accounting system simultaneously at current and constant prices, which differs from the others in its approach to the problem. First, it is based on the method originally developed for balancing the Italian 1992 Symmetric Input-Output Table (SIOT, hereafter) (Nicolardi, 1998; 2000) that had its foundation in

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Byron (1978). The method proposed by Nicolardi is more flexible than Byron's technique and allows the balancing of extremely large accounting systems. Second, the new method defined in this work utilizes the predictor-corrector method to solve the nonlinearity of the balancing system. Finally, it allows the direct control of the magnitude of the variations of the deflators, which the balancing procedure normally causes, while in the literature the problem is managed in a roundabout way.

The paper is organized as follows. In Section 2, the methodology of balancing at current prices is presented. Section 3 introduces the new method of simultaneous balancing at current and constant prices. In Section 4, some issues of the definition and calculation of variances are examined. Section 5 demonstrates an application of the method of the simultaneous balancing. Section 6 presents some concluding remarks.

2. Methodological overview

Stone approached the balancing of an accounting system that differently includes reliable estimates of the aggregates as a problem of constrained estimates obtained as a weighted linear combination of the initial estimates of the same aggregates. By considering a vector $\mathbf{x}$ of s accounting data, it is possible to express a system of k accounting equations in the form $\mathbf{Gx} = \mathbf{h}$, where $\mathbf{G}$ is a $(k \times s)$ matrix of constraints and $\mathbf{h}$ is a vector of known values. In particular, when the accounting constraints are not satisfied $\mathbf{Gx} \neq \mathbf{h}$ and the system is unbalanced. In this latter case, starting from an initial vector of unbalanced estimates of $\mathbf{x}$, say $\hat{\mathbf{x}}$, Stone suggested obtaining a vector $\tilde{\mathbf{x}}$ that satisfies the accounting constraints in $\mathbf{G}$ through a generalized constrained estimator defined as follows:

$$\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{VG}'(\mathbf{GVG}')^{-1}(\mathbf{G}\hat{\mathbf{x}} - \mathbf{h}) \tag{1}$$

where $\mathbf{V}$ is a prior estimate of the covariance matrix of $\hat{\mathbf{x}}$ and contains the *information* on the significance level of those estimates, and permits a reasoned distribution of the accounting residuals between them. Subsequently, Byron (1978) proposed an alternative approach to estimate $\tilde{\mathbf{x}}$ that can be used to solve large econometric systems. He considered a constrained quadratic loss function defined as follows:

$$Z = \frac{1}{2}(\tilde{\mathbf{x}} - \hat{\mathbf{x}})'\mathbf{V}^{-1}(\tilde{\mathbf{x}} - \hat{\mathbf{x}}) + \lambda(\mathbf{G}\hat{\mathbf{x}} - \mathbf{h}) \tag{2}$$

in which λ is the vector of Lagrange multipliers. In (2), $\tilde{\mathbf{x}}$ has to be as close as possible, in a quadratic loss sense, to $\hat{\mathbf{x}}$ and satisfies, at the same time, the accounting constraints $\mathbf{G}\tilde{\mathbf{x}} = \mathbf{h}$. The first order conditions on (2) are:

$$\begin{aligned} \tilde{\lambda} &= (\mathbf{GVG}')^{-1}(\mathbf{G}\hat{\mathbf{x}} - \mathbf{h}) \quad (a) \\ \tilde{\mathbf{x}} &= \hat{\mathbf{x}} - \mathbf{VG}'\tilde{\lambda} \qquad\qquad\quad (b) \end{aligned} \tag{3}$$

Although (3) is equivalent to (1), it allows the estimation problem to be solved under less restrictive conditions than (1), and, at the same time, the complexity of the estimation process to be reduced so that $\tilde{\mathbf{x}}$ can be worked out through a smaller linear equation system. Furthermore, Byron used the conjugate gradient method to estimate large accounting systems by applying (3). Recalling that (3a) is a system of linear equations and the matrix $(\mathbf{GVG}')$ is symmetric positive definite and the result of the product of three sparse matrices, the same relation (3a) can be rewritten as follows:

$$(\mathbf{GVG}')\,\lambda = \theta \;\; or \;\; \mathbf{H}\,\lambda = \theta \tag{4}$$

The solution to (4) is obtained by using the following iterative procedure that is exactly the conjugate gradient algorithm:

$$\begin{aligned} \pi_0 = \rho_0 = \theta - \mathbf{H}\,\lambda_0 \\ \alpha_i = {\rho'}_i\,\rho_i / {\pi'}_i\,\mathbf{H}\,\pi_i \\ \lambda_{i+1} = \lambda_i + \alpha_i\,\pi_i \\ \rho_{i+1} = \rho_i - \alpha_i\,\mathbf{H}\,\pi_i \\ \beta_i = {\rho'}_{i+1}\,\rho_{i+1} / {\rho'}_i\,\rho_i \\ \pi_{i+1} = \rho_{i+1} + \beta_i\,\pi_i \end{aligned} \tag{5}$$

where π and ρ are the gradient-based direction vectors, λ_0 is a vector of initial values for λ, and *i* and *i+1* refer to the iteration count. The iterative process is stopped when the condition $max\,(|\theta - \mathbf{H}\,\lambda_i|) < \varepsilon$ is verified, with ϵ sufficiently small. Substituting λ_i for $\hat{\lambda}$ in (3b), $\tilde{\mathbf{x}}$ is estimated.

The convergence process can be accelerated by a suitable normalization of $(\mathbf{GVG}')$ (Byron, 1978). Ralston and Wilf (1960) proved that the conjugate gradient algorithm converges in *k* iterations, where *k* is the size of λ, and in far less *k* if the problem is scaled in an appropriate way. A suitable method to accelerate the convergence process is to normalize the matrix $(\mathbf{GVG}')$ by means of a suitable diagonal matrix $\mathbf{W}$ whose elements are given by:

$$w_{ii} = \left(\sum\nolimits_j g_{ij}\mathbf{V}_{jj}\,g_{ij}\right)^{-\frac{1}{2}}; \quad w_{ij} = 0 \tag{6}$$

when $\mathbf{V}$ is assumed to be diagonal.[1] Therefore, (4) can be rewritten as follows:

$$\mathbf{W}(\mathbf{GVG}')\mathbf{W}\,\lambda^* = \mathbf{W}\theta \tag{7}$$

where $\lambda^* = \mathbf{W^{-1}}\lambda$ and the matrix $\mathbf{W}(\mathbf{GVG}')\mathbf{W}$ has unit elements on the diagonal, while all the off-diagonal elements are less then one in absolute value.

As a consequence, the iterative procedure (5) has to be reformulated as follows (Byron, 1978):

$$\begin{aligned} \pi_{\mathbf{0}} = \rho_{\mathbf{0}} = \mathbf{W}\,\theta - \mathbf{W}\,\mathbf{H}\,\lambda_0 \\ \alpha_i = {\rho'}_i\,\rho_{\mathbf{i}} / {\pi'}_i\,\mathbf{W}\,\mathbf{H}\,\pi_i \\ \lambda_{i+1} = \lambda_i + \alpha_i\,\pi_i \\ \rho_{i+1} = \rho_i - \alpha_i\,\mathbf{W}\,\mathbf{H}\,\pi_i \\ \beta_i = {\rho'}_{i+1}\,\rho_{i+1} / {\rho'}_i\,\rho_i \\ \pi_{i+1} = \rho_{i+1} + \beta_i\,\pi_i \end{aligned} \tag{8}$$

Also applying the balancing method proposed by Byron, the problem of calculating and managing (**GVG**') remains significant. Nicolardi (1998, 2000) proposed a method of decomposing the accounting matrix into blocks to allow both a significant reduction of the quantity of data to memorize and the calculation of the product (**GVG**') without defining **G** element by element. The development of Nicolardi's method starts from the observation that in (8) (**GVG**') appears in the second and fourth relation within the following product:

$$\mathbf{WH}\pi = \mathbf{W}(\mathbf{GVG}')\,\pi \tag{9}$$

[1] The matrix **V** is normally used in a diagonal form, where the null covariances of the estimates are assumed. In practical terms, the inefficiency of the estimates, which this simplification yields, is negligible in most cases. Also, the methodologies utilized in compiling the national accounts do not usually allow the estimation of the correlations between the estimates of different flows.

In (9) it can be seen that determining the product starting with the last pair of elements and proceeding to the first, the subsequent results are always formed by vectors, that is:

$$\begin{aligned} \gamma_{(s)} &= \mathbf{G'}_{(s\times k)}\, \pi_{(k)} \\ \gamma'_{(s)} &= \mathbf{V}_{(s\times s)}\, \gamma_{(s)} \\ \gamma''_{(s)} &= \mathbf{G}_{(k\times s)}\, \gamma'_{(s)} \\ \gamma'''_{(s)} &= \mathbf{W}_{(k\times k)}\, \gamma''_{(s)} \end{aligned} \tag{10}$$

where the subscripts are the sizes of the different matrices and vectors. As **V** and **W** are diagonal matrices and can be stored in vectorial form, it can easily be seen how the quantity of data to be memorized to obtain (9) is significantly reduced.

In (10), however, the calculation of $(\mathbf{G}'\pi)$ and $(\mathbf{G}\gamma')$ remained a problem because of the very large size that **G** can reach. To resolve this problem it was observed that the accounting matrices often have a block configuration, so that it is possible to reformulate the above products through a linear combination of a suitable decomposition of π and γ' in blocks without having to determine **G** (Nicolardi, 1998; 2000). A simple balancing scheme as defined below in (11a), where a_{ij}, b_{ij} and c_{ij} are respectively the *ij*th elements of the three generic $(n\times m)$ accounting data matrices **A**, **B** and **C**, can easily be rewritten in a compact form (11b) as follows:

$$\begin{aligned} &\begin{cases} a_{ij} = b_{ij} + c_{ij} \;\forall i,j \\ \sum_j a_{ij} = \sum_j b_{ij} + \sum_j c_{ij} \;\forall i \\ \sum_i a_{ij} = \sum_i b_{ij} + \sum_i c_{ij} \;\forall j \\ \sum_{i,j} a_{ij} = \sum_{i,j} b_{ij} + \sum_{i,j} c_{ij} \end{cases} \qquad (a) \\ &\begin{cases} \mathbf{A} - \mathbf{B} - \mathbf{C} = \mathbf{0} \\ \mathbf{A}\,\mathbf{i}_m - \mathbf{B}\,\mathbf{i}_m - \mathbf{C}\,\mathbf{i}_m = \mathbf{0} \\ \mathbf{i}'_n\,\mathbf{A} - \mathbf{i}'_n\,\mathbf{B} - \mathbf{i}'_n\,\mathbf{C} = \mathbf{0} \\ \mathbf{i}'_n\mathbf{A}\,\mathbf{i}_m - \mathbf{i}'_n\,\mathbf{B}\,\mathbf{i}_m - \mathbf{i}'_n\,\mathbf{C}\,\mathbf{i}_m = \mathbf{0} \end{cases} \qquad (b) \end{aligned} \tag{11}$$

where $\mathbf{i}_n$ and $\mathbf{i}_m$ are vectors of ones of size *n* and *m* respectively.[2] In (11b) only q matrices, vectors and scalars are formed by initial data, while the remainder come from their transformations, e.g. $\mathbf{A}\,\mathbf{i}_m$. By means of the formulation (11b) it is now possible to calculate the vectors γ and γ'' in (10) without explicitly using **G**.

In particular, in order to compute the product $\mathbf{G}'\pi$ in the first of (10), π has to be decomposed into *k* blocks $\Im_i$ (i.e. matrices, vectors and scalars), where k is equal to the number of the macro-equations in (11b). Each of the *k* blocks has the same sizes as the data blocks, or their transformations, which enter each macro-equation. The decomposition of the vector π is worked out by using its own data and filling the data arrays sequentially from $\Im_1$ to $\Im_k$. The generic array $\Im_i$ has to be filled row by row when it is a matrix.[3] Through the application of the scheme (11b), $k = 4$ blocks with sizes, respectively, ($n\times m$), ($n\times 1$), ($1\times m$) and (1×1) are obtained. The solution of the above product $\mathbf{G}'\pi$ is finally given by *q* blocks $\Re_i$, where *q* is the number of the accounting data blocks in (11b), that are calculated as a linear combination of the blocks $\Im_j$, where *j* corresponds to the macro-equations in which each of the initial *q* data matrices of the scheme (11b), ordered according to the sequential position in the balancing scheme and without repetition, is present. In the linear

[2]In (11a) and (11b) the balancing equations 2-4 are already implicit in the first equation, but they are normally used explicitly so as to be able to express the constraint conditions on the marginal distributions of the accounting matrices.

[3]Filling a matrix row by row by using the data in an array can be considered the inverse operation of the matrix row vectorization.

combination, each array $\Im_j$ has the same algebraic sign of the *i*th accounting data block of ς in the *j*th equation, where ς is the ordered vector of initial matrices.

For instance, when the scheme (11b) is used, $q = 3$ since the only matrices of initial data are **A**, **B** and **C**, and ς = (**A**, **B**, **C**). If a ($q \times l_1$) matrix $\mathbf{\Lambda}$ is specified, where each row *i* contains the identification numbers of the macro-equations where the *i*th matrix in ς, or its transformations, is present, the *q* arrays $\Re_i$ can be worked out as follows:

$$\Re_i = \left({}^{el}\sum_{j=1}^{l_1} \delta_{ij}\, \Im_{\mathbf{\Lambda}_{ij}} \right) \qquad i = 1, \ldots, q$$

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \tag{12}$$

$$\begin{cases} \Re_1 = +\Im_1 + \Im_2 \mathbf{i'}_m + \mathbf{i}_n \Im_3 + \Im_4 \mathbf{i}_n \mathbf{i'}_m \\ \Re_2 = -\Im_1 - \Im_2 \mathbf{i'}_m - \mathbf{i}_n \Im_3 - \Im_4 \mathbf{i}_n \mathbf{i'}_m \\ \Re_3 = -\Im_1 - \Im_2 \mathbf{i'}_m - \mathbf{i}_n \Im_3 - \Im_4 \mathbf{i}_n \mathbf{i'}_m \end{cases}$$

where l_1 is equal to the maximum number of arrays $\Im_j$ used to calculate each $\Re_i$ in the first of (12), $\mathbf{\Lambda}_{ij}$ denotes the *ij*th element of the array $\mathbf{\Lambda}$, δ_{ij} is a term that is equal to $+1$ or -1 depending on whether the sign of $\Im_j$ is positive or negative, and the superscript *el* indicates that all the sums are computed by means of an *ad hoc* element-by-element sum because of the fact that the arrays $\Im_j$ can have different sizes and it is, therefore, not possible to apply the common rules of matrix algebra. The following are thus defined:

- the sum of a matrix **A** and a column vector **b** yields the matrix **C** = **A** + **b i**′;
- the sum of a matrix **A** and a row vector **b** yields the matrix **C** = **A** + **i b**;
- the sum of a matrix **A** and a scalar *b* yields the matrix **C** = **A** + *b* (**i i**′);
- the sum of a vector **a** and a scalar *b* yields the vector **c** = **a** + *b* **i**.

The sequential row vectorization of the *q* blocks yields γ.

To determine $\gamma^{''}$ in the third of (10), $\gamma^{'}$ has to be decomposed into *q* blocks $\Im_i$, where *q* is equal to the number of the elements in ς. Each of the arrays $\Im_i$ has the same size as the *i*th element in ς. The decomposition of the vector $\gamma^{'}$ is, thus, obtained through the same method of decomposing π. By applying the balancing scheme (11b), $q = 3$ arrays $\Im_i$ are defined, with sizes respectively equal to $(n \times m)$, $(n \times m)$, $(n \times m)$.

The solution of the product $\mathbf{G}\gamma^{'}$ is given by *k* blocks $\Re_i$, where *k* is the number of the macro-equations in (11b) and is equal to 4 in this work. Each array $\Re_i$ is obtained through the linear combination of all the arrays $\Im_j$, where the index *j* denotes the positions that the accounting data blocks, which enter the *i*th macro-equation in (11b), have in ς. Each array $\Im_j$ indispensably has the same algebraic sign of the *j*th element of ς in the *j*th macro-equation. If a ($k \times l_2$) matrix $\mathbf{\Lambda}$ is specified, where each row *i* contains the position of each *j*th data matrix in ς in each *j*th equation of (11b), the *k* arrays $\Re_i$ can be worked out as follows:

$$\Re_i = \left(\sum_{j=1}^{l_2} \delta_{ij} \, \Im^{tr}_{\mathbf{\Lambda}_{ij}} \right) \quad i = 1, \ldots, k$$

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \tag{13}$$

$$\begin{cases} \Re_1 = +\Im_1 - \Im_2 - \Im_3 \\ \Re_2 = +\Im_1 \mathbf{i}_m - \Im_2 \mathbf{i}_m - \Im_3 \mathbf{i}_m \\ \Re_3 = +\mathbf{i}'_n \Im_1 - \mathbf{i}'_n \Im_2 - \mathbf{i}'_n \Im_3 \\ \Re_4 = +\mathbf{i}'_n \Im_1 \mathbf{i}_m - \mathbf{i}'_n \Im_2 \mathbf{i}_m - \mathbf{i}'_n \Im_3 \mathbf{i}_m \end{cases}$$

where l_2 is equal to the maximum number of arrays $\Im_j$ used to work out each $\Re_i$ in (13), $\mathbf{\Lambda}_{ij}$ denotes the *ij*th element of the array $\mathbf{\Lambda}$, δ_{ij} has the same meaning as in (12), and the superscript *tr* indicates that the array $\Im_j$ has to be transformed as the *j*th element of ς of the *i*th macro-equation in (11b).

The sequential row vectorization of the *k* blocks yields γ''.

3. A Method for the Simultaneous Balancing of an Accounting System at Current and Constant Prices

In the literature, it is possible to find some methods to solve the problem of balancing an accounting system at simultaneously current and constant prices. They are essentially based on the Stone's balancing method described in Section 2, and use an *ad hoc* transformation for variables and the definition of a suitable matrix of variances and covariances to solve the main problem of the simultaneous balancing, which is related to the resolution of a nonlinear system of simultaneous equations (Weale, 1988).

For the sake of brevity and without losing in generality, only the first equation of the balancing system (11) is considered. Let $\mathbf{A}^c$, $\mathbf{B}^c$ and $\mathbf{C}^c$ indicate the accounting data blocks at current prices, $\mathbf{A}^k$, $\mathbf{B}^k$ and $\mathbf{C}^k$ denote the accounting data blocks at constant prices, and $\mathbf{A}^i$, $\mathbf{B}^i$ and $\mathbf{C}^i$ be three sets of appropriate price deflators for $\mathbf{A}^c$, $\mathbf{B}^c$ and $\mathbf{C}^c$. A suitable scheme to balance the above accounting data blocks simultaneously at current and constant prices is defined as follows:

$$\begin{cases} \mathbf{A}^c - \mathbf{B}^c - \mathbf{C}^c = \mathbf{0} \\ \mathbf{A}^k - \mathbf{B}^k - \mathbf{C}^k = \mathbf{0} \\ (\mathbf{A}^c \circ \mathbf{A}^i) - \mathbf{A}^k = \mathbf{0} \\ (\mathbf{B}^c \circ \mathbf{B}^i) - \mathbf{B}^k = \mathbf{0} \\ (\mathbf{C}^c \circ \mathbf{C}^i) - \mathbf{C}^k = \mathbf{0} \end{cases} \tag{14}$$

where $(\mathbf{A}^c \circ \mathbf{A}^i)$, $(\mathbf{B}^c \circ \mathbf{B}^i)$ and $(\mathbf{C}^c \circ \mathbf{C}^i)$ are Hadamard products. Clearly, the last three equations are nonlinear and the system cannot be balanced using the method described in Section 2.

This work delineates a new method to simultaneously balance an accounting system at current and constant prices, which differs from the others in the literature in the use of the predictor-corrector method as part of the balancing method for the nonlinear accounting system (14). As is well known, the predictor-corrector method is a mathematical algorithm, commonly used in numerical analysis, that proceeds in two steps to estimate an objective quantity: the predictor step, and the corrector step. In the prediction step a rough approximation of the quantity is obtained, while in the corrector step the initial approximation is refined using another means (Allgower and Georg, 2003; Luenberger, 1984).

In this work, the predictor-corrector method is used to define an iterative procedure that permits to obtain by successive approximations the simultaneous balancing at current and constant prices of the accounting system. In order to apply the predictor-corrector method, the balancing scheme (14) has been divided in three distinct groups of equations: the first group contains the equations to balance the accounting data at current prices (i.e. the first equation in (14)), the second group contains the equations to reconciliate the accounting data at current prices and constant prices (i.e. the last three equations in (14)), and the third group contains the equations to balance the accounting data at constant prices (i.e. the second equation in (14)).

The nonlinear equations of the second group above have successively been linearized by means of a logarithmic transformation. The iterative procedure for simultaneously balancing a general accounting system as (14) at current and constant prices can be defined as follows:

$$\begin{array}{ll} \left\{\mathbf{A}^c - \mathbf{B}^c - \mathbf{C}^c = \mathbf{0}\right. & (Step\,1) \\[1em] \left\{\begin{array}{l} \log\left(\mathbf{A}^c\right) + \log\left(\mathbf{A}^i\right) - \log\left(\mathbf{A}^k\right) = \mathbf{0} \\ \log\left(\mathbf{B}^c\right) + \log\left(\mathbf{B}^i\right) - \log\left(\mathbf{B}^k\right) = \mathbf{0} \\ \log\left(\mathbf{C}^c\right) + \log\left(\mathbf{C}^i\right) - \log\left(\mathbf{C}^k\right) = \mathbf{0} \end{array}\right. & (Step\,2) \\[1em] \left\{\mathbf{A}^k - \mathbf{B}^k - \mathbf{C}^k = \mathbf{0}\right. & (Step\,3) \end{array} \tag{15}$$

where each group of equations represents a linear system.

The general solution to the problem, therefore, lies in applying a suitable iterative method of estimation for *Steps 1-3* until a balanced accounting system is obtained, at both current and constant prices. At each step, the starting values are the balanced estimates worked out in the previous step. In the practical application of the iterative procedure, however, *Step 2* has to be utilized twice: the first time to obtain the estimates of the flows at constant prices to use in *Step 3*, the second to obtain the estimates of the flows at current prices to use in *Step 1* of the next iteration.

If the whole balancing process is iterated a sufficient number of times, the residuals of the accounting systems described in (15) tend to 0, and the simultaneous balancing of the accounting system at both current and constant prices can be obtained. From a practical point of view, the iterative balancing procedure is stopped when all the residuals of the three balancing systems in (15) are in absolute value less then a sufficiently small value ϵ. However, it can be noted that the residuals worked out through *Step 2* are smaller than those obtained from the other two steps. Therefore, the value of ϵ of *Step 2* has to be suitably smaller than that used for the stop condition in *Step 1* and *Step 3*.

A worthwhile aspect of this method is that, through the definition of a suitable matrix of variances for the deflators, it is able to control the variations induced in the price system by the balancing procedure. This aspect of the method is particularly useful when some deflators are already published and cannot be changed in the balancing process, as, for instance, the general consumer price index.

4. The Structure of Variance System

In the application of Stone's balancing procedure, a relevant aspect that needs to be faced is the calibration of the system of variances, and this is for two important reasons.

First, as pointed out in Section 2, the matrix $\mathbf{V}$ contains the information on the level of reliability of each initial estimate in $\hat{\mathbf{x}}$, and this is on the basis of those levels of reliability that residuals are apportioned within the system in a reasoned way. In balancing the accounting system, in fact, the initial estimates in $\hat{\mathbf{x}}$ that experience higher variances would be adjusted to a greater extent than those with lower variances, while the estimates with variances equal to 0 would not be changed at all.[4]

Second, the balancing procedure converges very slowly or does not converge at all when the system of variances is misspecified.

A relatively simple way to determine $\mathbf{V}$ is to provide each estimate in $\hat{\mathbf{x}}$ with a weight between 0 and 1 that expresses the relative degree of reliability of the estimates, and then multiply each estimate in $\hat{\mathbf{x}}$ by its own weight. Thus, the 0-weighting estimates will assume null variances, while the other estimates will assume values of variances according to the relative reliability of the different accounting items.

To apply the balancing method at both current and constant prices proposed in Section 3, three structures of variances have to be calculated, one for each step of (15). Moreover, it has to be noted that at each step of any iteration of the balancing procedure the values in each block of data change because of the balancing process itself, so that it is necessary to re-estimate the matrices of variances whenever new estimates of the accounting data are worked out.

In this work, to provide a suitable solution to accelerate the re-estimation process of the structure of variances, a structure of weights for each block of the estimates of the accounting data $\mathbf{A}^c$, $\mathbf{B}^c$, $\mathbf{C}^c$, $\mathbf{A}^k$, $\mathbf{B}^k$, $\mathbf{C}^k$, $\mathbf{A}^i$, $\mathbf{B}^i$ and $\mathbf{C}^i$ was worked out once at the beginning of the balancing process. Therefore, the new variances of each block of the balanced accounting data worked out at each step of (15) were calculated by multiplying the new estimates by the relative structure of weights as follows:

$$\begin{array}{lll} V\left(\mathbf{A}^c\right) = \mathbf{A}^c \circ \mathbf{P}\left(\mathbf{A}^c\right); & V\left(\mathbf{A}^k\right) = \mathbf{A}^k \circ \mathbf{P}\left(\mathbf{A}^k\right); & V\left(\mathbf{A}^i\right) = \mathbf{A}^i \circ \mathbf{P}\left(\mathbf{A}^c\right) \\ V\left(\mathbf{B}^c\right) = \mathbf{B}^c \circ \mathbf{P}\left(\mathbf{B}^c\right); & V\left(\mathbf{B}^k\right) = \mathbf{B}^k \circ \mathbf{P}\left(\mathbf{B}^k\right); & V\left(\mathbf{B}^i\right) = \mathbf{B}^i \circ \mathbf{P}\left(\mathbf{B}^c\right) \\ V\left(\mathbf{A}^c\right) = \mathbf{A}^c \circ \mathbf{P}\left(\mathbf{A}^c\right); & V\left(\mathbf{C}^k\right) = \mathbf{C}^k \circ \mathbf{P}\left(\mathbf{C}^k\right); & V\left(\mathbf{C}^i\right) = \mathbf{C}^i \circ \mathbf{P}\left(\mathbf{C}^c\right) \end{array} \tag{16}$$

where $\mathbf{P}\left(\cdot\right)$ represents the weights for the relative blocks of accounting data in the brackets.

A further problem that is faced within the balancing process is the correct definition of the variance matrices in *Step 2* of (15) because the variance matrices of the accounting data at current and constant prices have a different magnitude from that of the price index variance matrices. This involves a misspecification of the whole system of variances of *Step 2*, and, consequently, the balancing procedure converges very slowly or does not converge at all. In fact, the relative values of the implied deflators' variances with respect to the current and constant accounting flows' variances are very close to zero, and the implied deflators remain substantially unchanged in the balancing process, also when their variances are not nil in absolute value. To obtain the convergence of the balancing algorithm, an appropriate normalization of the system of variances is needed.

In this work, a normalization based on the ratio between the maximum log-values in each complete set of data, that is the current price and constant price data sets, and the implied deflator data set, has been applied. In particular, let $\mathbf{M}^c$, $\mathbf{M}^k$, $\mathbf{M}^i$ and $\mathbf{M}$ indicate respectively the whole set of data at current prices, the whole set of data at constant prices, the whole set of implied deflators and the data set encompassing all the previous data sets. A vector of three elements, say $\mathbf{v}^r$, is thus defined as follows:

$$\mathbf{v}^r = \left(\frac{{}_{ml}\mathbf{M}}{{}_{ml}\mathbf{M}^c}; \frac{{}_{ml}\mathbf{M}}{{}_{ml}\mathbf{M}^k}; \frac{{}_{ml}\mathbf{M}}{{}_{ml}\mathbf{M}^i}\right) \tag{17}$$

[4] It is important to recall that in this work the matrix **V** is diagonal, so that the covariances of the estimates are not considered.

where the subscript *ml* indicates the maximum log value of the corresponding data set. The three ratios in $\mathbf{v}^r$ are used in *Step 2* to normalize the variances of respectively $\mathbf{M}^c$, $\mathbf{M}^k$ and $\mathbf{M}^i$ as follows:[5]

$$\begin{aligned}&\bar{\mathbf{V}}\left(\log(\mathbf{M}^c)\right) = \mathbf{V}\left(\log(\mathbf{M}^c)\right)\,|v_1^r|\,;\bar{\mathbf{V}}\left(\log(\mathbf{M}^k)\right) = \mathbf{V}\left(\log(\mathbf{M}^k)\right)\,|v_2^r|\,;\\&\bar{\mathbf{V}}\left(\log(\mathbf{M}^i)\right) = \mathbf{V}\left(\log(\mathbf{M}^i)\right)\,|v_3^r|\end{aligned} \tag{18}$$

5. An application to the Italian 2005 SIOT

The simultaneous balancing procedure at current and constant prices was tested on the Italian 2005 SIOT at basic prices.[6] This matrix, published by the Italian National Institute of Statistics (ISTAT, hereafter), is compiled with the total flows of 59 branches of production at current prices only. An *ad hoc* estimate of the 2005 SIOT with 2000 as price-base year was, then, worked out. For this, where they were available, the flows at constant prices of the national accounts were used, or the price indices of both input and output also published by ISTAT. Table 1 shows the unbalanced 2005 SIOT at current and constant prices in an aggregated form, according to the $A6$ classification in 1995 ESA.

The contents of the table by rows are organized as follows: Output by branches at basic prices in rows 2-13; Intermediate consumption at basic prices (ICBP) in rows 14-15; Taxes less subsidies on products (NT) in rows 16-17; Value added at basic prices (VABP) in rows 18-19; Imports cif (ICIF) in rows 20-21; Total supply at basic prices (TSBP) in rows 22-23. By columns Table 1 is organized as follows: Input by branches at basic prices in columns 3-8; Intermediate uses at basic prices (IUBP) in column 9; Final consumption expenditure (FCE) in column 10; Gross capital formation (GCF) in column 11; Exports fob (EFOB) in column 12; Total uses at basic prices (TUBP) in column 13. Finally, the last column of the table reports the vector $\mathbf{G}\hat{\mathbf{x}}$ of residuals (Res.).

Table 2 contains the A6 classification codes (column 1) and the corresponding description (column 2), which are used in Tables 1 and 3.

The balancing procedure for the SIOT at current and constant prices was divided in 3 steps, as in (15), in each of which the block balancing method was used.

The system of balancing equations was defined according to the ESA recommendation in compiling and balancing national SUTs and IOTs (Eurostat, 2008).

For *Steps 1* and *3*, 7458 equations were utilized, with a vector $\hat{\mathbf{x}}$ of 11464 elements, while for *Step 2* the number of equations was 11464 with a vector $\hat{\mathbf{x}}$ of 34392 elements.

In the same iteration, as stop criterion of the iterative procedure for the accounting system at both current and constant prices the condition $\varepsilon = 0.1$ was placed for *Step 1* and *Step 3*, while for *Step 2* the stop criterion was $\varepsilon = 0.001$.

In accordance with the method described in Section 4, weights between 0 and 1 were attributed to the initial estimates, which were worked out as depicted above, by considering the different level of reliability of each item.[7] Specifically, the weights assigned to the estimates, which are all the items at current prices and some deflators, and are obtained directly from the national accounts, were lower than the weights assigned to the estimates

[5] One of the ratios in $\mathbf{v}^r$ will be equal to 1. This depends on the formula that is used to calculate the data at constant prices based on the data at current prices. If scheme (14) is used, the element equal to 1 is $\mathbf{v}_1^r$.

[6] All the data used in this work are from the online ISTAT database.

[7] The structure of weights depicted in this Section is the result of an accurate calibration process to obtain a highly significant estimate of the SIOT balanced flows, and reasonable Market share coefficients. Weights were fixed through a subjective method that considered both the method developed in this paper to work out the initial flow estimates and the previous experiences in the field of national accounting balancing.

Table 1: 2005 SIOT at current price (c) and price-base year 2000 (k). Unbalanced flows. Basic prices. Billions of Euro

Br.	A1	A2	A3	A4	A5	A6	IUBP	FCE	GCF	EFOB	TUBP	Res.
							Current prices					
A1	5.3	28.6	0.1	6.4	0.5	0.5	41.2	11.3	0.2	4.3	57.6	1.2
A2	8.2	442.0	49.1	112.6	22.3	28.6	663.6	220.3	98.7	265.3	1245.9	5.7
A3	0.2	8.7	17.0	12.7	8.2	4.6	52.1	7.3	128.1	1.0	187.1	3.8
A4	3.3	124.4	21.5	132.1	31.8	18.9	328.0	316.6	29.8	49.2	720.3	-6.0
A5	1.3	68.1	16.9	109.3	101.6	36.0	333.0	164.7	24.5	19.8	550.9	7.4
A6	0.2	10.2	1.9	9.4	4.2	23.5	49.7	341.7	1.1	1.2	394.9	5.9
ICBP	18.3	682.1	107.8	385.6	169.7	111.3	1499.8	1084.1	280.5	343.5	3209.4	-14.4
NT	0.5	10.0	1.6	10.2	6.5	9.5	38.1					
VABP	28.4	270.3	77.4	298.1	347.9	268.5	1314.7					
ICIF	9.8	290.7	1.1	34.4	24.1	2.2	362.3					
TSBP	56.4	1240.2	183.3	726.3	543.6	388.9	3223.8					
							2000 base-year prices					
A1	5.0	27.1	0.1	6.0	0.4	0.5	39.9	10.0	0.2	3.9	53.2	-1.5
A2	7.4	401.6	43.6	102.3	19.5	25.4	605.0	193.0	87.2	238.8	1115.9	-33.8
A3	0.2	7.5	14.6	11.4	6.9	3.9	43.7	6.4	113.1	0.9	160.4	3.9
A4	2.9	113.8	19.2	120.6	28.3	17.0	300.5	280.0	26.3	44.3	641.4	-9.0
A5	1.1	60.5	14.7	96.4	86.7	31.2	284.3	146.1	21.6	17.8	472.6	11.8
A6	0.2	8.9	1.7	8.3	3.5	20.2	42.0	301.3	1.0	1.1	341.5	8.3
ICBP	16.6	629.1	94.7	345.6	142.4	97.3	1320.5	945.2	247.7	309.1	2818.1	-37.8
NT	0.5	9.0	1.4	8.9	5.4	8.2	33.1					
VABP	29.8	248.2	61.1	263.0	292.5	226.3	1139.5					
ICIF	9.0	266.9	1.0	31.6	22.1	2.0	332.6					
TSBP	54.7	1149.6	156.5	650.4	460.8	333.2	2855.9					

Table 2: A6 NACE 1.1 Branch classification.

Code	Description
1	Agriculture, hunting and forestry; fishing and operation of fish hatcheries and fish farms
2	Industry, including energy
3	Construction
4	Wholesale and retail trade, repair of motor vehicles and household goods, hotels and restaurants; transport and communications
5	Financial, real-estate, renting and business activities
6	Other service activities

worked out by means of indirect methods, and are the estimates at constant prices and the estimates of the remaining implied deflators. Furthermore, the weights assigned to the totals were lower than the weights assigned to the other flows in the SIOT.

The matrices of variances were then obtained through the product of each estimate for the respective weight, as in (16). The vector of normalization $\mathbf{v}^r$ was subsequently calculated as depicted in Section 4.

A further problem in *Step 2* is related to the necessity of calculating the logarithms of the estimates in $\hat{\mathbf{x}}$. The problem is due to the fact that there may be negative values in the SIOT (e.g. Change in inventories), for which it is not possible to calculate the value of the logarithm. In this work, the criterion adopted was to consider the absolute value of those negative values and then inverted the operation in the relative balancing equation.

Table 3 reports the balanced 2005 SIOT at current and constant prices in an aggregated form as in Table 1.

Table 3: 2005 SIOT at current price (c) and price base year 2000 (k). Balanced flows. Basic prices. Billions of Euro.

Branches	A1	A2	A3	A4	A5	A6	IUBP	FCE	GCF	EFOB	TUBP
						Current prices					
A1	5.3	28.7	0.1	6.4	0.5	0.5	41.3	11.4	0.2	4.3	57.3
A2	8.3	444.7	49.2	114.0	22.5	28.7	667.5	222.0	98.9	266.3	1254.6
A3	0.2	8.8	17.3	13.0	8.4	4.7	52.4	7.5	125.2	1.0	186.1
A4	3.3	124.3	21.3	133.0	32.0	18.9	332.8	316.7	29.4	49.6	728.6
A5	1.3	68.9	17.0	111.3	102.9	36.5	338.0	167.8	24.1	19.8	549.7
A6	0.2	10.3	1.9	9.5	4.3	23.4	49.6	342.5	1.1	1.3	394.5
ICBP	18.5	685.8	106.7	387.1	170.6	112.8	1481.5	1067.9	279.0	342.2	3170.8
NT	0.5	9.8	1.6	10.0	6.4	9.4	37.7				
VABP	28.4	270.2	76.7	297.3	348.9	270.2	1291.7				
ICIF	9.8	288.8	1.0	34.1	23.8	2.2	359.8				
TSBP	57.3	1254.6	186.1	728.6	549.7	394.5	3170.8				
						2000 base-year prices					
A1	5.0	27.8	0.1	6.3	0.5	0.5	40.2	10.4	0.2	4.0	54.7
A2	7.3	408.4	44.5	104.8	20.0	26.2	611.3	197.5	90.2	240.8	1139.8
A3	0.2	7.1	14.3	10.7	6.5	3.7	42.5	6.5	108.9	0.9	158.7
A4	2.8	112.4	19.2	119.8	27.8	16.8	298.8	278.4	26.3	43.9	647.4
A5	1.0	58.8	14.7	93.4	84.7	31.1	283.7	145.7	21.0	17.4	468.0
A6	0.1	8.4	1.6	7.9	3.2	19.6	40.9	297.3	1.0	1.1	340.3
ICBP	16.4	622.9	94.4	342.9	142.7	98.0	1317.3	935.8	247.6	308.1	2808.8
NT	0.4	8.8	1.4	8.6	5.3	8.2	32.7				
VABP	29.3	245.4	61.8	264.1	297.8	232.0	1130.4				
ICIF	8.5	262.8	1.0	31.9	22.2	2.1	328.4				
TSBP	54.7	1139.8	158.7	647.4	468.0	340.3	2808.8				

6. Conclusions

In this paper, a suitable procedure to balance a general system of economic accounts simultaneously at both current and constant prices has been introduced. The balancing method was tested on the Italian 2005 SIOT at basic prices. Comparison of the balanced and non-balanced 59 branch SIOT shows that the simultaneous balancing procedure yields reasonable changes in the initial estimates from an economic and accounting point of view. In fact, the balancing system revalued the flows at current prices by an average of 3%. The balancing process also yielded greater modifications to the deflators and to the flows at constant prices, as was to be expected given the indirect methods used for determining the initial estimates. Nevertheless, by analyzing in detail the variations yielded by the balancing procedure, it can be noted that the greatest relative variations involved the flows with negligible values, mainly in the matrix of intermediate consumption and the vector of the change in inventories. In all the other cases, the relative variation was less than 10%. Furthermore, the convergence of the balancing procedure was very rapid: 43 iterations were sufficient to obtain the balancing of the accounting system. These results suggest that the procedure proposed in this paper is suitable to solve the still relevant problem of simultaneous balancing at current and constant prices of the national accounting systems. Further future developments could emerge mainly from the search for stronger conditions for the convergence of the iterative process, and also for more efficient criteria of recalculation and management of the variance matrices in the different steps of the procedure.

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